Phew!

(a common phrase heard in this course)

On to the weak interactions

Several differences between $W^{\pm}$ and Z bosons and QED/gluons, most noticeably the fact that W and Z bosons are quite massive

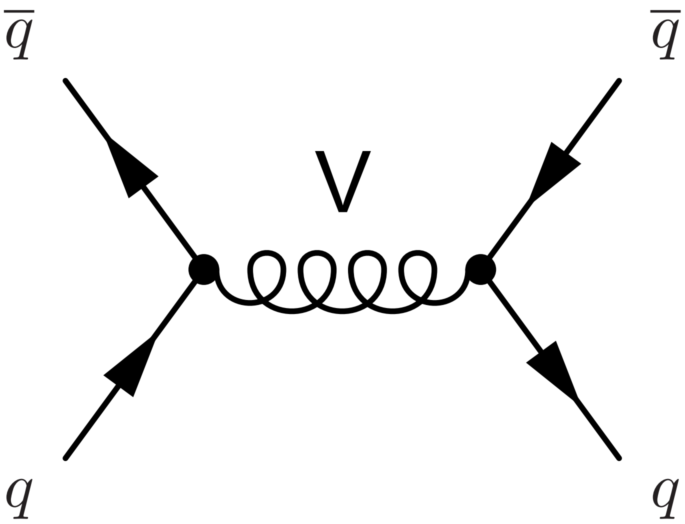
m_W = 80.4 GeV
m_Z = 91.2 GeV

Because they are massive, they can have three spin polarizations (vector boson!), not just two. So we impose Lorentz condition but not the Coulomb gauge

$\epsilon^{\mu} p_{\mu} = 0$ Lorentz condition

~~$\epsilon^0 = 0 \rightarrow \epsilon \cdot \mathbf{p} = 0$ Coulomb gauge~~

Changes to matrix element calculation for weak theory



Weak vector boson propagator:

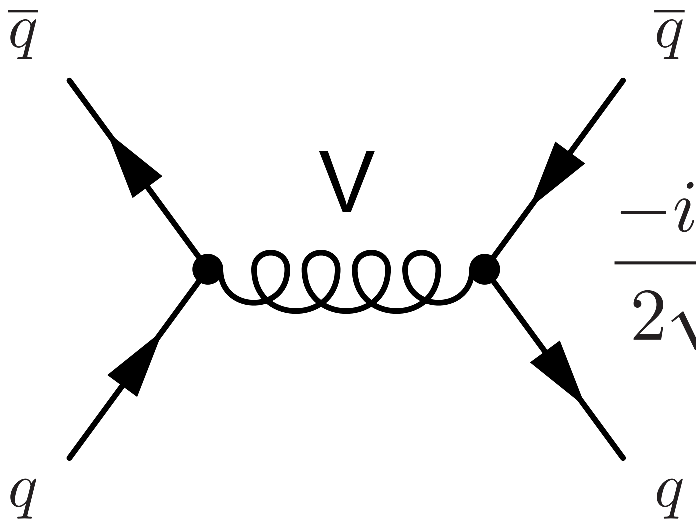
$$\frac{-i(g_{\mu\nu} - q_\mu q_\nu / M^2)}{q^2 - M^2 + iM\Gamma}$$

Where last term in denominator is the decay width of the new object

For $q^2 << M^2$ and large M we can often reduce this to:

$$\frac{ig_{\mu\nu}}{M^2}$$

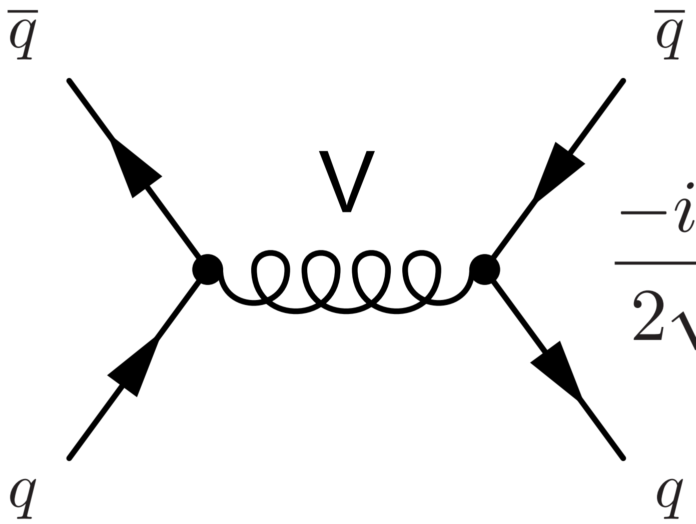
Changes to matrix element calculation for weak theory



$$\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$

What are the two types of terms that we're adding here? Anyone remember?

