Fraunhofer vs Fresnel approximation

- Consider an incoming wave *f(x,y)* that propagates in free space (*d*)

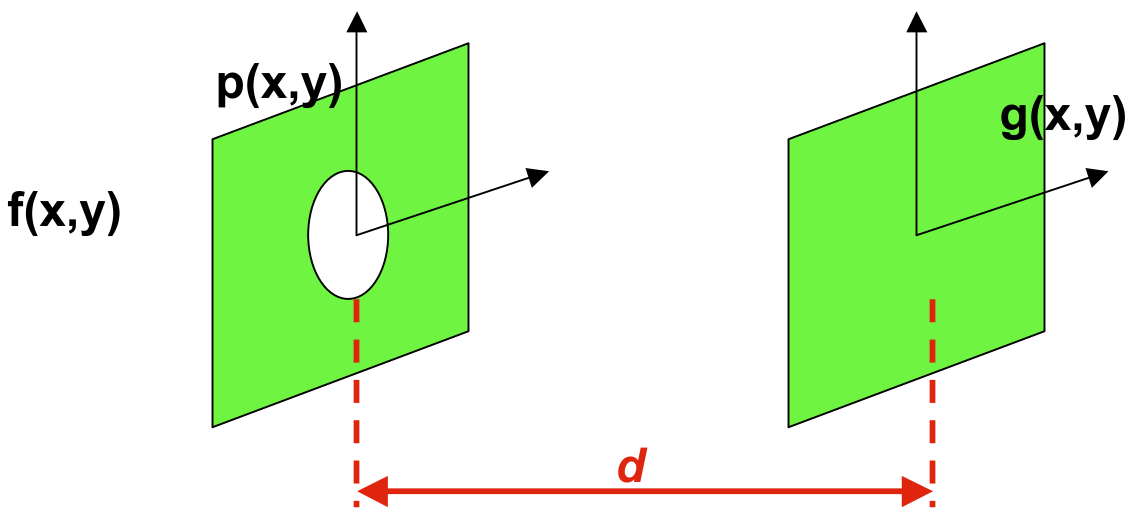
$$g(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx'dy' f(x',y')h(x-x',y-y')$$
$$= h_0 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx'dy' f(x',y')e^{-ik\frac{(x-x')^2+(y-y')^2}{2d}}$$

- Now assume the function is confined to $x'^2+y'^2 \ll b^2$
- We also consider *d* to be large enough so that $N'_F = b^2/(\lambda d)$ is small
- Then

$$g(x,y) = h_0 e^{-i\pi\frac{x^2+y^2}{\lambda d}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx'dy' f(x',y')e^{+2i\pi\frac{xx'+yy'}{\lambda d}}$$

Diffraction

- Consider an incoming wave *f(x,y)*
- The wave is intercepted by an aperture with transmission *p(x,y)* also called "pupil function"
- Then propagates a distance *d*
- The final complex amplitude of the wave is *g(x,y)*



Fraunhofer diffraction

- In Fraunhofer diffraction, the complex wave amplitude downstream of the aperture is computed using the Fraunhofer approximation

$$g(x,y) = h_0 e^{-i\pi\frac{x^2+y^2}{\lambda d}} F\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)$$

- This is valid if the Fresnel number $N'_F = b^2/(\lambda d)$ is <<1

- Consider an incoming plane wave $U(x,y) = I_i^{1/2}$

- Downstream of the aperture with transmission *p(x,y)* we have

$$f(x,y) = I_i^{1/2} p(x,y)$$

- And after a drift of length *d*

$$g(x,y) = I_i^{1/2} h_0 P\left(\frac{x}{\lambda d}, \frac{x}{\lambda d}\right)$$ where $$P(\nu_x,\nu_y) = \int\int_{-\infty}^{+\infty} dxdy p(x,y) e^{2i\pi[\nu_x x+\nu_y y]}.$$

Fraunhofer diffraction: rectangular aperture I

- Consider an incoming wave intercepted by a rectangular aperture of size D_x and D_y. What is the intensity of the diffraction pattern?

