Helmholtz Equation

- Consider the function U to be complex and of the form:

$$U(\vec{r},t) = U(\vec{r})\exp(2\pi\upsilon t)$$

- Then the wave equation reduces to

$$\nabla^2 U(\vec{r}) + k^2 U(\vec{r}) = 0$$

where

$$k \equiv \frac{2\pi\upsilon}{c} = \frac{\omega}{c}$$

Helmholtz equation

Plane wave

- The wave

$$U(\overrightarrow{r}) = Ae^{i\overrightarrow{k}.\overrightarrow{r}}$$

  is a solution of the Helmholtz equations.

- Consider the wavefront, e.g., the points located at a constant phase, usually defined as phase=$2\pi q$.

- For the present case the wavefronts are decribed by $\overrightarrow{k}.\overrightarrow{r} = 2\pi q$

  which are equation of planes separated by λ.

- The optical intensity is proportional to $|U|^2$ and is $|A|^2$ (a constant)

Spherical and paraboloidal waves

- A spherical wave is described by

$$U(r) = \frac{A}{r} e^{ikr}$$

and is solution of the Helmholtz equation.

- In spherical coordinate, the Laplacian is given by

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

- The wavefront are spherical shells $kr = 2\pi q$

- Considering $r \simeq z + \frac{x^2+y^2}{2z}$ give the paraboloidal wave:

$$U(r) = \frac{A}{z} e^{-ikz} \; e^{-ik\frac{x^2+y^2}{2z}}.$$

The paraxial Helmholtz equation

- Start with Helmholtz equation

$$[\nabla^2 + k^2]U = 0.$$

- Consider the wave

Complex amplitude

Complex envelope

$$U(\overrightarrow{r}) = A(\overrightarrow{r})e^{-ikz},$$

which is a plane wave (propagating along z) transversely modulated by the complex "amplitude" A.

- Assume the modulation is a slowly varying function of z (slowly here mean slow compared to the wavelength)
- A variation of A can be written as

$$\delta A = \frac{\partial A}{\partial z}\delta z \ll A \qquad \delta z \sim \lambda$$

- So that

$$\frac{\partial A}{\partial z} \ll A/\lambda \sim kA$$

The paraxial Helmholtz equation

- So $$\frac{\partial^2 A}{\partial^2 z} \ll k\frac{\partial A}{\partial z} \ll k^2 A$$
- Expand the Laplacian

$$\nabla^2 = [\nabla_\perp^2 + \partial_z^2]A(\overrightarrow{r})e^{ikz}$$

Transverse Laplacian

- The longitudinal derivative is

$$\partial_z^2[A(\overrightarrow{r})e^{-ikz}] = [\partial_z^2 A - 2ik\partial_z A - k^2 A]e^{-ikz}$$

- Plug back in Helmholtz equation

$$\nabla_\perp^2[A(\overrightarrow{r})e^{-ikz}] + \cancel{\partial_z^2 A(\overrightarrow{r})} - 2ik\partial_z A(\overrightarrow{r}) - k^2 A]e^{-ikz} + k^2 A(\overrightarrow{r})e^{-ikz} = 0$$

- Which finally gives the paraxial Helmholtz equation (PHE):

$$\nabla_\perp^2 A(\overrightarrow{r}) - 2ik\partial_z A(\overrightarrow{r}) = 0$$

Gaussian Beams I

- The paraboloid wave is solution of the PHE

$$A(\overrightarrow{r}) = \frac{A_0}{z} e^{-ik\frac{x^2+y^2}{2z}}.$$

- Doing the change $z \rightarrow q(z) = z - \xi$ shifted paraboloid wave (which is still a solution of PHE)

- If ξ complex, the wave is of Gaussian type and we write

$$q(z) = z + iz_0$$

where z_0 is the Rayleigh range

- We also introduce $\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{i\lambda}{\pi W^2(z)}$

Wavefront curvature

Beam width

Gaussian Beams II

- R and W can be related to z and z_0:

$$R(z) = z + \frac{z^2}{z_0^2} = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

$$W^2(z) \ = \ \frac{\lambda z_0}{\pi}\left[1 + \left(\frac{z}{z_0}\right)^2\right] \equiv W_0^2(z)\left[1 + \left(\frac{z}{z_0}\right)^2\right]$$

