Index ellipsoid I

- The energy density associated to an electromagnetic wave is $U = \frac{1}{2}\mathbf{E}\cdot\mathbf{D}$
- Remember that

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \epsilon_o \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

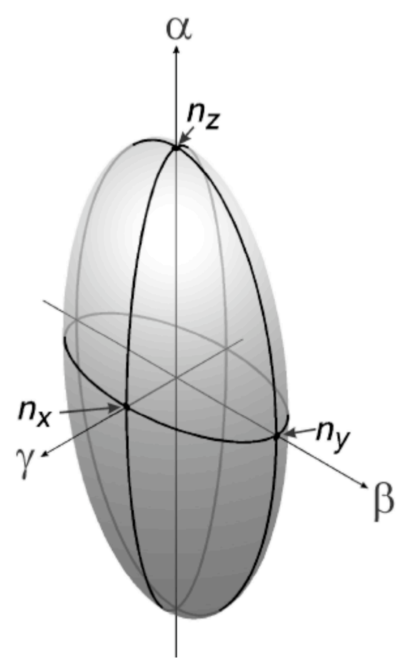
- So the energy density is

$$U = \frac{1}{2}\left[\frac{D_x^2}{\epsilon_o n_x^2} + \frac{D_y^2}{\epsilon_o n_y^2} + \frac{D_z^2}{\epsilon_o n_z^2}\right]$$

- Which can be rewritten

$$\frac{\left(\frac{D_x}{\sqrt{2\epsilon_o U}}\right)^2}{n_x^2} + \frac{\left(\frac{D_y}{\sqrt{2\epsilon_o U}}\right)^2}{n_y^2} + \frac{\left(\frac{D_z}{\sqrt{2\epsilon_o U}}\right)^2}{n_z^2} = 1$$

$$\frac{\alpha^2}{n_x^2} + \frac{\beta^2}{n_y^2} + \frac{\gamma^2}{n_z^2} = 1$$



This defines an ellipsoid

Index ellipsoid II

- The ellipsoid in (α,β,γ) intersect the axis at

$$\alpha = \pm n_x,\ \beta = \pm n_y,\ \gamma = \pm n_z$$

  is called
  - Index ellispoid
  - Optical indicatrix
  - Ellipsoid of wave normals

- Correspondence between (α,β,γ) and (x,y,z)?
  A wave propagating along the x-axis $\Rightarrow$ D vector is in (y,z) plane
  $\Rightarrow$ Dx=0 $\Rightarrow$ α=0
  similarly a wave propagating along y (resp. z) $\Rightarrow$ β (resp. γ) =0

Propagation along a principal axis

- Nothing “exciting”: a linearly-polarized plane wave with E-field aligned with one of the principal axis *i* propagates with phase velocity c/n_i.

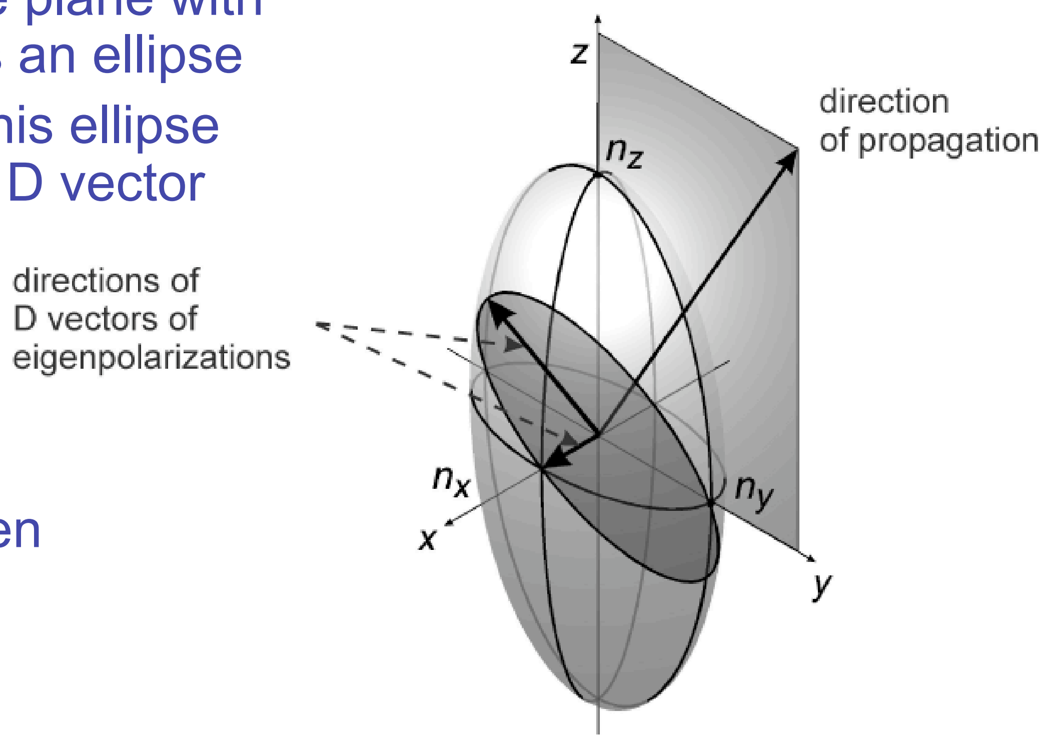
Propagation along an arbitrary direction