Changes to matrix element calculation for weak theory

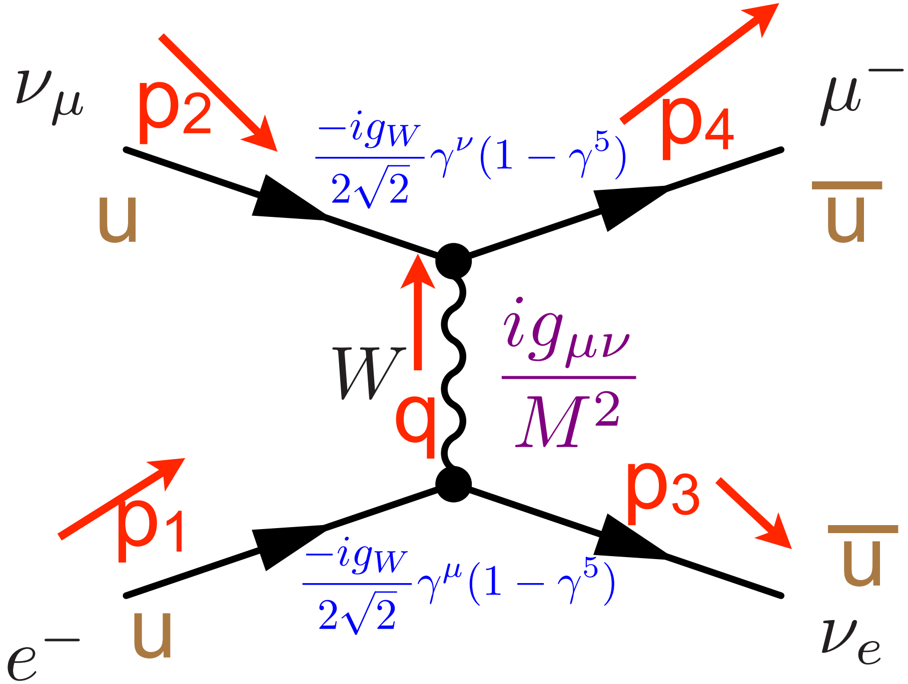


$$\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$

We are adding a vector to an axial vector, which violates parity! It violates it... maximally, as we'll see. (V-A) coupling

Inverse muon decay

$$(2\pi)^4\delta^4(p_2+q-p_4)$$



$$i$$

$$\frac{1}{(2\pi)^4}d^4q$$

You might get this from a muon neutrino beam hitting a tank of water

$$(2\pi)^4\delta^4(p_1-q-p_3)$$

Assume small momentum transfer

$$\mathcal{M}=\int i[\overline{u}(3)][-i\frac{g_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)][u(1)]\frac{ig_{\mu\nu}}{M^2}[\overline{u}(4)][-i\frac{g_W}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)][u(2)]$$
$$(2\pi)^4\delta^4(p_1-q-p_3)(2\pi)^4\delta^4(p_2+q-p_4)\frac{d^4q}{(2\pi)^4}$$

Inverse muon decay

$$\mathcal{M} = \int i[\overline{u}(3)][-i\frac{g_W}{2\sqrt{2}}\gamma^{\mu}(1-\gamma^5)][u(1)]\frac{ig_{\mu\nu}}{M^2}[\overline{u}(4)][-i\frac{g_W}{2\sqrt{2}}\gamma^{\nu}(1-\gamma^5)][u(2)]$$

$$(2\pi)^4\delta^4(p_1 - q - p_3)(2\pi)^4\delta^4(p_2 + q - p_4)\frac{d^4q}{(2\pi)^4}$$

$$\mathcal{M} = [\overline{u}(3)][\frac{g_W}{2\sqrt{2}}\gamma^{\mu}(1-\gamma^5)][u(1)]\frac{1}{M_W^2}[\overline{u}(4)][\frac{g_W}{2\sqrt{2}}\gamma_{\mu}(1-\gamma^5)][u(2)]$$

$$\mathcal{M} = \frac{g_W^2}{8M_W^2}[\overline{u}(3)\gamma^{\mu}(1-\gamma^5)u(1)][\overline{u}(4)\gamma_{\mu}(1-\gamma^5)u(2)]$$

Apply Casimir’s trick:

$$\sum_{spins}[\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^* = \mathrm{Trace}[\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2(\not{p}_a + m_a)]$$

Inverse muon decay

$$\mathcal{M} = \frac{g_W^2}{8M_W^2}[\overline{u}(3)\gamma^\mu(1-\gamma^5)u(1)][\overline{u}(4)\gamma_\mu(1-\gamma^5)u(2)]$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{64M_W^4}[\overline{u}(3)\gamma^\mu(1-\gamma^5)u(1)][\overline{u}(4)\gamma_\mu(1-\gamma^5)u(2)][\overline{u}(3)\gamma^\nu(1-\gamma^5)u(1)]^*[\overline{u}(4)\gamma_\nu(1-\gamma^5)u(2)]^*$$

Apply Casimir's trick:

$$\sum_{spins}[\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^* = \text{Trace}[\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2(\not{p}_a + m_a)]$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{64M_W^4}Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_1)\overline{\gamma^\nu(1-\gamma^5)}(\not{p}_3 + m_3)]Tr[\gamma_\mu(1-\gamma^5)(\not{p}_2 + m_2)\overline{\gamma_\nu(1-\gamma^5)}(\not{p}_4 + m_4)]$$

Neutrino masses ~0 ($m_2 = m_3 = 0$)

$$|\mathcal{M}|^2 = \frac{g_W^4}{64M_W^4}Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_e)\overline{\gamma^\nu(1-\gamma^5)}(\not{p}_3)]Tr[\gamma_\mu(1-\gamma^5)(\not{p}_2)\overline{\gamma_\nu(1-\gamma^5)}(\not{p}_4 + m_\mu)]$$

Inverse muon decay and some gamma matrix math

$$|\mathcal{M}|^2 = \frac{g_W^4}{64M_W^4} Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_e)\overline{\gamma^\nu(1-\gamma^5)}(\not{p}_3)]Tr[\gamma_\mu(1-\gamma^5)(\not{p}_2)\overline{\gamma_\nu(1-\gamma^5)}(\not{p}_4 + m_\mu)]$$

$$\overline{\gamma}^\mu = \gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\gamma^{5\dagger} = -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger}$$

$$\gamma^{5\dagger} = -i(-\gamma^3)(-\gamma^2)(-\gamma^1)(\gamma^0)$$

$$\gamma^{5\dagger} = i\gamma^3\gamma^2\gamma^1\gamma^0$$

$$\gamma^{5\dagger} = -i\gamma^2\gamma^3\gamma^1\gamma^0 = i\gamma^2\gamma^1\gamma^3\gamma^0$$

$$\gamma^{5\dagger} = -i\gamma^2\gamma^1\gamma^0\gamma^3 = i\gamma^1\gamma^2\gamma^0\gamma^3$$

$$\gamma^{5\dagger} = -i\gamma^2\gamma^1\gamma^0\gamma^3 = i\gamma^2\gamma^0\gamma^1\gamma^3$$

$$\gamma^{5\dagger} = -i\gamma^0\gamma^2\gamma^1\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$$

$$\overline{\gamma_\mu\gamma^5} = \gamma^0(\gamma_\mu\gamma^5)^\dagger\gamma^0$$

$$\overline{\gamma_\mu\gamma^5} = \gamma^0\gamma^{5\dagger}\gamma_\mu^\dagger\gamma^0 = \gamma^0\gamma^5\gamma_\mu^\dagger\gamma^0$$

$$\overline{\gamma_\mu\gamma^5} = -\gamma^5\gamma^0\gamma_\mu^\dagger\gamma^0 = -\gamma^5\gamma_\mu = \gamma_\mu\gamma^5$$

