Index ellipsoid I

- The energy density associated to an electromagnetic wave is

$$U = \frac{1}{2} \mathbf{E} \cdot \mathbf{D}$$

- Remember that

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

- So the energy density is

$$U = \frac{1}{2} \left[ \frac{D_x^2}{\varepsilon_0 n_x^2} + \frac{D_y^2}{\varepsilon_0 n_y^2} + \frac{D_z^2}{\varepsilon_0 n_z^2} \right]$$

- Which can be rewritten

$$\frac{\left( \frac{D_x}{\sqrt{2\varepsilon_0 U}} \right)^2}{n_x^2} + \frac{\left( \frac{D_y}{\sqrt{2\varepsilon_0 U}} \right)^2}{n_y^2} + \frac{\left( \frac{D_z}{\sqrt{2\varepsilon_0 U}} \right)^2}{n_z^2} = 1$$

$$\frac{\alpha^2}{n_x^2} + \frac{\beta^2}{n_y^2} + \frac{\gamma^2}{n_z^2} = 1$$

This defines an ellipsoid
Index ellipsoid II

• The ellipsoid in $(\alpha, \beta, \gamma)$ intersect the axis at

\[
\alpha = \pm n_x, \quad \beta = \pm n_y, \quad \gamma = \pm n_z
\]

is called
  – Index ellipsoid
  – Optical indicatrix
  – Ellipsoid of wave normals

• Correspondence between $(\alpha, \beta, \gamma)$ and $(x, y, z)$?
A wave propagating along the x-axis $\Rightarrow$ D vector is in $(y, z)$ plane
$\Rightarrow$ $Dx=0 \Rightarrow \alpha=0$
similarly a wave propagating along y (resp. z) $\Rightarrow \beta$ (resp. $\gamma$) =0
Propagation along a principal axis

- Nothing “exciting”: a linearly-polarized plane wave with E-field aligned with one of the principal axis $i$ propagates with phase velocity $c/n_i$. 
Propagation along an arbitrary direction

- If the wave travels in the crystal with an arbitrary direction, the normal modes associated to the wave are linearly polarized.

- These modes can be found from the index ellipsoid

  - Start with \( \mathbf{k} \times \mathbf{H} = -\omega \mathbf{D} \)
    \( \mathbf{k} \times \mathbf{E} = \omega \mu_0 \mathbf{H} \)

  - Combining gives \( \mathbf{k} \times (\mathbf{k} \times \epsilon^{-1} \mathbf{D}) = -\omega^2 \mu_0 \mathbf{D} \)

  - Finally \( -\mathbf{u} \times (\mathbf{u} \times \eta \mathbf{D}) = \frac{1}{n^2} \mathbf{D} \) with \( \eta = \epsilon_0 \epsilon^{-1} \)

Can be viewed as the projection of \( \eta \mathbf{D} \) on a plane orthogonal to \( \mathbf{u} \)
Index ellipsoid

• How to find the
  – 1: given the propagation direction of the ray draw the corresponding ray in the \((\alpha, \beta, \gamma)\) plane.
  – 2: draw a plane normal to the ray and containing the origin
  – 3: the intersection of the plane with the index ellipsoid gives an ellipse
  – 4: the principal axis of this ellipse give the direction of the D vector of the two linear eigenvectors. The length of the semi-minor and semi-major axis gives the index of refraction along the eigen polarizations.
Uniaxial crystals

- A wave traveling with angle $\theta$ wrt the ordinary axis give an index ellipse of the form

$$\frac{1}{n^2(\theta)} = \frac{\cos^2 \theta}{n_o^2} + \frac{\sin^2 \theta}{n_e^2}$$

- Normal modes have refractive indexes $n_o$ and $n_e(\theta)$

- Ordinary wave has index $n_o$ regardless of $\theta$ and $E || D$

- Extraordinary wave mode has refractive index $n_e(\theta)$ and $E$ and $D$ are not generally parallel

P. Piot, PHYS 630 – Fall 2008
Vector directions

- Optical wave characterized by $k$, $E$, $D$, $H$, and $B$ with power flow given by $S$.

$$\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^*$$

$S \perp E$ and $H$

$$\vec{k} \times \vec{H} = -\omega \vec{D}$$

$D \perp k$ and $H$

$$\vec{k} \times \vec{E} = \omega \mu \vec{H}$$

$H \perp k$ and $E$

- $D$, $E$, $k$, $S$ are in the same plane $\perp$ to $(B$ and $H)$
Disperssion relation

• From Maxwell’s equations

\[ \overrightarrow{k} \times (\overrightarrow{k} \times \overrightarrow{E}) - \mu \varepsilon \omega^2 \overrightarrow{E} = 0 \]
\[ M \overrightarrow{E} = 0 \]

• Where the matrix M is

\[ M = \begin{bmatrix}
  n_1^2k_0^2 - k_2^2 - k_3^2 & k_1k_2 & k_1k_3 \\
  k_2k_1 & n_2^2k_0^2 - k_1^2 - k_3^2 & k_2k_3 \\
  k_3k_1 & k_3k_2 & n_3^2k_0^2 - k_1^2 - k_2^2
\end{bmatrix} \]

• Non trivial solution is determinant of M vanishes i.e.

\[ |M| = 0 \Rightarrow \omega(\overrightarrow{k}) \]

• \( \omega(k) \) is the dispersion relation
k-surface

- $\omega(k)$ generally describes a centro-symmetric surface comprising two sheets

- The group velocity is $\vec{v} = \nabla_k \omega(\vec{k})$

- The Poynting vector is $||$ to the group velocity
Special case of uniaxial crystal

• Take $n_1 = n_2 = n_o$ and $n_3 = n_e$ the equation for the $k$-surface becomes

\[
\begin{align*}
(k^2 - n_0 k_0^2) \left( \frac{k_1^2 + k_2^2}{n^2_e} + \frac{k_3^2}{n^2_o} - k^2_o \right) &= 0
\end{align*}
\]
Double refraction

- Explore refraction at the interface of a isotropic and anisotropic media
- Snell’s law applies
  \[ \sin \theta_1 = n(\theta_a + \theta) \sin \theta \]

  where \( \theta_a + \theta \) is the angle of the refracted wave with respect to the crystal optical axis.
- Case of a uni-axial crystal two refracted waves
  \[
  \sin \theta_1 = n_0 \sin \theta_0 \\
  \sin \theta_1 = n(\theta_a + \theta_e) \sin \theta_e
  \]
Polarizer

- Linearly polarized light beam can be produced via multiple reflection
- Or using anisotropic crystal arranged as a Wollaston prism
- Glan-Foucault prism
Waveplate

• Waveplate can be used to manipulate the polarization of an incoming wave

• The phase shift between the two optical directions is

\[ \Gamma = \frac{2\pi \Delta n L}{\lambda} \]

• The corresponding Jones matrix is

\[
T = \begin{bmatrix}
1 & 0 \\
0 & e^{-i\Gamma}
\end{bmatrix}
\]