Gaussian Beams III

- Expliciting A in U gives

$$
\begin{aligned}
U(\vec{r}) &= \frac{A_0}{q(z)} e^{-ikz} e^{-ik\frac{\rho^2}{2q(z)}} \\
&= A_0 \left[\frac{1}{R(z)} - \frac{i\lambda}{\pi W(z)^2}\right] e^{-\frac{\rho^2}{W(z)^2}} e^{-ikz-ik\frac{\rho^2}{2R(z)}} \\
&= A_0 \left[\frac{1}{z[1+(z_0/z)^2]} - \frac{i}{z_0[1+(z/z_0)^2]}\right] e^{-\frac{\rho^2}{W(z)^2}} e^{-ikz-ik\frac{\rho^2}{2R(z)}} \\
&= A_0 \left[\frac{z}{z^2+z_0^2} - \frac{iz_0}{z^2+z_0^2}\right] e^{-\frac{\rho^2}{W(z)^2}} e^{-ikz-ik\frac{\rho^2}{2R(z)}} \\
&= \frac{A_0}{\sqrt{z^2+z_0^2}} \frac{z-iz_0}{\sqrt{z^2+z_0^2}} e^{-\frac{\rho^2}{W(z)^2}} e^{-ikz-ik\frac{\rho^2}{2R(z)}} \\
&= \frac{A_0}{iz_0\sqrt{1+(z/z_0)^2}} \frac{iz+z_0}{\sqrt{z^2+z_0^2}} e^{-\frac{\rho^2}{W(z)^2}} e^{-ikz-ik\frac{\rho^2}{2R(z)}}
\end{aligned}
$$

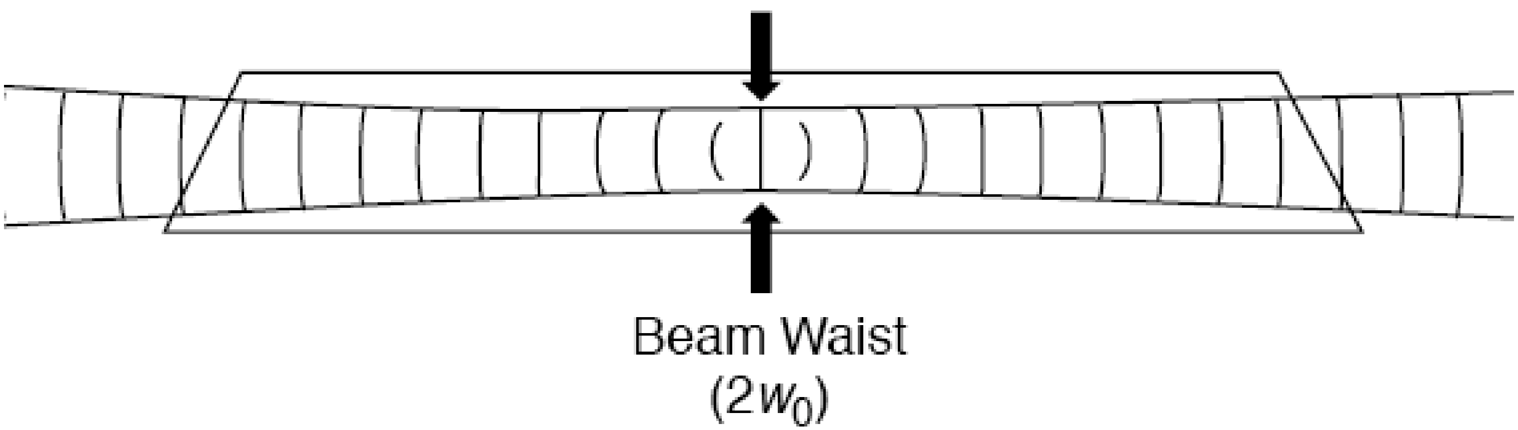
Gaussian Beams IV

- Introducing the phase $\zeta \equiv atan(z/z_0)$ finally get

$$U(\vec{r}) = A_1 \frac{W_0}{W(z)} e^{-\frac{\rho^2}{W(z)^2}} e^{-ikz - ik\frac{\rho^2}{2R(z)} + i\zeta(z)}$$