Inverse muon decay

$$|\mathcal{M}|^2 = \frac{g_W^4}{64M_W^4} Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_e)\overline{\gamma^\nu(1-\gamma^5)}(\not{p}_3)]Tr[\gamma_\mu(1-\gamma^5)(\not{p}_2)\overline{\gamma_\nu(1-\gamma^5)}(\not{p}_4 + m_\mu)]$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{64M_W^4} Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_e)(\gamma^\nu - \gamma^\nu\gamma^5)(\not{p}_3)]Tr[\gamma_\mu(1-\gamma^5)(\not{p}_2)(\gamma_\nu - \gamma_\nu\gamma^5)(\not{p}_4 + m_\mu)]$$

Focus on first trace

$$Tr[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 - \gamma^\mu \not{p}_1 \gamma^\nu \gamma^5 \not{p}_3 + \gamma^\mu m_e \gamma^\nu \not{p}_3 - \gamma^\mu m_e \gamma^\nu \gamma^5 \not{p}_3 - \gamma^\mu \gamma^5 \not{p}_1 \gamma^\nu \not{p}_3 + \gamma^\mu \gamma^5 \not{p}_1 \gamma^\nu \gamma^5 \not{p}_3 - \gamma^\mu \gamma^5 m_e \gamma^\nu \not{p}_3 + \gamma^\mu \gamma^5 m_e \gamma^\nu \gamma^5 \not{p}_3]$$

Reminders:

$$Tr(\gamma^\mu\gamma^\nu\gamma^\sigma) = 0$$

$$Tr(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda\gamma^\beta) = 0$$

$$Tr(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda\gamma^\beta\gamma^\alpha\gamma^\delta) = 0$$

Any odd number

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$Tr(\gamma^5) = Tr(\gamma^5\gamma^\mu\gamma^\nu) = 0$$

Inverse muon decay

$$Tr[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 - \gamma^\mu \not{p}_1 \gamma^\nu \gamma^5 \not{p}_3 - \gamma^\mu \gamma^5 \not{p}_1 \gamma^\nu \not{p}_3 + \gamma^\mu \gamma^5 \not{p}_1 \gamma^\nu \gamma^5 \not{p}_3]$$

Use: $\gamma^\mu \gamma^5 = -\gamma^5 \gamma^\mu \qquad \gamma^5 \gamma^5 = 1$

$$Tr[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 - \gamma^\mu \not{p}_1 \gamma^\nu \gamma^5 \not{p}_3 - \gamma^\mu \gamma^5 \not{p}_1 \gamma^\nu \not{p}_3 + \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3]$$

$$2Tr[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 + \gamma^5 \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3]$$

-1 for even permutation of 0123
+1 for odd permutation of 0123
0 for repeated indices

Use:

$$Tr(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma}$$

Finally getting to the traces:

$$2Tr[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 + \gamma^5 \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3]$$

Commute for nicer ordering

$$2Tr[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 - \gamma^5 \gamma^\mu \gamma^\nu \not{p}_1 \not{p}_3]$$

Then we get:

$$8(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} p_1 \cdot p_3 - i\epsilon^{\mu\nu\lambda\sigma} p_{1\lambda} p_{3\sigma})$$

And
second trace must be:

$$8(p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} - g_{\mu\nu} p_2 \cdot p_4 - i\epsilon_{\mu\nu\alpha\beta} p_2^\alpha p_4^\beta)$$

So total matrix element (almost there)

Finally:

$$|\mathcal{M}|^2 = \frac{g_W^4}{64M_W^4} 8(p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu}p_2 \cdot p_4 - i\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta) \times$$
$$8(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}p_1 \cdot p_3 - i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma})$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4} (p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu}p_2 \cdot p_4 - i\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta) \times$$
$$(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}p_1 \cdot p_3 - i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma})$$

So total matrix element

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}\left(p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu}p_2 \cdot p_4 - i\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta\right)\times$$

$$\left(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}p_1 \cdot p_3 - i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\right)$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_2 \cdot p_4)(p_1 \cdot p_3) + (p_2 \cdot p_3)(p_1 \cdot p_4) + (p_1 \cdot p_2)(p_3 \cdot p_4) -$$
$$(p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4) + 4(p_1 \cdot p_3)(p_2 \cdot p_4)$$
$$-i\epsilon_{\mu\nu\lambda\sigma}p_2^\lambda p_4^\sigma\left(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}p_1 \cdot p_3\right)$$
$$-i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\left(p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu}p_2 \cdot p_4\right)$$
$$-\epsilon_{\mu\nu\lambda\sigma}p_2^\lambda p_4^\sigma \epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma})$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_2 \cdot p_4)(p_1 \cdot p_3) + (p_2 \cdot p_3)(p_1 \cdot p_4) + (p_1 \cdot p_2)(p_3 \cdot p_4) -$$
$$(p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4) + 4(p_1 \cdot p_3)(p_2 \cdot p_4)$$
$$-i\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta\left(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}p_1 \cdot p_3\right)$$
$$-i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\left(p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu}p_2 \cdot p_4\right)$$
$$-\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta \epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma})$$