- We have

$$p(x,y) = \begin{cases} 1 \text{ if } -D_x/2 < x < D_x/2 \text{ and } -D_y/2 < x < D_y/2 \\ 0 \text{ elsewhere} \end{cases}$$

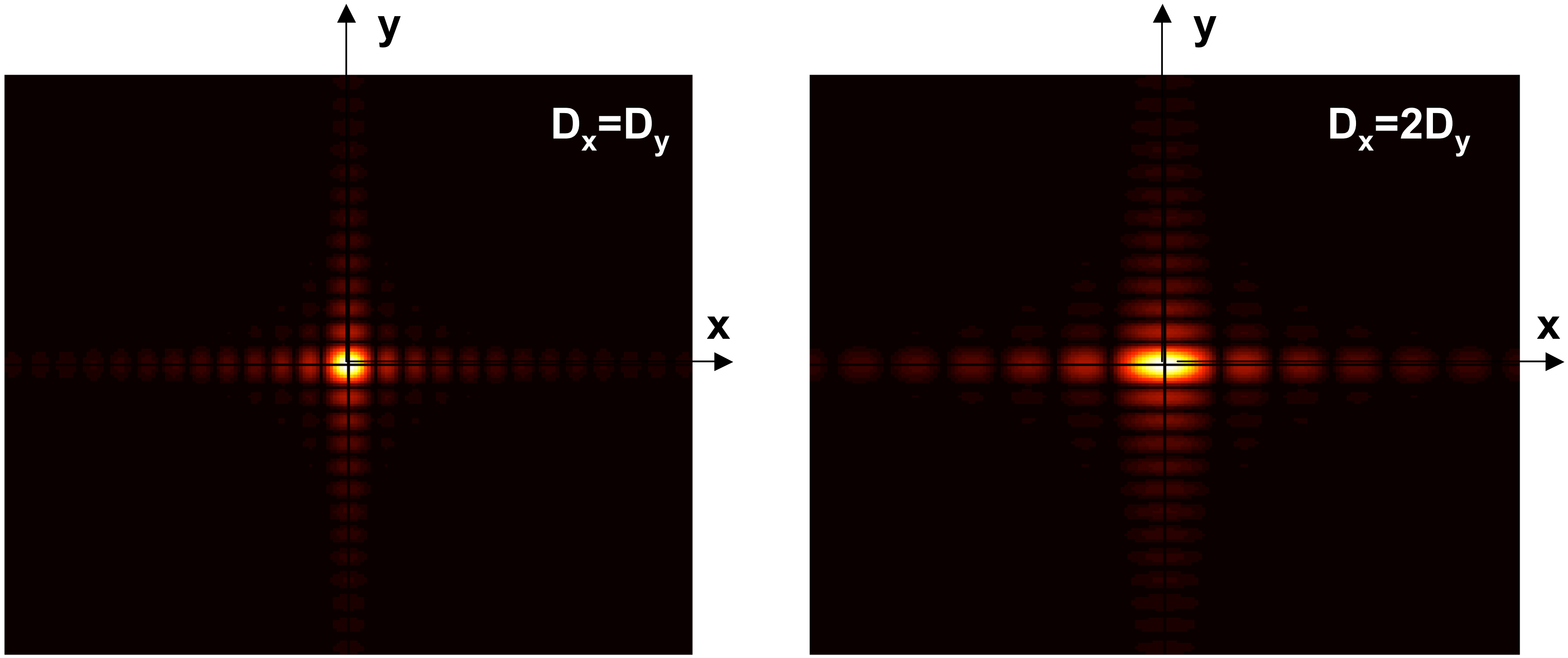
- Using the previous slides

$$\begin{aligned}
P(\nu_x,\nu_y) &= \int_{-D_x/2}^{+D_x/2} \int_{-D_y/2}^{+D_y/2} e^{2i\pi[\nu_x x + \nu_y y]} dx dy \\
&= \int_{-D_x/2}^{+D_x/2} e^{2i\pi\nu_x x} dx \int_{-D_y/2}^{+D_y/2} e^{2i\pi\nu_y y} dy \\
&= \int_{0}^{+D_x/2} 2\cos(2\pi\nu_x x) dx \int_{0}^{+D_y/2} 2\cos(2\pi\nu_y y) dy \\
&= 4 \left[\frac{\sin(2\pi\nu_x x)}{2\pi\nu_x}\right]_0^{D_x/2} \left[\frac{\sin(2\pi\nu_y y)}{2\pi\nu_y}\right]_0^{D_y/2} \\
&= D_x D_y \frac{\sin(\pi\nu_x D_x)}{\pi\nu_x D_x} \frac{\sin(2\pi\nu_y D_y)}{\pi\nu_y D_y}
\end{aligned}$$