where $A_1 \equiv A_0/(iz_0)$

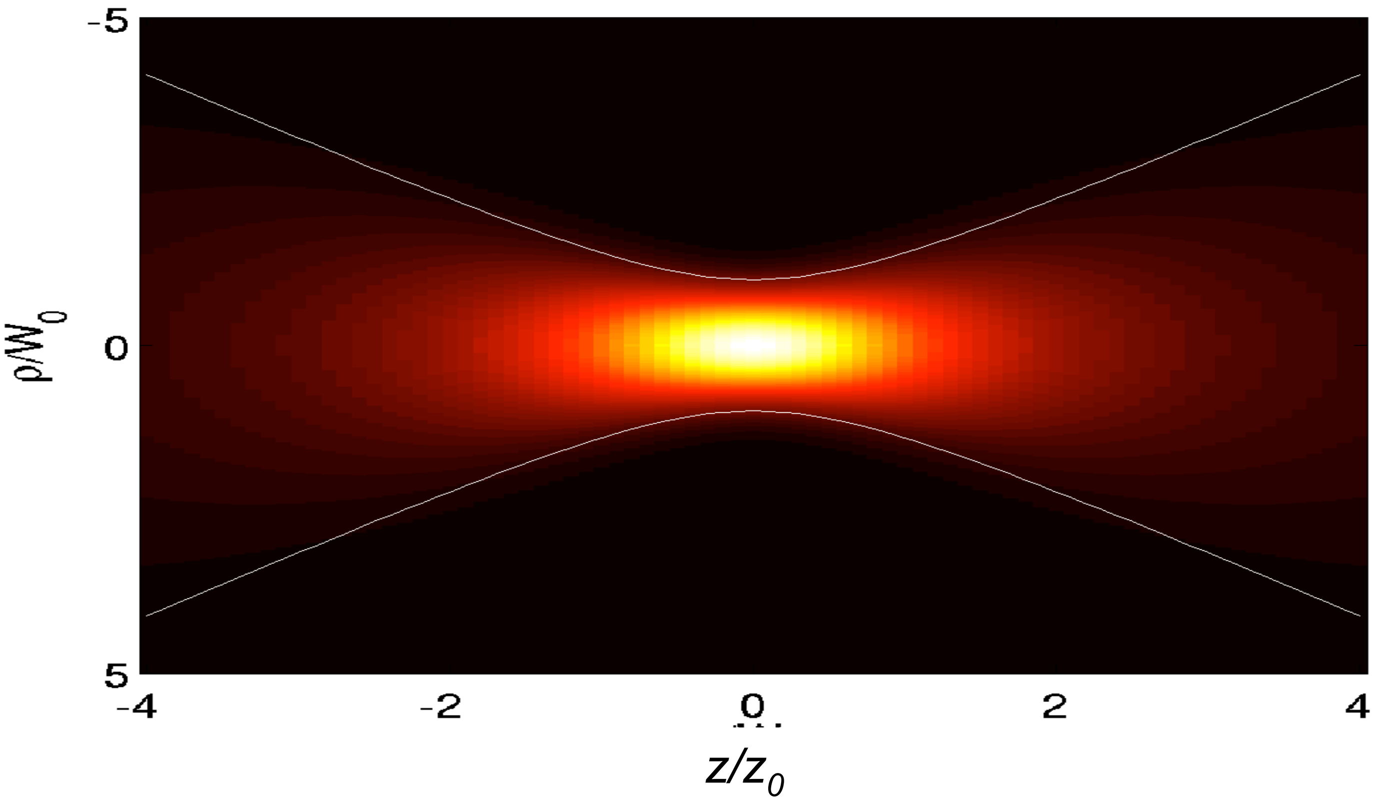
- This equation describes a Gaussian beam.