Simplifying the mess

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}(2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3)$$

$$-i\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta \left(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} p_1 \cdot p_3\right)$$

$$-i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\left(p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu}p_2 \cdot p_4\right)$$

$$-\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta \epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma})$$

First terms in () the same, but epsilons have opposite sign! These all cancel

Look at terms like:

$$i\epsilon_{01\alpha\beta}p_2^\alpha p_4^\beta \left(p_1^0 p_3^1 + p_1^1 p_3^0\right) +$$

$$i\epsilon_{10\alpha\beta}p_2^\alpha p_4^\beta \left(p_1^1 p_3^0 + p_1^0 p_3^1\right) + \ldots$$

Simplifying the mess

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}(2(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)$$
$$+i\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta\left(g^{\mu\nu}p_1\cdot p_3\right)$$
$$+i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\left(g_{\mu\nu}p_2\cdot p_4\right)$$
$$-\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma})$$

Terms like

$$+i\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta\left(g^{\mu\nu}p_1\cdot p_3\right)$$

Epsilon only non-zero when μ!=ν, but term in () only non-zero when μ==ν

Simplifying the mess

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}(2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3)$$

$$-\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\lambda\sigma} p_2^\alpha p_4^\beta p_{1\lambda} p_{3\sigma})$$

What is? $\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\lambda\sigma}$

$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\lambda\sigma} = A[\delta_\alpha^\lambda \delta_\beta^\sigma - \delta_\beta^\lambda \delta_\alpha^\sigma]$$

α must be λ or σ for terms to be non-zero. And it is anti-symmetric in λ and σ

To find A, let's try λ=2, σ=3

$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu 23} = A[\delta_\alpha^2 \delta_\beta^3 - \delta_\beta^2 \delta_\alpha^3]$$

Clear?

$$\epsilon_{\mu\nu 23}\epsilon^{\mu\nu 23} = A[\delta_2^2 \delta_3^3 - \delta_3^2 \delta_2^3] = A$$

$$\epsilon_{\mu\nu 23}\epsilon^{\mu\nu 23} = \epsilon_{0123}\epsilon^{0123} + \epsilon_{1023}\epsilon^{1023} = +1(-1) + -1(+1) = -2 = A$$

Simplifying the mess

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}(2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3)$$
$$-\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\lambda\sigma} p_2^\alpha p_4^\beta p_{1\lambda} p_{3\sigma})$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}(2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3)$$
$$+2p_2^\alpha p_4^\beta p_{1\lambda} p_{3\sigma}(\delta_\alpha^\lambda \delta_\beta^\sigma - \delta_\beta^\lambda \delta_\alpha^\sigma))$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}(2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) + 2(p_1 \cdot p_2)(p_3 \cdot p_4) - 2(p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$|\mathcal{M}|^2 = \frac{4g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4))$$

Almost done

$$|\mathcal{M}|^2 = \frac{4g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4))$$

Finally, we want to average over initial spins, but massless neutrino has only one spin configuration, so divide by 2:

$$|\mathcal{M}|^2 = \frac{2g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4))$$

As always, pick a frame

$$|\mathcal{M}|^2 = \frac{2g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4))$$

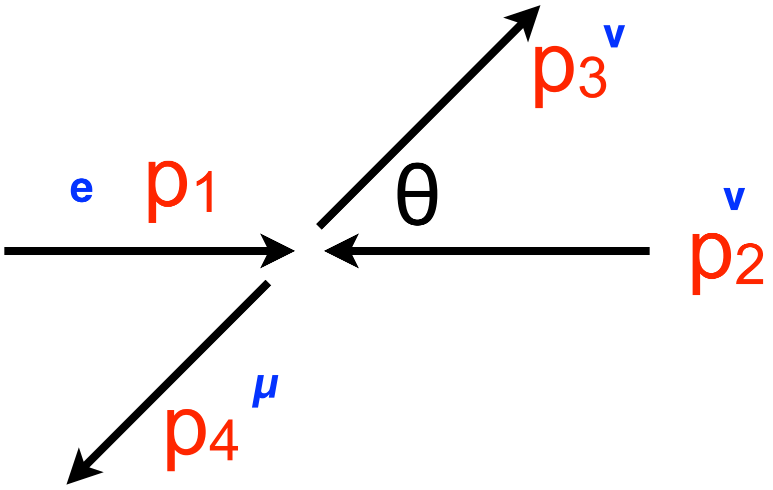
$|\mathbf{p}_1|=|\mathbf{p}_2|=p_i$

$|\mathbf{p}_3|=|\mathbf{p}_4|=p_f$

$E_1+E_2 = 2E$

$2E = E_3 + E_4$

Assume massless electron. Pick center of mass frame where $E_1 = E_2 = E$



$$|\mathbf{p}_i| = E$$

$$|\mathbf{p}_f| = \sqrt{E_4^2 - m_\mu^2} = E_3$$

$$2E = \sqrt{E_4^2 - m_\mu^2} + E_4$$

$$2E - E_4 = \sqrt{E_4^2 - m_\mu^2}$$

$$4E^2 + E_4^2 - 4EE_4 = E_4^2 - m_\mu^2$$

$$4E^2 - 4EE_4 = -m_\mu^2$$

$$4EE_4 = 4E^2 + m_\mu^2$$

$$E_4 = E + m_\mu^2/4E$$

$$E_3 = E - m_\mu^2/4E$$

$$p_1 \cdot p_2 = E^2 + \mathbf{p}_i^2 = 2E^2$$

$$p_3 \cdot p_4 = E_3E_4 + \mathbf{p}_f^2 = E_3E_4 + E_3^2$$

$$p_3 \cdot p_4 = (E + m_\mu^2/4E)(E - m_\mu^2/4E) + (E - m_\mu^2/4E)^2$$

As always, pick a frame

$$|\mathcal{M}|^2 = \frac{2g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4))$$