Fraunhofer diffraction: rectangular aperture II

- We finally obtain

$$I(x,y) = I_0 \mathrm{sinc}^2\left(\frac{\pi D_x x}{\lambda d}\right)\mathrm{sinc}^2\left(\frac{\pi D_y y}{\lambda d}\right)$$



Fraunhofer diffraction: circular aperture I

- Now we take a circular aperture of radius a

$$p(x,y)=\begin{cases}1 \text{ if } x^2+y^2<a\\ 0 \text{ elsewhere}\end{cases}.$$

- The Fourier transform of the transmission function is

$$\begin{aligned} P(\nu_x,\nu_y) &= \int_{y^-(x)}^{y^+(x)}\int_{-\infty}^{+\infty} e^{2i\pi[\nu_x x+\nu_y y]}dxdy \\ &= \int_0^a\int_0^{2\pi} e^{2i\pi[\nu_x r\cos\theta+\nu_y r\sin\theta]}rdrd\theta \end{aligned}$$

- Here we change to cylindrical coordinates because of the cylindrical symmetry. Introducing

$$\nu_\rho \equiv \sqrt{\nu_x^2+\nu_y^2}$$

$$\alpha \equiv \arctan(\nu_y/\nu_x)$$

Fraunhofer diffraction: circular aperture II

- We can write

$$
\begin{aligned}
P(\nu_\rho) &= \int\int e^{2i\pi r\nu_\rho[\cos\theta\cos\alpha+\sin\theta\cos\alpha]} r dr d\theta \\
&= \int\int e^{2i\pi r\nu_\rho \cos\theta'} r dr d\theta' \\
&= \frac{1}{2\pi}\int_0^a r dr\, J_0(2\pi\nu_\rho r)
\end{aligned}
$$

- So finally

$$
\begin{aligned}
P(\nu_\rho) &= \frac{1}{2\pi\nu_\rho^2}\int_0^{2\pi\nu_\rho a} \rho d\rho J_0(\rho) \\
&= \frac{1}{2\pi\nu_\rho^2} 2\pi\nu_\rho a J_1(2\pi\nu_\rho a)
\end{aligned}
$$

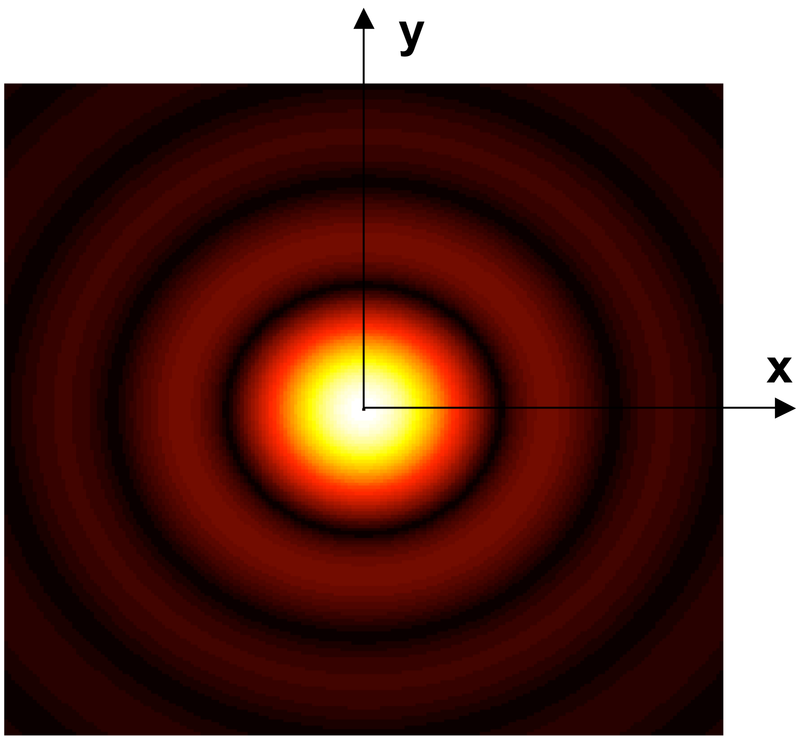
And the diffraction pattern intensity is