Intensity distribution of a Gaussian Beam

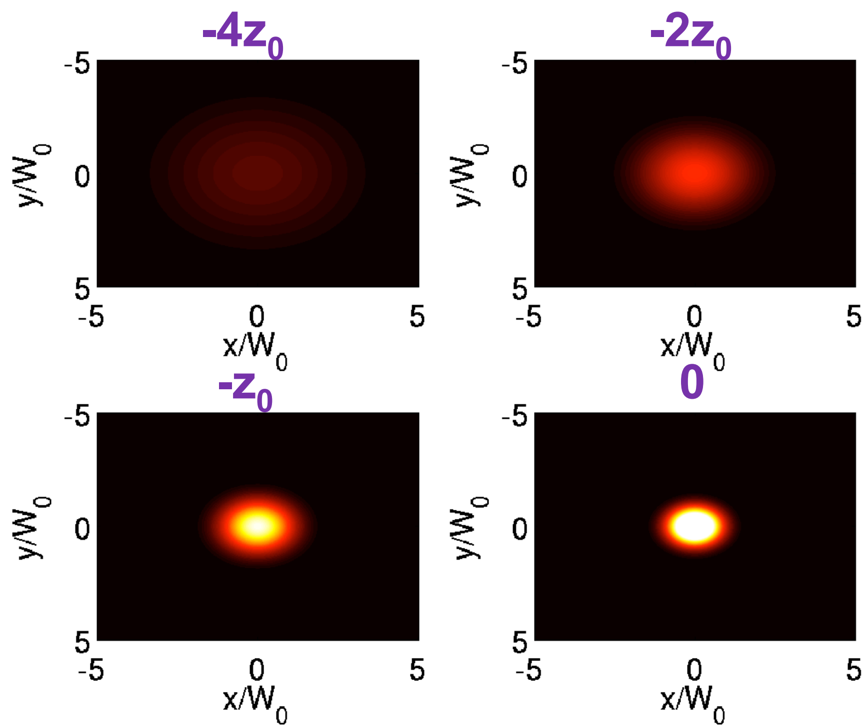
- The optical intensity is given by

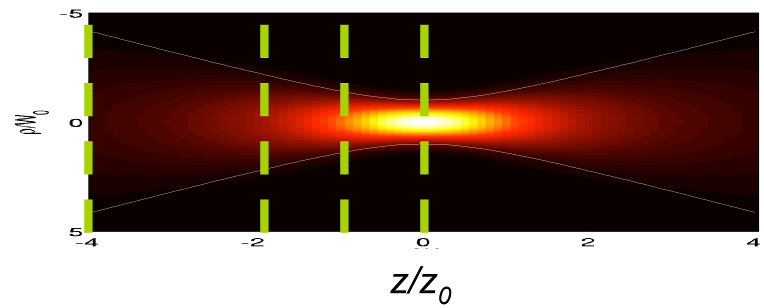
$$I(\overrightarrow{r}) = |U(\overrightarrow{r}|^2 = I_0 \left(\frac{W_0}{W(z)} \right)^2 e^{-\frac{2\rho^2}{W(z)^2}}$$