$$p_1 \cdot p_2 = E^2 + \mathbf{p}_i^2 = 2E^2$$

$$p_3 \cdot p_4 = E_3 E_4 + \mathbf{p}_f^2 = E_3 E_4 + E_3^2$$

$$p_3 \cdot p_4 = (E + m_\mu^2/4E)(E - m_\mu^2/4E) + (E - m_\mu^2/4E)^2$$

$$p_3 \cdot p_4 = E^2 - m_\mu^4/(16E^2) + E^2 + m_\mu^4/(16E^2) - m_\mu^2/2$$

$$p_3 \cdot p_4 = 2E^2 - m_\mu^2/2$$

$$|\mathcal{M}|^2 = \frac{2g_W^4}{M_W^4}(2E^2)(2E^2 - m_\mu^2/2)$$

$$|\mathcal{M}|^2 = \frac{8E^4 g_W^4}{M_W^4}(1 - \frac{m_\mu^2}{4E^2})$$

Differential cross section

$$|\mathcal{M}|^2 = \frac{8E^4 g_W^4}{M_W^4}(1 - \frac{m_\mu^2}{4E^2})$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{|M|^2}{(E_1 + E_2)^2}\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{|M|^2}{(2E)^2}\frac{E - m_{\mu^2}/4E}{E}$$

$$\frac{d\sigma}{d\Omega} = \frac{8E^4 g_W^4}{M_W^4}(1 - \frac{m_\mu^2}{4E^2})\frac{1}{64\pi^2}\frac{1}{(2E)^2}\frac{E - m_{\mu^2}/4E}{E}$$

Total cross section

$$\frac{d\sigma}{d\Omega} = \frac{8E^4 g_W^4}{M_W^4}(1 - \frac{m_\mu^2}{4E^2})\frac{1}{64\pi^2}\frac{1}{(2E)^2}\frac{E - m_{\mu^2}/4E}{E}$$

$$\frac{d\sigma}{d\Omega} = \frac{E^2 g_W^4}{32\pi^2 M_W^4}(1 - \frac{m_\mu^2}{4E^2})\frac{E - m_{\mu^2}/4E}{E}$$

$$\frac{d\sigma}{d\Omega} = \frac{E^2 g_W^4}{32\pi^2 M_W^4}(1 - \frac{m_\mu^2}{4E^2})(1 - m_{\mu^2}/4E^2)$$

$$\frac{d\sigma}{d\Omega} = \frac{E^2 g_W^4}{32\pi^2 M_W^4}(1 - \frac{m_\mu^2}{4E^2})^2$$

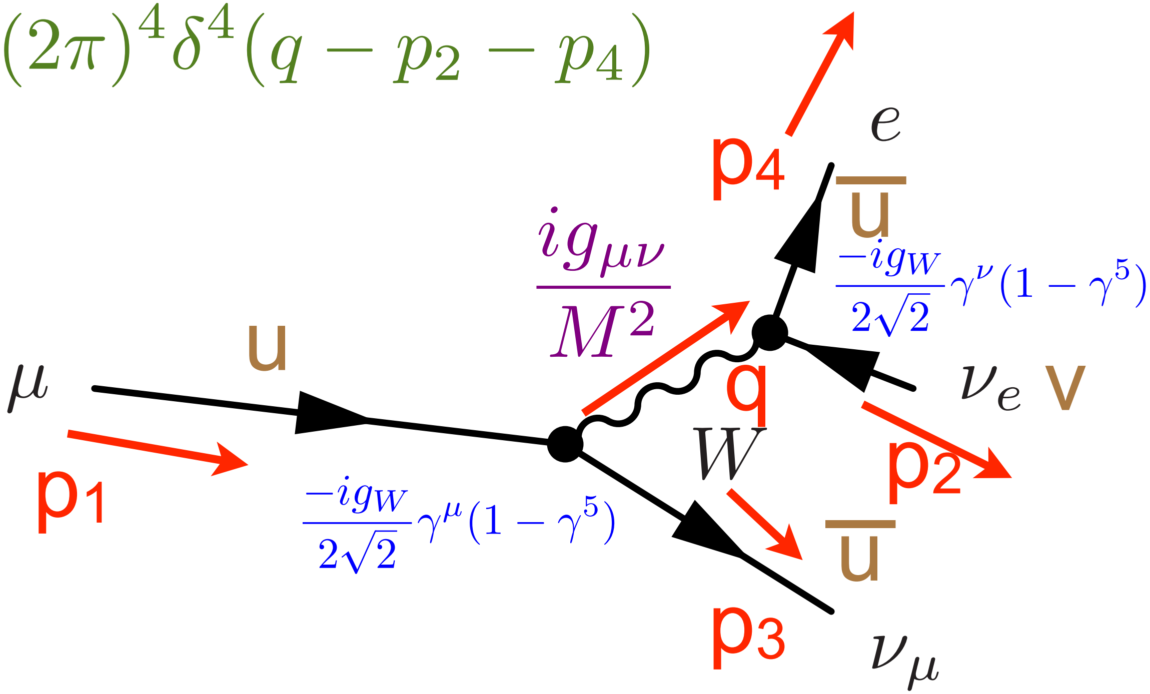
$$\boxed{\sigma = \frac{E^2 g_W^4}{8\pi m_W^4}\left(1 - \frac{m_\mu^2}{4E^2}\right)^2}$$

Again, let’s take a breather

Any suggestions, other than… PHEW?