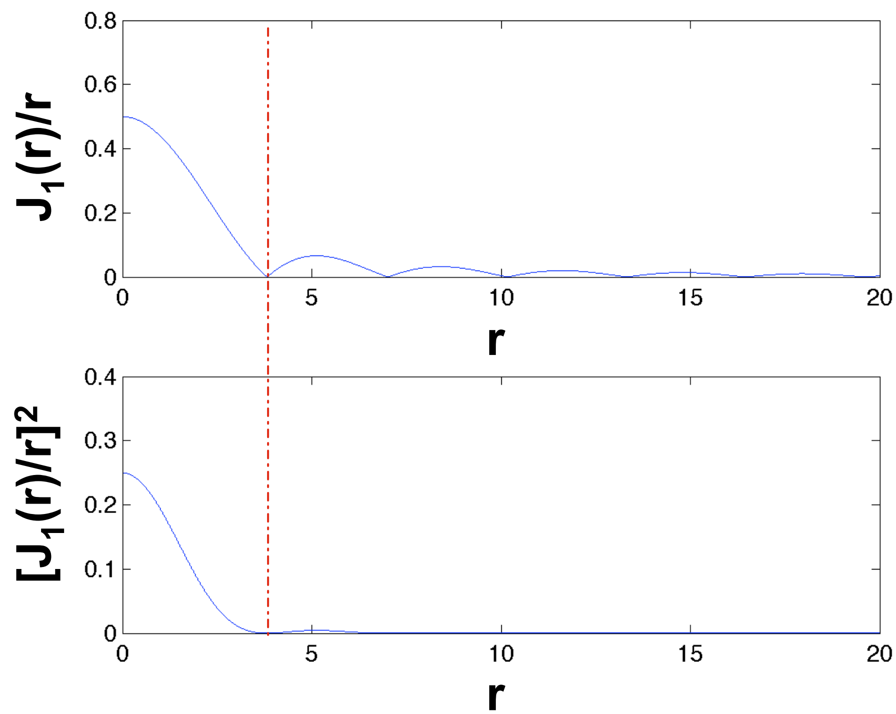
$$
I(\rho) = I_0 \left[\frac{J_1\left(\frac{2\pi\rho a}{\lambda d}\right)}{\frac{2\pi\rho a}{\lambda d}} \right]^2
$$

Fraunhofer diffraction: circular aperture III

- Diffraction pattern intensity

$$I(\rho) = I_0 \left[\frac{J_1\left(\frac{2\pi\rho a}{\lambda d}\right)}{\frac{2\pi\rho a}{\lambda d}} \right]^2$$





Fresnel diffraction I

- In Fresnel diffraction, the complex wave amplitude downstream of the aperture is computed using the Fresnel approximation

$$
\begin{aligned}
g(x,y) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx'dy' f(x',y')h(x-x',y-y') \\
&= h_0 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx'dy' f(x',y') e^{-ik\frac{(x-x')^2+(y-y')^2}{2d}}
\end{aligned}
$$

- The intensity is given by

$$
\begin{aligned}
I(x,y) &= |g(x,y)|^2 \\
&= \frac{I_i}{(\lambda d)^2}\left|\int\int p(x',y') e^{-i\pi\frac{(x-x')^2+(y-y')^2}{\lambda d}} dx'dy'\right|^2
\end{aligned}
$$