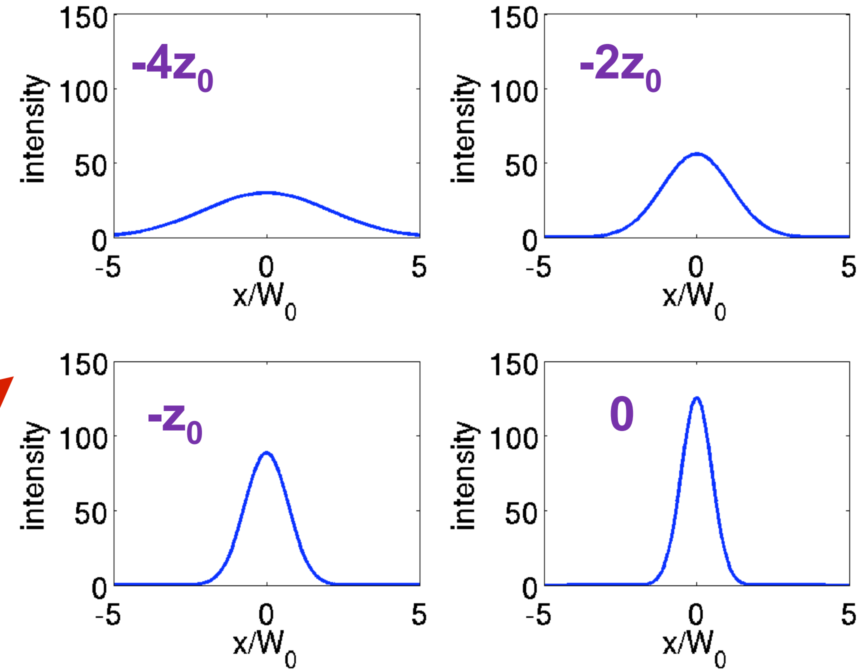


Intensity distribution

- Transverse intensity distribution at different z locations





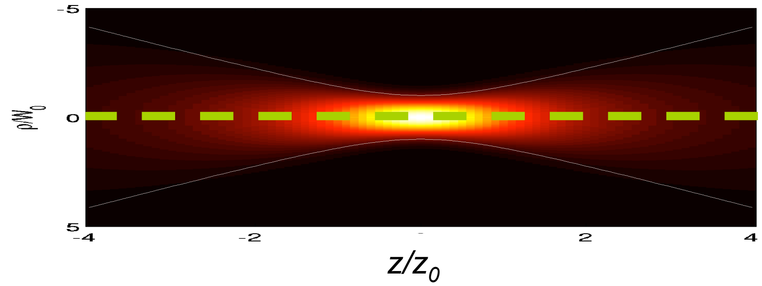


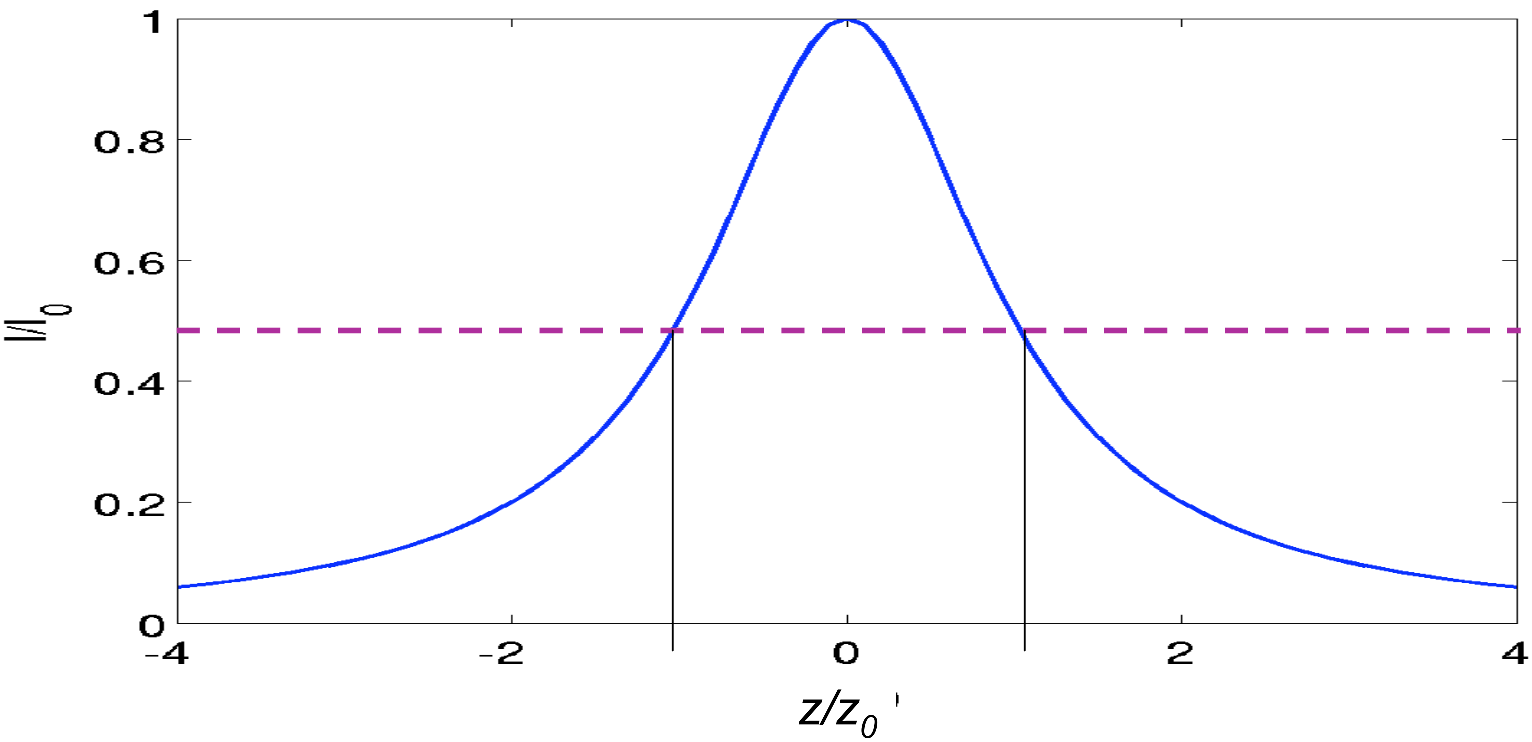
- Corresponding “profiles”