Muon decay (something more tangible)

$$(2\pi)^4\delta^4(q - p_2 - p_4)$$



$$(2\pi)^4\delta^4(p_1 - q - p_3)$$

$$i$$

$$\frac{1}{(2\pi)^4}d^4q$$

Assume small momentum transfer

$$\mathcal{M} = \int i[\overline{u}(3)][-i\frac{g_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)][u(1)]\frac{ig_{\mu\nu}}{M^2}[\overline{u}(4)][-i\frac{g_W}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)][v(2)]$$

$$(2\pi)^4\delta^4(p_1 - q - p_3)(2\pi)^4\delta^4(q - p_2 - p_4)\frac{d^4q}{(2\pi)^4}$$

But we evaluated this already

u(2) became v(2), which changes the sign on the mass term of a neutrino, which is zero anyway

One of the delta functions is also different, but it gets canceled. So at least this part is simple :)

$$|\mathcal{M}|^2 = \frac{2g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4))$$

Let's evaluate in muon rest frame, and assume zero electron mass (valid vs muon mass) so:

$$p_1 = (m_\mu, \mathbf{0}) \rightarrow (p_1 \cdot p_2) = m_\mu E_2$$

$$p_1 = p_2 + p_3 + p_4 \rightarrow (p_3 + p_4)^2 = (p_1 - p_2)^2$$

$$(p_1 - p_2)^2 = p_1^2 + p_2^2 - 2(p_1 \cdot p_2) = m_\mu^2 - 2m_\mu E_2$$

$$(p_3 + p_4)^2 = p_3^2 + p_4^2 + 2(p_3 \cdot p_4) = 2(p_3 \cdot p_4) = m_\mu^2 - 2m_\mu E_2$$

$$p_3 \cdot p_4 = m_\mu^2/2 - m_\mu E_2$$

But we evaluated this already

$$|\mathcal{M}|^2 = \frac{2g_W^4}{M_W^4}(m_\mu E_2)(m_\mu^2/2 - m_\mu E_2))$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}(m_\mu E_2)(m_\mu^2 - 2m_\mu E_2))$$

$$|\mathcal{M}|^2 = \frac{g_W^4 m_\mu^2}{M_W^4}|\mathbf{p}_2|(m_\mu - 2|\mathbf{p}_2|))$$

Way way back we found the decay rate for any particle from Fermi's Golden Rule:

$$d\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 ... - p_n) \times$$

$$\prod_{j=2}^{n} \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

If decay products are massless, that simplifies things a bit. Here n=4 in total, so...

$$\mathrm{d}\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 ... - p_n) \times$$
$$\prod_{j=2}^{n} \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

$$\mathrm{d}\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 - p_4) \times$$
$$\frac{d^3\mathbf{p}_2}{(2\pi)^3 2|\mathbf{p_2}|} \frac{d^3\mathbf{p}_3}{(2\pi)^3 2|\mathbf{p_3}|} \frac{d^3\mathbf{p}_4}{(2\pi)^3 2|\mathbf{p_4}|}$$

Let's work it out...

$$\delta^4(p_1 - p_2 - p_3 - p_4) = \delta(E_1 - E_2 - E_3 - E_4)\delta^3(\mathbf{p_1} - \mathbf{p_2} - \mathbf{p_3} - \mathbf{p_4})$$

$$\mathbf{p_1} = 0$$

$$E_1 = m_\mu$$

$$E_2 = |\mathbf{p_2}|$$

$$E_3 = |\mathbf{p_3}|$$

$$E_4 = |\mathbf{p_4}|$$

$$\delta^4(p_1 - p_2 - p_3 - p_4) = \delta(m_\mu - |\mathbf{p_2}| - |\mathbf{p_3}| - |\mathbf{p_4}|)\delta^3(\mathbf{p_2} + \mathbf{p_3} + \mathbf{p_4})$$

$$\mathrm{d}\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta(m_\mu - |\mathbf{p}_2| - |\mathbf{p}_3| - |\mathbf{p}_4|)\delta^3(\mathbf{p_2} + \mathbf{p_3} + \mathbf{p_4}) \times$$

$$\frac{d^3\mathbf{p}_2}{(2\pi)^3 2|\mathbf{p_2}|}\frac{d^3\mathbf{p}_3}{(2\pi)^3 2|\mathbf{p_3}|}\frac{d^3\mathbf{p}_4}{(2\pi)^3 2|\mathbf{p_4}|}$$

with $$|\mathcal{M}|^2 = \frac{g_W^4 m_\mu^2}{M_W^4}|\mathbf{p}_2|(m_\mu - 2|\mathbf{p}_2|)$$

Let's work it out...

$$d\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta(m_\mu - |\mathbf{p}_2| - |\mathbf{p}_3| - |\mathbf{p}_4|)\delta^3(\mathbf{p_2} + \mathbf{p_3} + \mathbf{p_4})\times$$

$$\frac{d^3\mathbf{p}_2}{(2\pi)^3 2|\mathbf{p_2}|}\frac{d^3\mathbf{p}_3}{(2\pi)^3 2|\mathbf{p_3}|}\frac{d^3\mathbf{p}_4}{(2\pi)^3 2|\mathbf{p_4}|}$$

p3 integral gives

$$d\Gamma = \frac{|\mathcal{M}|^2}{16(2\pi)^5 m_1}\int \delta(m_\mu - |\mathbf{p}_2| - |\mathbf{p}_2 + \mathbf{p}_4| - |\mathbf{p}_4|)\times$$

$$\frac{d^3\mathbf{p}_2 d^3\mathbf{p}_4}{|\mathbf{p_2}||\mathbf{p_4}||\mathbf{p_2} + \mathbf{p_4}|}$$

Two integrals left. Let's point **p4** along z axis so:

$$d^3\mathbf{p}_2 = |\mathbf{p}_2|^2 d|\mathbf{p}_2| \sin\theta d\phi d\theta$$

Let's work it out...