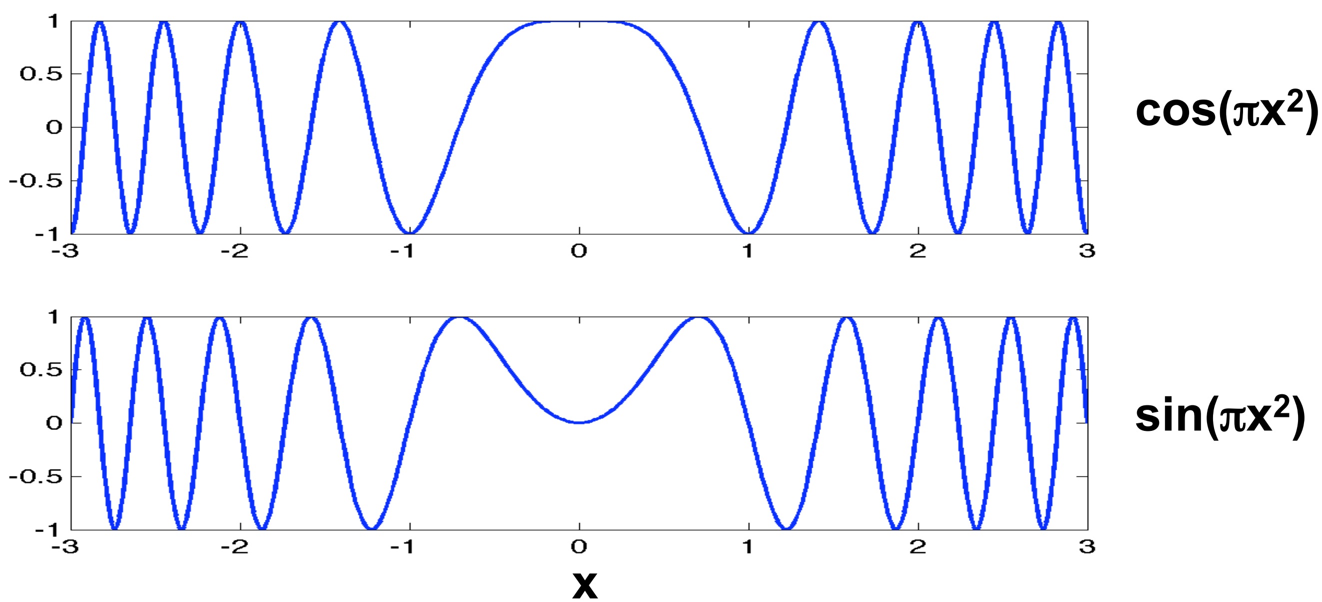
Fresnel diffraction II

- Written in a normalized coordinate system

size ~ $a/[\lambda d]^{1/2}$

$$I(X,Y) = \frac{I_i}{(\lambda d)^2}\left|\int\int p(X',Y')e^{-i\pi(X-X')^2+(Y-Y')^2}dX'dY'\right|^2$$

- This is the convolution of the transmission function *p(X,Y)* of the considered aperture with the function $\exp -i\pi(X^2+Y^2)$



Fresnel diffraction III

- In the equation

size ~ $a/[\lambda d]^{1/2}$

$$I(X,Y) = \frac{I_i}{(\lambda d)^2} \left| \int\int p(X',Y') e^{-i\pi(X-X')^2+(Y-Y')^2} dX'dY' \right|^2$$

The result of this convolution is governed by the Fresnel number $N_F=a^2/(\lambda d)$

- If N_F is large the convolution is going to yield a function similar to $p(X,Y)$.
- In the limit $N_F \rightarrow \infty$, ray optics is applicable ($\lambda \rightarrow 0$) and the pattern is the shadow of the aperture
- In the opposite limit Fresnel diffraction pattern converge to the Fraunhofer pattern.