Intensity distribution (cnt'd)

- On-axis intensity as a function of z is given by

$$I(r=0,z) = I_0\left(\frac{W_0}{W(z)}\right)^2 = \frac{I_0}{1+(z/z_0)^2}$$

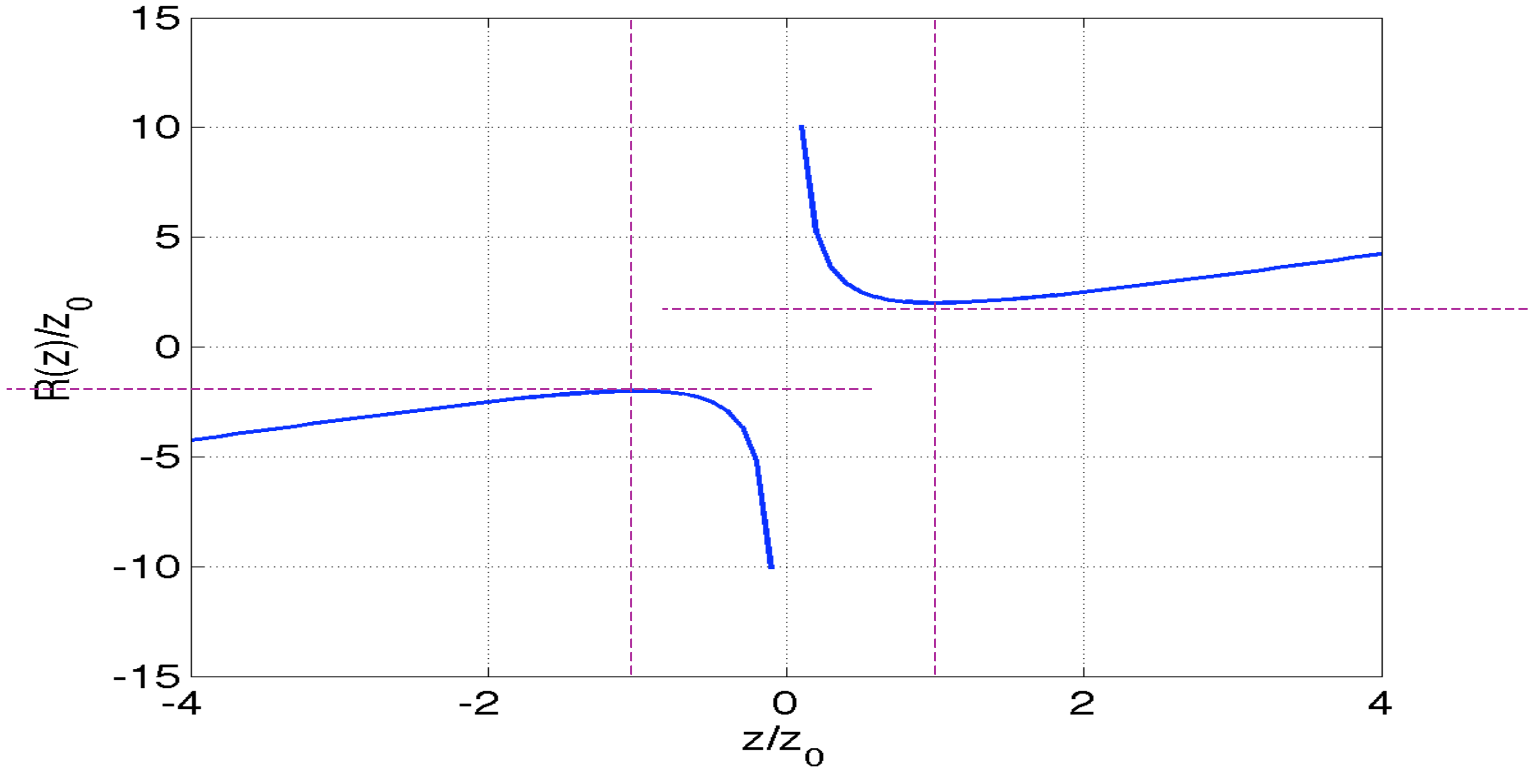