- If the wave travels in the crystal with an arbitrary direction, the normal modes associated to the wave are linearly polarized.
- These modes can be found from the index ellipsoid
- Start with
$$\begin{aligned}\vec{k} \times \vec{H} &= -\omega \vec{D} \\ \vec{k} \times \vec{E} &= \omega \mu_0 \vec{H}\end{aligned}$$
- Combining gives $\vec{k} \times (\vec{k} \times \epsilon^{-1} \vec{D}) = -\omega^2 \mu_0 \vec{D}$
- Finally $-\hat{u} \times (\hat{u} \times \eta \vec{D}) = \frac{1}{n^2} \vec{D}$ with $\eta = \epsilon_0 \epsilon^{-1}$

Can be viewed as the projection of ηD on a plane orthogonal to *u*

Index ellipsoid

- How to find the
  - 1: given the propagation direction of the ray draw the corresponding ray in the (α,β,γ) plane.
  - 2: draw a plane normal to the ray and containing the origin
  - 3: the intersection of the plane with the index ellipsoid gives an ellipse
  - 4: the principal axis of this ellipse give the direction of the D vector of the two linear eigenvectors. The length of the semi-minor and semi-major axis gives the index of refraction along the eigen polarizations.



Uniaxial crystals

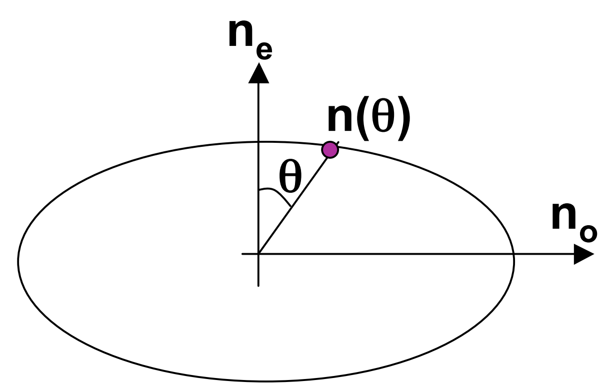
- A wave traveling with angle θ wrt the ordinary axis give an index ellipse of the form

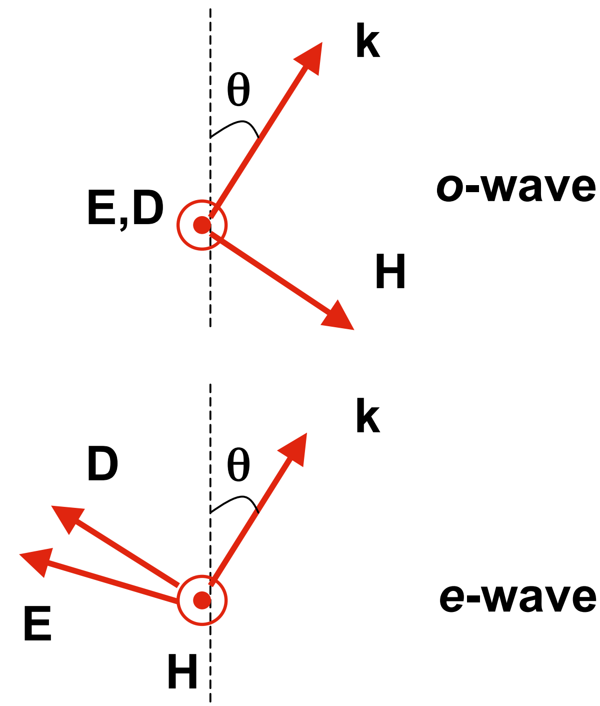
$$\frac{1}{n^2(\theta)} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