Fresnel diffraction: slit aperture

- Consider a slit infinitely long in the y-direction then

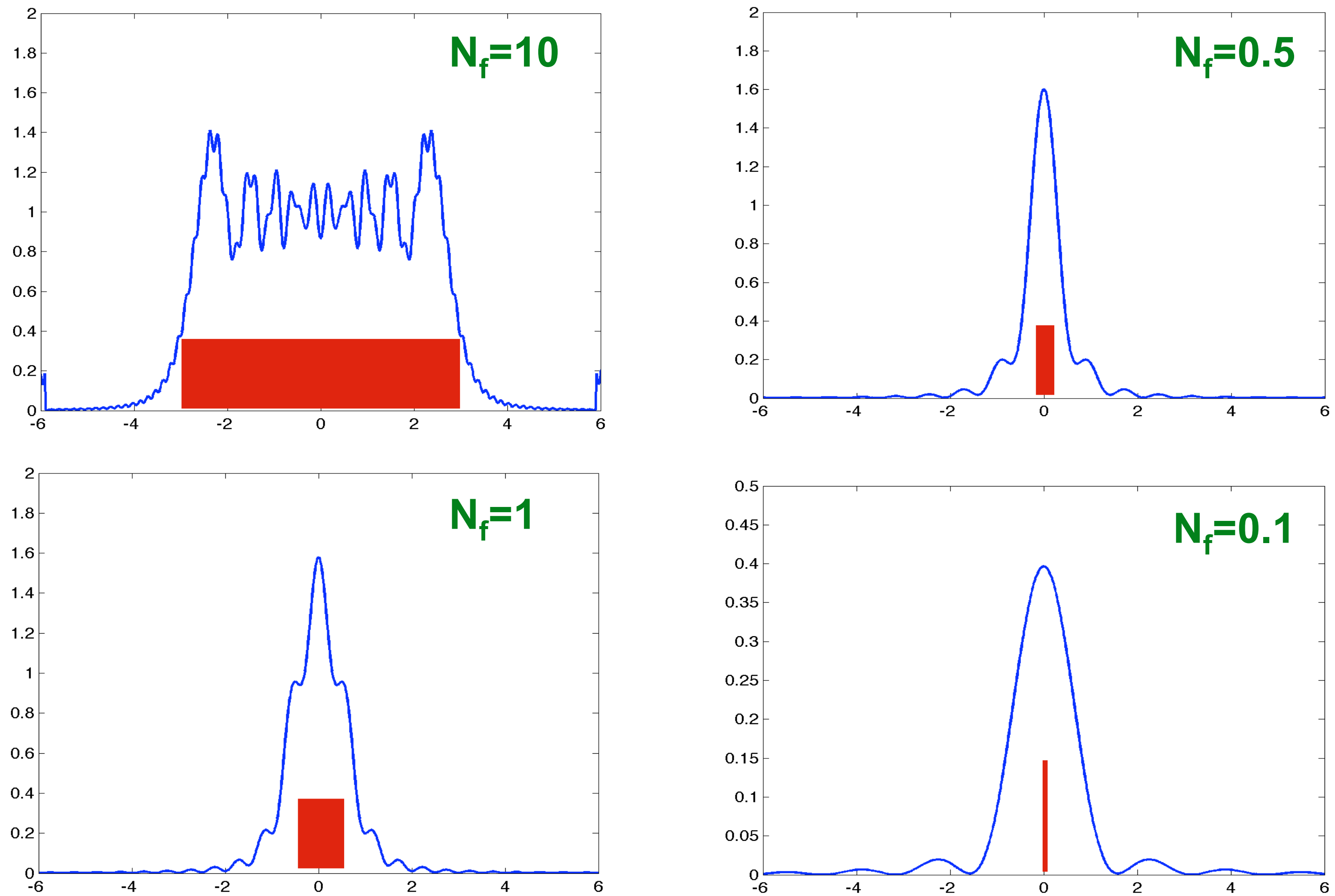
$$p(x,y) = \begin{cases} 1 \text{ if } |X| < \frac{a}{\lambda d} = N_F^{1/2} \\ 0 \text{ elsewhere} \end{cases}$$

- From $I(X,Y) = I_i |g(X)|^2$ we need to compute g(X)

$$\begin{aligned} g(X) &= \int_{-\sqrt{N_f}}^{+\sqrt{N_f}} e^{-i\pi(X-X')^2} dX' \\ &= \int_{X-\sqrt{N_f}}^{X+\sqrt{N_f}} e^{-i\pi(X')^2} dX' \end{aligned}$$

Fresnel diffraction: slit aperture II

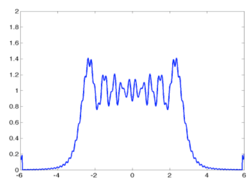
- Fresnel patterns for different Fresnel number



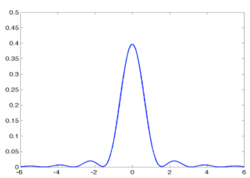
Summary

- In the order of increasing distance from the aperture, diffraction pattern is

  - A **shadow** of the aperture.



  - A **Fresnel** diffraction pattern, which is a convolution ot the "normalized" aperture function with $\exp[-i\pi(X^2+Y^2)]$.



  - A **Fraunhofer** diffraction pattern, which is the squared-absolute value of the Fourier transform of the aperture function. The far field has an angular divergence proportional to λ/D where D is the diameter of the aperture.