Wavefront radius

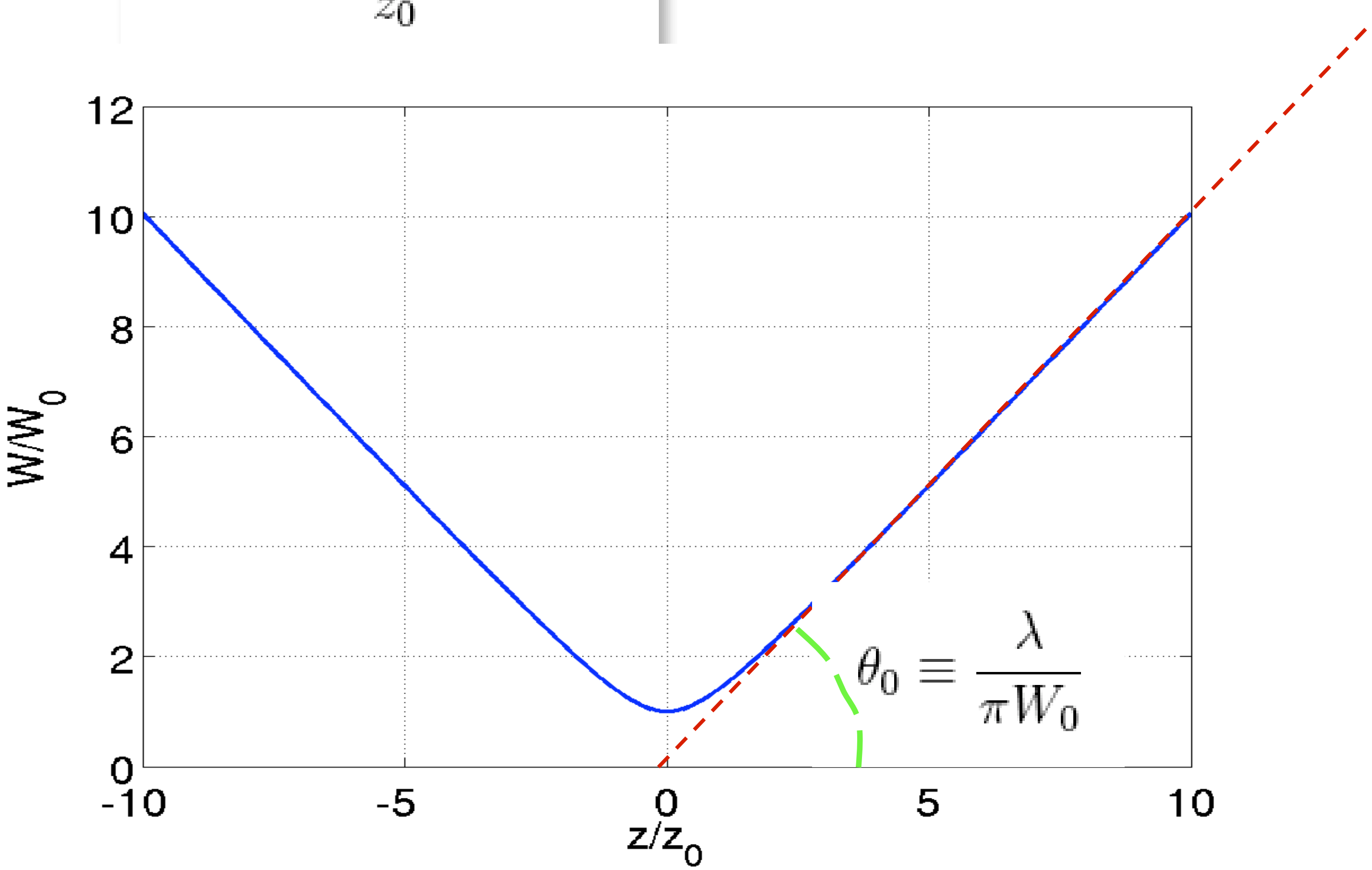
- The curvature of the wavefront is given by

$$R(z) = z + \frac{z^2}{z_0^2} = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$