- Normal modes have refractive indexes n_o and $n_e(\theta)$

- Ordinary wave has index n_o regardless of θ and E||D

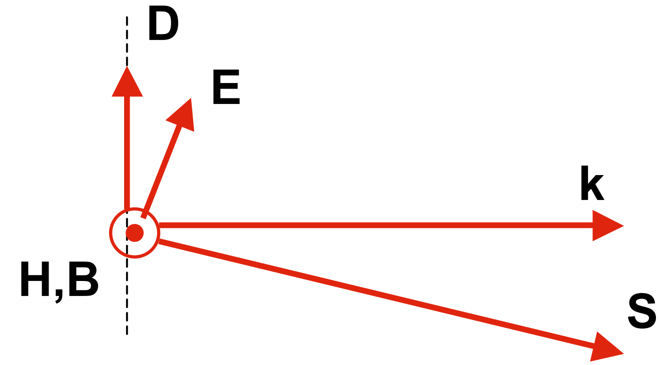
- Extraordinary wave mode has refractive index $n_e(\theta)$ and E and D are not generally parallel





Vector directions

- Optical wave characterized by k, E, D, H, and B with power flow given by S.
- $$\overrightarrow{S} = \frac{1}{2}\overrightarrow{E} \times \overrightarrow{H}^*$$
  S ⊥ E and H
- $$\overrightarrow{k} \times \overrightarrow{H} = -\omega\overrightarrow{D}$$
  D ⊥ k and H
- $$\overrightarrow{k} \times \overrightarrow{E} = \omega\mu\overrightarrow{H}$$
  H ⊥ k and E



- D, E, k, S are in the same plane ⊥ to (B and H)

Dispersion relation

- From Maxwell's equations

$$\begin{aligned} \overrightarrow{k} \times (\overrightarrow{k} \times \overrightarrow{E}) - \mu\epsilon\omega^2 \overrightarrow{E} &= 0 \\ M\overrightarrow{E} &= 0 \end{aligned}$$

- Where the matrix M is

$$M = \begin{bmatrix} n_1^2 k_0^2 - k_2^2 - k_3^2 & k_1 k_2 & k_1 k_3 \\ k_2 k_1 & n_2^2 k_0^2 - k_1^2 - k_3^2 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & n_3^2 k_0^2 - k_1^2 - k_2^2 \end{bmatrix}$$

