Alternating & Direct Currents

- AC versus DC signals
- AC characterization
- Mathematical tools:
  - Complex number
  - Complex representation of an AC signal
- Resistor in an AC circuit
- Capacitors
- Reactance and Impedance
- RC circuits
- High and low-pass filters
Alternating Current (AC) versus Direct Current (DC)

• With AC it is possible to build electric generators, motors and power distribution systems that are far more efficient than DC.
• AC is used predominately across the world of high power
Alternating Current (AC): waveforms

- AC signal are periodic:

\[ S(t + T) = S(t) \]

\[ f = \frac{1}{T} = \frac{\omega}{2\pi} \]

\textbf{UNITS:}

- $f$: in Hertz (Hz)
- $\omega$: in rad.s$^{-1}$

Heinrich Rudolf Hertz (1857-1894)
Can an AC waveform be characterized by few parameters?

- Peak-to-peak (PP) \( PP = \max(S) - \min(S) \)
- Peak \( P = \max(S) \)
- Average \( \langle S \rangle = \frac{1}{T} \int_t^{t+T} S(t) dt \)
- Practical Average \( AVG = \frac{1}{T} \int_t^{t+T} |S(t)| dt \)
- Root-mean-square \( RMS \equiv \sqrt{\langle S^2 \rangle - \langle S \rangle^2} = \left[ \frac{1}{T} \int_t^{t+T} S^2(t) dt \right]^{1/2} \)

**True average value of all points (considering their signs) is zero!**

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Alternating Current (AC): characterization

• For some analytical waveform, there exits relation between the different parameters
• Take a sinusoidal waveform with amplitude 1 then

\[ (RMS)^2 = \frac{1}{T} \int_{t}^{t+T} \sin^2(\omega t) \, dt = \frac{1}{\omega} \int_{0}^{2\pi} \frac{1}{\omega} \sin^2(\phi) \, d\phi \]

\[ \Rightarrow (RMS) = \frac{\sqrt{2}}{2} \]

\[ (AVG) = \frac{1}{T} \int_{t}^{t+T} |\sin(\omega t)| \, dt = \frac{1}{\omega} \int_{0}^{2\pi} \frac{1}{\omega} |\sin(\phi)| \, d\phi \]

\[ = 2 \cdot \frac{1}{2\pi} \int_{0}^{\pi} \sin(\phi) \, d\phi = \frac{2}{\pi} \]

RMS = 0.707 (Peak)
AVG = 0.637 (Peak)
P-P = 2 (Peak)

RMS = Peak
AVG = Peak
P-P = 2 (Peak)

RMS = 0.577 (Peak)
AVG = 0.5 (Peak)
P-P = 2 (Peak)

RMS = ???
AVG = ???
P-P = 2 (Peak)
Alternating Current (AC): characterization

- It matters what waveform is considered

- For instance for the same peak value, a square waveform will result in higher power than a triangular waveform.
Alternating Current (AC): mathematical tools

- In the following we will consider sinusoidal-type waveform (in principle any waveform can be synthesized as a series of sine wave (Fourier).

- We will write (in real notation)

\[ S(t) = S_0 \cos(\omega t + \phi) \]

- It often better to use complex notation:

\[ S(t) = \Re[S_0 e^{i(\omega t + \phi)}] \]

- And will often do calculation in complex notation and at the end recall that our physical signal is the real part of the complex results.

- We can associate a vector in the complex plane to this complex number.
Resistor in an AC circuit

\[ V_R = RI_R \]

- \( R \) is a real number. So in the complex plane, all quantities are along real axis.
- Current and Voltage are said to be **in phase**.
- When instantaneous value of current is zero, corresponding instantaneous value of voltage is zero.
- Note power > 0 at all time ⇒ resistor always dissipates energy.