Beam width and divergence

- Beam width is given by $W^2(z) = \frac{\lambda z_0}{\pi}\left[1+\left(\frac{z}{z_0}\right)^2\right] \equiv W_0^2(z)\left[1+\left(\frac{z}{z_0}\right)^2\right]$
- For large z $W(z) \simeq \frac{W_0}{z_0} z \equiv \theta_0 z$

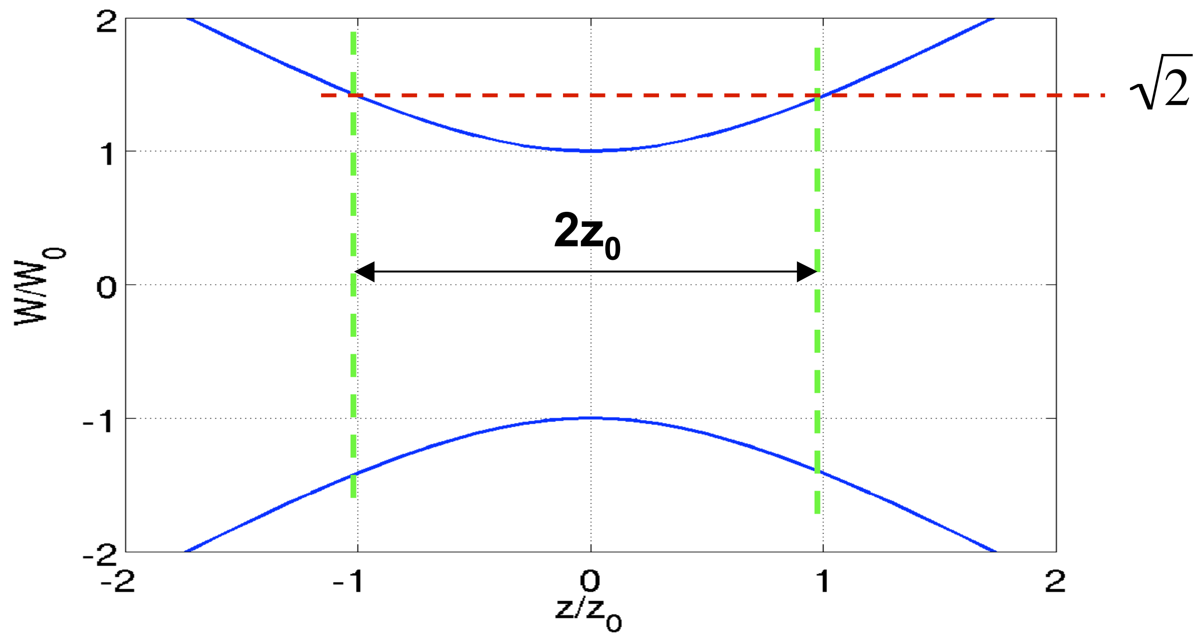


$$\theta_0 \equiv \frac{\lambda}{\pi W_0}$$

Depth of focus

- A depth of focus can be defined from the Rayleigh range

$$2z_0 = \frac{2\pi W_0^2}{\lambda}.$$



Phase

- The argument as three terms

$$\varphi(\overrightarrow{r}) = kz - \zeta(z) + \frac{k\rho^2}{2R(z)}$$

Phase associated to plane wave

Guoy phase shift

Spherical distortion of the wavefront

- On axis (ρ=0) the phase still has the "Guoy shift"

$$\varphi(\overrightarrow{r}) = kz - \zeta(z)$$

Varies from $-\pi/2$ to $+\pi/2$

- At z_0 the Guoy shift is $\pi/4$

Summary

- At z_0
  - Beam radius is sqrt(2) the waist radius
  - On-axis intensity is 1/2 of intensity at waist location
  - The phase on beam axis is retarded by $\pi/4$ compared to a plane wave
  - The radius of curvature is the smallest.

- Near beam waist
  - The beam may be approximated by a plane wave (phase ~kz).

- Far from the beam wait
  - The beam behaves like a spherical wave (except for the phase excess introduced by the Guoy phase)