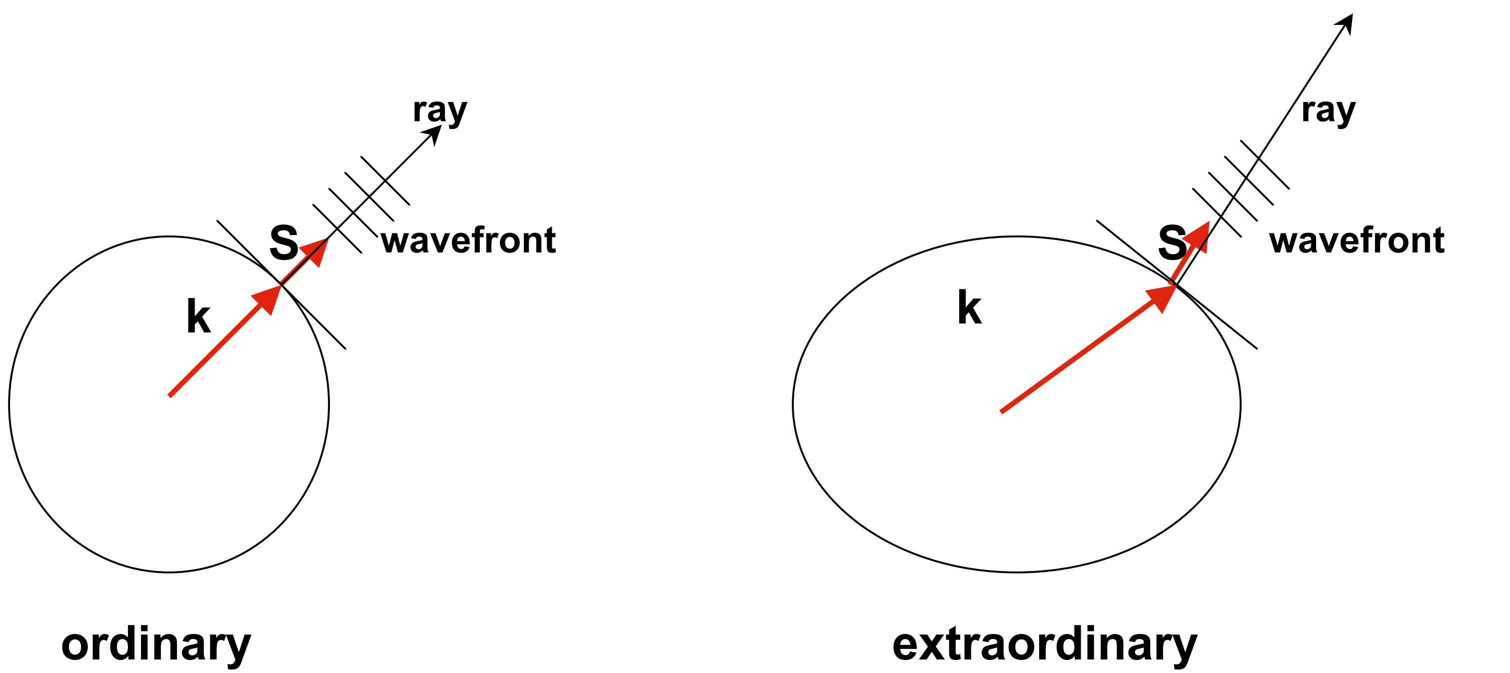
- Non trivial solution is determinant of M vanishes i.e.

$$|M| = 0 \Rightarrow \omega(\overrightarrow{k})$$

- $\omega(k)$ is the dispersion relation

k-surface

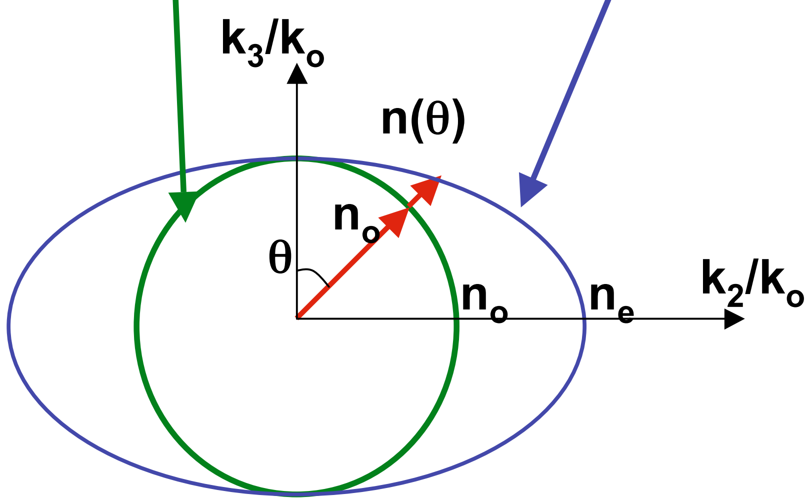
- $\omega(k)$ generally describes a centro-symmetric surface comprising two sheets
- The group velocity is $\vec{v} = \vec{\nabla}_k \omega(\vec{k})$
- The Poynting vector is || to the group velocity