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Capacitors: voltage versus current relation

- Current induced by electric displacement:

\[
\vec{J} = \frac{\partial \vec{D}}{\partial t} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t}
\]

- Assume a simple model of two plate separated by a small distance. Gauss’s law gives:

\[
\oint \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0} \implies E = \frac{Q}{\varepsilon_0 A}
\]

\[
\Rightarrow V = \frac{QL}{\varepsilon_0 A} \equiv \frac{Q}{C}
\]

\[
\vec{J} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \implies \frac{\partial V}{\partial t} = \frac{Jd}{\varepsilon_0} = \frac{Id}{A \varepsilon_0} = \frac{1}{C} I \iff I = C \frac{\partial V}{\partial t}
\]

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Capacitors: technical aspects

- Unit for Capacitance is Farad (in honor to Faraday)

- Capacitor symbol:

- Real world capacitors also introduce a resistance (we will ignore this effect)
Capacitor

- A capacitor either acts as a load or as a source.

- A capacitor can therefore store energy.
Capacitor in an AC circuits

- Capacitors do not behave the same as resistors
- Resistors allow a flow of e- proportional to the voltage drop
- Capacitors oppose change by drawing or supplying current as they charge or discharge.

\[ I_C = C \frac{dV_C}{dt} = i \omega CV_C \]
Reactance and Impedance

• The general linear relation between V and I is of the form

\[ Z \equiv \frac{V}{I} \]

Z is called **impedance**.

• For a resistor \( Z = R \) is a **real number**.

• For a capacitor \( Z = \frac{-i}{\omega C} \) is an **imaginary number**

• Generally \( Z \) will be a **complex number** (if \( V \) and \( I \) are written in their complex forms)

• For instance if a circuit has both capacitor(s) and resistor(s) we expect \( Z \) to generally be a complex number

• For a capacitor the quantity \( X_C = \frac{-1}{\omega C} \) is called reactance and is in Ohm (Ω)

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Example: Impedance of a series RC Circuits

• Let’s compute the total impedance of the RC circuit:

\[ V_T = V_C + V_R = \frac{-i}{\omega C} I + RI = (R - \frac{i}{\omega C})I \]

\[ \Rightarrow Z = R - \frac{i}{\omega C} = R + iX_C \]

• The impedance can be written as:

\[ Z = \sqrt{R^2 + \frac{1}{\omega^2 C^2}} e^{i\Xi}, \text{ with } \tan \Xi = -\frac{1}{\omega RC} \]

• NA: \( Z = 5 - 26.52i \) or \(|Z| = 29.99\) and \( \Xi = -79.325 \) degree
Example: Impedance of a parallel RC Circuits

Let’s compute the total impedance of the RC circuit:

\[ I = I_C + I_R = \left( i\omega C + \frac{1}{R} \right) V \]

\[ \Rightarrow Z = \left( i\omega C + \frac{1}{R} \right)^{-1} = \frac{1}{1 + \frac{1}{X_C + \frac{1}{R}}} \]

• NA: \( Z = 4.83 - 0.91i \) or \(|Z| = 4.91 \) and \( \varphi = -10.68 \) degree
General Analysis of an RC series circuits

Let’s write the ODE for the current

\[ V = V_R + V_C = RI + \int \frac{I}{C} dt \]

\[ \Leftrightarrow \frac{dI}{dt} + \frac{1}{RC} I = \frac{1}{R} \frac{dV}{dt} \]

How do we solve?
Solving the differential equation for the RC series circuit

- Previous equation is of the form:
  \[ y'(t) + \alpha y(t) = f(t), \quad y(0) = y_0 \]

- First find the solution for the homogeneous equation
  \[ y_h = D e^{-\alpha t} \]

- Then find a particular solution of the inhomogeneous equation
  \[ y_p(t) = g(t) e^{-\alpha t} \]

\[ f(t) = (g(t) e^{-\alpha t})' + \alpha g(t) e^{-\alpha t}, \quad \Rightarrow \quad f(t) = g'(t) e^{-\alpha t} \]

\[ g(t) = \int_0^t f(s)e^{\alpha s} \, ds \]