$$d\Gamma = \frac{|\mathcal{M}|^2}{16(2\pi)^5 m_1} \int \delta(m_\mu - |\mathbf{p}_2| - |\mathbf{p}_2 + \mathbf{p}_4| - |\mathbf{p}_4|) \times$$

$$\frac{d^3\mathbf{p}_2 d^3\mathbf{p}_4}{|\mathbf{p_2}||\mathbf{p_4}||\mathbf{p_2} + \mathbf{p_4}|}$$

$$d^3\mathbf{p_2} = |\mathbf{p}_2|^2 d|\mathbf{p}_2| \sin\theta d\phi d\theta$$

$$|\mathbf{p}_2 + \mathbf{p}_4|^2 = |\mathbf{p}_2|^2 + |\mathbf{p}_4|^2 + 2|\mathbf{p}_2||\mathbf{p}_4| \cos\theta \equiv u^2$$

Define u here (to replace θ)

$$2udu = -2|\mathbf{p}_2||\mathbf{p}_4| \sin\theta d\theta$$

ϕ integral gives 2π

$$d\Gamma = \frac{|\mathcal{M}|^2}{16(2\pi)^4 m_1} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|} \frac{d|\mathbf{p}_2|}{|\mathbf{p}_2 + \mathbf{p}_4|} |\mathbf{p}_2| \int \delta(m_\mu - |\mathbf{p}_2| - |\mathbf{p}_2 + \mathbf{p}_4| - |\mathbf{p}_4|) \sin\theta d\theta$$

Let’s work it out...

$$u = |\mathbf{p}_2 + \mathbf{p}_4|$$

$$2u\,du = -2|\mathbf{p}_2||\mathbf{p}_4|\sin\theta\, d\theta$$

$$d\Gamma = \frac{|\mathcal{M}|^2}{16(2\pi)^4 m_1}\frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|}\frac{d|\mathbf{p}_2|}{|\mathbf{p}_2+\mathbf{p}_4|}|\mathbf{p}_2|\int \delta(m_\mu - |\mathbf{p}_2| - |\mathbf{p}_2+\mathbf{p}_4| - |\mathbf{p}_4|)\sin\theta\, d\theta$$

$$d\Gamma = \frac{-|\mathcal{M}|^2}{16(2\pi)^4 m_1}\frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2}d|\mathbf{p}_2|\int \delta(m_\mu - |\mathbf{p}_2| - u - |\mathbf{p}_4|)du$$

Limits of Integration:

$$u = \sqrt{|\mathbf{p}_2|^2 + |\mathbf{p}_4|^2 + 2|\mathbf{p}_2||\mathbf{p}_4|\cos\theta}$$

$$\theta = 0 \rightarrow u_+ = \sqrt{|\mathbf{p}_2|^2 + |\mathbf{p}_4|^2 + 2|\mathbf{p}_2||\mathbf{p}_4|} = \sqrt{(|\mathbf{p}_2| + |\mathbf{p}_4|)^2}$$

$$u_+ = |\mathbf{p}_2| + |\mathbf{p}_4|$$

$$\theta = \pi \rightarrow u_- = \sqrt{|\mathbf{p}_2|^2 + |\mathbf{p}_4|^2 - 2|\mathbf{p}_2||\mathbf{p}_4|} = \sqrt{(|\mathbf{p}_2| - |\mathbf{p}_4|)^2}$$

$$u_- = |(|\mathbf{p}_2| - |\mathbf{p}_4|)|$$

Let's work it out...

$$d\Gamma = \frac{|\mathcal{M}|^2}{16(2\pi)^4 m_1} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} d|\mathbf{p}_2| \int_{u_-}^{u^+} \delta(m_\mu - |\mathbf{p}_2| - u - |\mathbf{p}_4|) du$$

Removed minus sign for limit swap

$$u_+ = |\mathbf{p}_2| + |\mathbf{p}_4|$$
$$u_- = |(|\mathbf{p}_2| - |\mathbf{p}_4|)|$$

Does integral = 1 or 0? Does delta function contribute? Only if ...

$$u_- < (m_\mu - |\mathbf{p}_2| - |\mathbf{p}_4|) < u_+$$
$$|\mathbf{p}_2| - |\mathbf{p}_4| \leqslant u_-$$
$$|\mathbf{p}_4| - |\mathbf{p}_2| \leqslant u_-$$
$$|\mathbf{p}_2| - |\mathbf{p}_4| \leqslant m_\mu - |\mathbf{p}_4| - |\mathbf{p}_2|$$
$$|\mathbf{p}_4| - |\mathbf{p}_2| \leqslant m_\mu - |\mathbf{p}_4| - |\mathbf{p}_2|$$

Back to delta function

$$u_- < (m_\mu - |\mathbf{p}_2| - |\mathbf{p}_4|) < u_+$$

$$|\mathbf{p}_2| - |\mathbf{p}_4| \leqslant u_-$$

$$|\mathbf{p}_4| - |\mathbf{p}_2| \leqslant u_-$$

So we get from the u_- inequality:

$$|\mathbf{p}_2| - |\mathbf{p}_4| \leqslant m_\mu - |\mathbf{p}_4| - |\mathbf{p}_2|$$

$$|\mathbf{p}_4| - |\mathbf{p}_2| \leqslant m_\mu - |\mathbf{p}_4| - |\mathbf{p}_2|$$

Does integral = 1 or 0? Does delta function contribute? Only if ...

$$|\mathbf{p}_2| < m_\mu/2$$

$$|\mathbf{p}_4| < m_\mu/2$$

Back to delta function

$$u_- < (m_\mu - |\mathbf{p}_2| - |\mathbf{p}_4|) < u_+$$

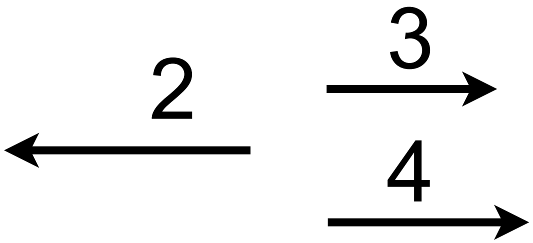
So we get from the u_+ inequality:

$$|\mathbf{p}_2| + |\mathbf{p}_4| \geqslant u_+$$
$$m_\mu - |\mathbf{p}_4| - |\mathbf{p}_2| < |\mathbf{p}_2| + |\mathbf{p}_4|$$