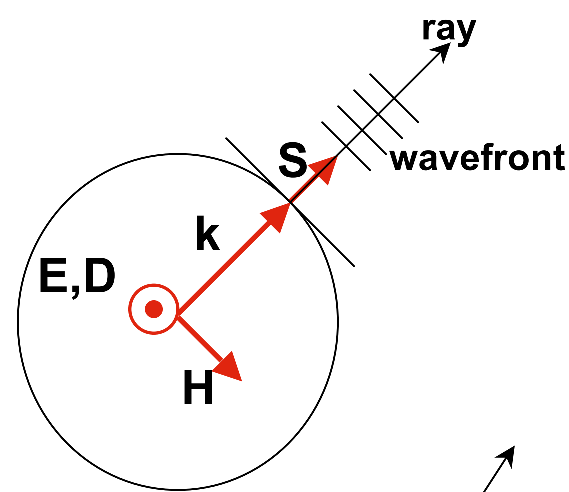
Special case of uniaxial crystal

- Take $n_1=n_2=n_o$ and $n_3=n_e$ the equation for the *k*-surface becomes

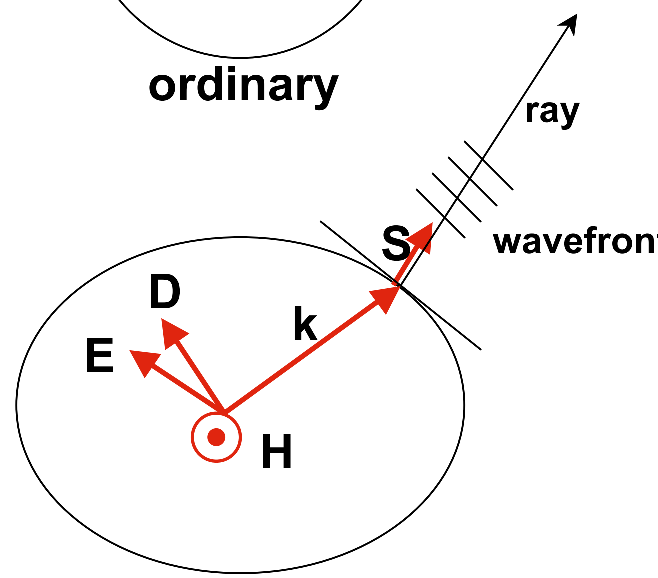
$$(k^2 - n_0 k_0^2)\left(\frac{k_1^2+k_2^2}{n_e^2}+\frac{k_3^2}{n_o^2}-k_o^2\right)=0$$



in y-z plane



ordinary



extraordinary

Double refraction

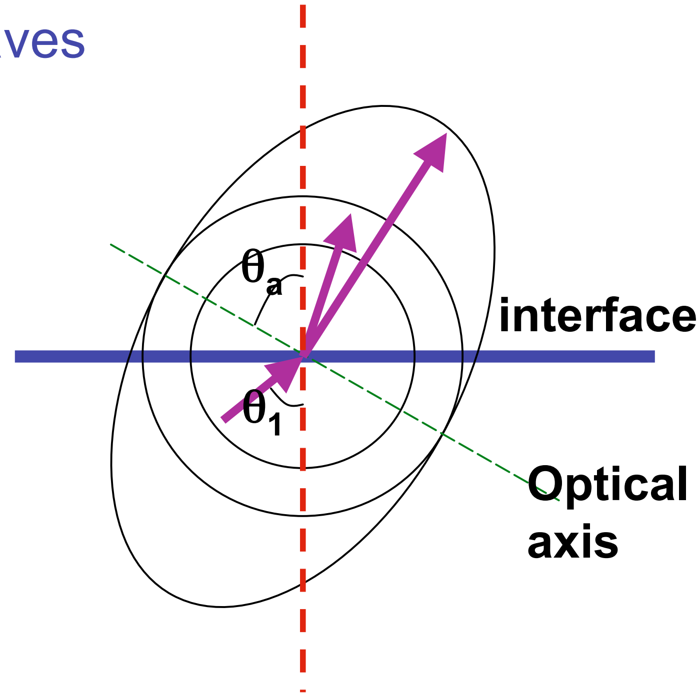
- Explore refraction at the interface of a iso-tropic and anisotropic media
- Snell's law applies

$$\sin\theta_1 = n(\theta_a + \theta)\sin\theta$$

where $\theta_a+\theta$ is the angle of the refracted wave with respect to the crystal optical axis.