- The general solution is of the form
  \[ y_g = y_h + y_p = e^{-\alpha t} \left( D + \int_0^t f(s)e^{\alpha s} \, ds \right) \]

- So finally we have
  \[ y(t) = e^{-\alpha t} \left( y_0 + \int_0^t f(s)e^{\alpha s} \, ds \right) \]
General Analysis of an RC series circuits

- Applying previous results to RC series circuits gives:

\[ I(t) = \frac{V_0}{R} e^{-\frac{t}{RC}} \left( 1 + \frac{\omega^2 + \frac{i\omega}{RC}}{1} \left( e^{i\omega t + \frac{t}{RC}} - 1 \right) \right) \]

- Or in real notations:

\[ I(t) = \frac{V_0}{R} \left( 1 - \frac{\omega^2}{\omega^2 + \frac{1}{R^2 C^2}} \right) e^{-\frac{t}{RC}} + \frac{V_0}{R} \frac{\omega^2}{\omega^2 + \frac{1}{R^2 C^2}} \cos \omega t - \frac{V_0}{R^2 C} \frac{\omega}{\omega^2 + \frac{1}{R^2 C^2}} \sin \omega t \]
General Analysis of an RC series circuits

\( f = 200 \text{ Hz} \)

\[ \frac{I}{(V/R)} \]

Time (sec)

\((R[\Omega], C[\mu F])\)
**RC series circuits as frequency filters: low pass**

- The voltage across capacitor is

\[ V_c = -\frac{i}{\omega C} I = -\frac{i}{\omega C} \frac{V}{Z} \]

\[ \Rightarrow V_c = \frac{1-iRC\omega}{1+R^2C^2\omega^2} V \]

- The gain \( A \) is defined as:

\[ A = |A| e^{i\Theta} \]

\[ A = \frac{V_c}{V} = \frac{1-iRC\omega}{1+R^2C^2\omega^2} \]

\[ \Rightarrow |A| = \frac{1}{\sqrt{1+R^2C^2\omega^2}} ; \Theta = \arctan(-RC\omega) \]

- Note the limits

\[ \lim_{\omega >> 1/RC} |A| = 0 ; \lim_{\omega << 1/RC} |A| = 1 \]

\[ \lim_{\omega >> 1/RC} \Theta = -90 ; \lim_{\omega << 1/RC} \Theta = 0 \]
RC series circuits as frequency filters: low pass

- Signal with frequencies below $1/RC$ are **unaltered**,
- Signal with frequency **above** $1/RC$ are **attenuated**

Gains with $\omega RC$:
- $0.707$ at $\omega RC = 1$

Phase with $\omega RC$:
- $-\pi/4$ at $\omega RC = 1$
RC series circuits as frequency filters: high pass

- The voltage across capacitor is

\[ V_R = RI = R \frac{V}{Z} \]

\[ \Rightarrow V_c = R \frac{1}{R - \frac{i}{C \omega}} V \]

- The gain \( A \) is defined as:

\[ A = \left| A \right| e^{i \Theta} \]

\[ A = \frac{V_c}{V} = \frac{R^2 C^2 \omega^2 + i RC \omega}{1 + R^2 C^2 \omega^2} \]

\[ \Rightarrow \left| A \right| = \frac{RC \omega}{\sqrt{1 + R^2 C^2 \omega^2}}; \Theta = \arctan \left( \frac{1}{RC \omega} \right) \]

- Note the limits

\[ \lim_{\omega \gg 1/RC} \left| A \right| = 1; \quad \lim_{\omega \ll 1/RC} \left| A \right| = RC \omega \]

\[ \lim_{\omega \gg 1/RC} \Theta = 0; \quad \lim_{\omega \ll 1/RC} \Theta = 90 \]
RC series circuits as frequency filters: high pass

- Signal with frequencies **above** 1/RC are **unaltered**,
- Signal with frequency **below** 1/RC are **attenuated**

\[
\frac{\pi}{4}
\]

\[
0.707
\]