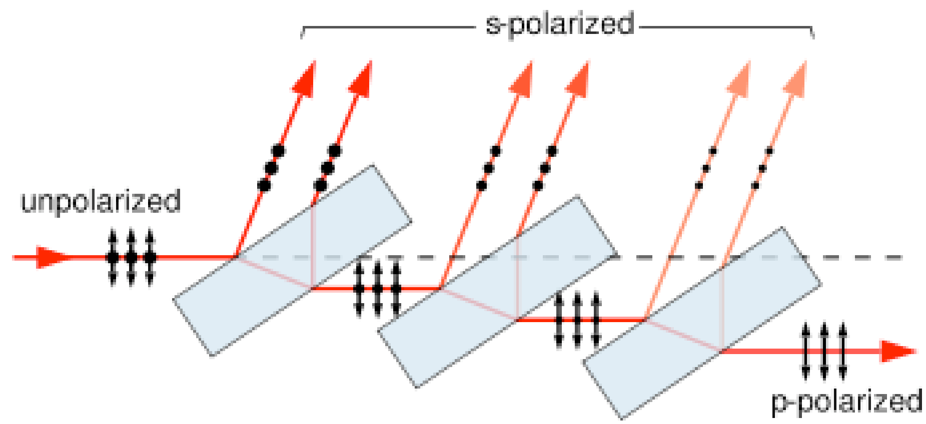
- Case of a uni-axial crystal two refracted waves

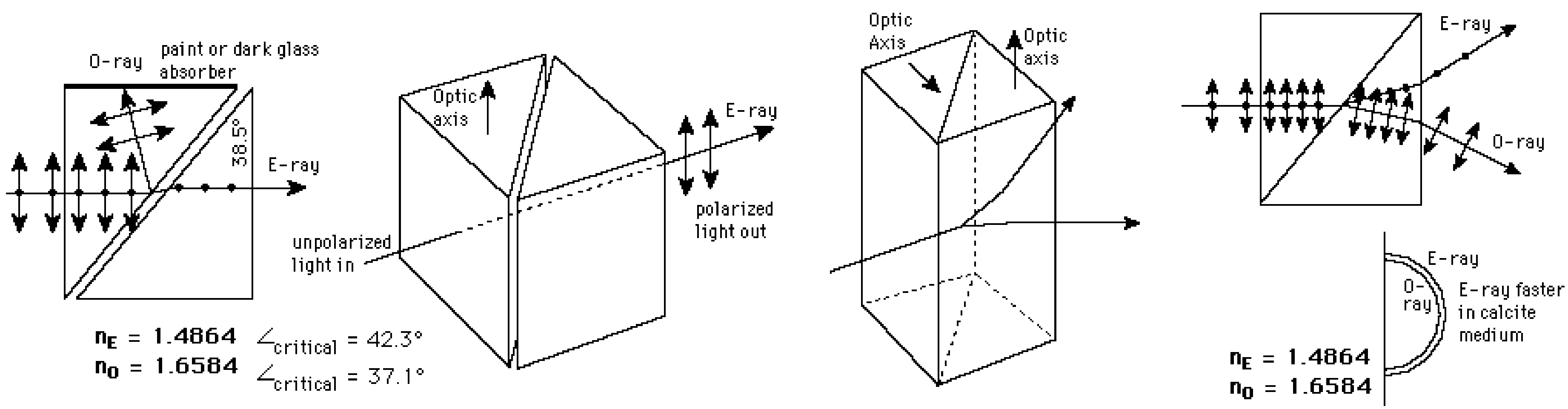
$$\begin{aligned}\sin\theta_1 &= n_0 \sin\theta_0 \\ \sin\theta_1 &= n(\theta_a+\theta_e)\sin\theta_e\end{aligned}$$



Polarizer

- Linearly polarized light beam can be produced via multiple reflection
- Or using anisotropic crystal arranged as a Wollaston prism
- Glan-Foucault prism





Waveplate

- Waveplate can be used to manipulate the polarization of an incoming wave
- The phase shift between the two optical directions is

$$\Gamma = \frac{2\pi\,\Delta n\, L}{\lambda}$$

- The corresponding Jones matrix is

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\Gamma} \end{bmatrix}$$