Does integral = 1 or 0? Does delta function contribute? Only if ...

$$|\mathbf{p}_2| + |\mathbf{p}_4| \geqslant u_+$$
$$(|\mathbf{p}_2| + |\mathbf{p}_4|) < m_\mu/2$$

So putting limits together

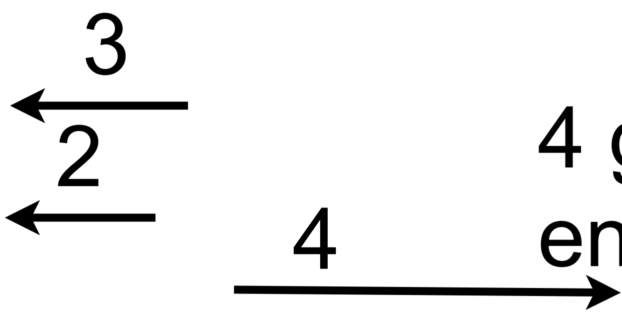


2 gets half of energy

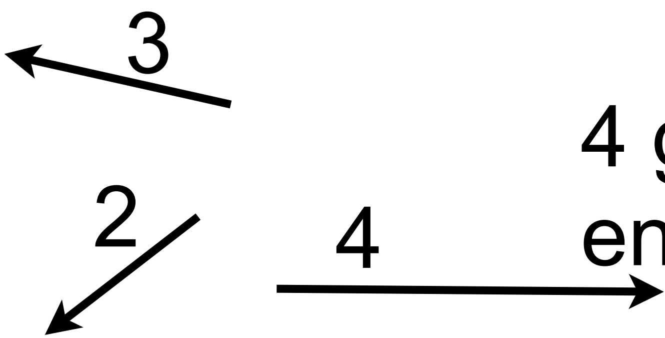
$$|\mathbf{p}_2| < m_\mu/2$$

$$|\mathbf{p}_4| < m_\mu/2$$

$$(|\mathbf{p}_2| + |\mathbf{p}_4|) < m_\mu/2$$



4 gets half of energy



4 gets less than half of energy, 2+3 get a bit more

Going back to the integral

$$d\Gamma = \frac{|\mathcal{M}|^2}{16(2\pi)^4 m_1} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} d|\mathbf{p}_2| \int_{u_-}^{u^+} \delta(m_\mu - |\mathbf{p}_2| - u - |\mathbf{p}_4|) du$$

Becomes ...

$$d\Gamma = \frac{|\mathcal{M}|^2}{16(2\pi)^4 m_1} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} d|\mathbf{p}_2|$$

with ...

$$|\mathcal{M}|^2 = \frac{g_W^4 m_\mu^2}{M_W^4} |\mathbf{p}_2|(m_\mu - 2|\mathbf{p}_2|))$$

Now be careful to include $|M|^2$ in integral

$$\Gamma = \frac{g_W^4 m_\mu^2}{16(2\pi)^4 m_1 M_W^4} \int |\mathbf{p}_2|(m_\mu - 2|\mathbf{p}_2|) \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} d|\mathbf{p}_2|$$

Evaluating integrals

$$\Gamma = \frac{g_W^4 m_\mu^2}{16(2\pi)^4 m_1 M_W^4} \int |\mathbf{p}_2|(m_\mu - 2|\mathbf{p}_2|) \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} d|\mathbf{p}_2|$$

$$|\mathbf{p}_2| < m_\mu/2$$

$$|\mathbf{p}_4| < m_\mu/2$$

$$(|\mathbf{p}_2| + |\mathbf{p}_4|) < m_\mu/2$$

So

$$\frac{m_\mu}{2} - |\mathbf{p}_4| < |\mathbf{p}_2| < \frac{m_\mu}{2}$$

$$\Gamma = \frac{g_W^4 m_\mu}{16(2\pi)^4 M_W^4} \int \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} \left[\frac{m_\mu x^2}{2} - \frac{2x^3}{3} \right]_{x=\frac{m_\mu}{2}-|\mathbf{p}_4|}^{x=\frac{m_\mu}{2}}$$

$$\Gamma = \frac{g_W^4 m_\mu}{16(2\pi)^4 M_W^4} \int_{|\mathbf{p}_4|=0}^{|\mathbf{p}_4|=\frac{m_\mu}{2}} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} \left(\frac{m_\mu |\mathbf{p}_4|^2}{2} - \frac{2}{3}|\mathbf{p}_4|^3 \right)$$

Evaluating integrals

$$\Gamma = \frac{g_W^4 m_\mu}{16(2\pi)^4 M_W^4} \int_{|\mathbf{p}_4|=0}^{|\mathbf{p}_4|=\frac{m_\mu}{2}} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} \left(\frac{m_\mu |\mathbf{p}_4|^2}{2} - \frac{2}{3}|\mathbf{p}_4|^3 \right)$$

$$d^3\mathbf{p}_4 = 4\pi|\mathbf{p}_4|^2 d|\mathbf{p}_4|$$

For massless electron, E_e = $|\mathbf{p}_4|$

$$\Gamma = \frac{g_W^4 m_\mu}{64\pi^3 M_W^4} \int_{E=0}^{E=\frac{m_\mu}{2}} \left(\frac{m_\mu E^2}{2} - \frac{2}{3}E^3 \right) dE$$

$$\frac{d\Gamma}{dE} = \frac{g_W^4 m_\mu}{64\pi^3 M_W^4} \left(\frac{m_\mu E^2}{2} - \frac{2}{3}E^3 \right)$$

A nice plot from Griffiths

$$\frac{d\Gamma}{dE} = \frac{g_W^4 m_\mu}{64\pi^3 M_W^4}\left(\frac{m_\mu E^2}{2} - \frac{2}{3}E^3\right)$$

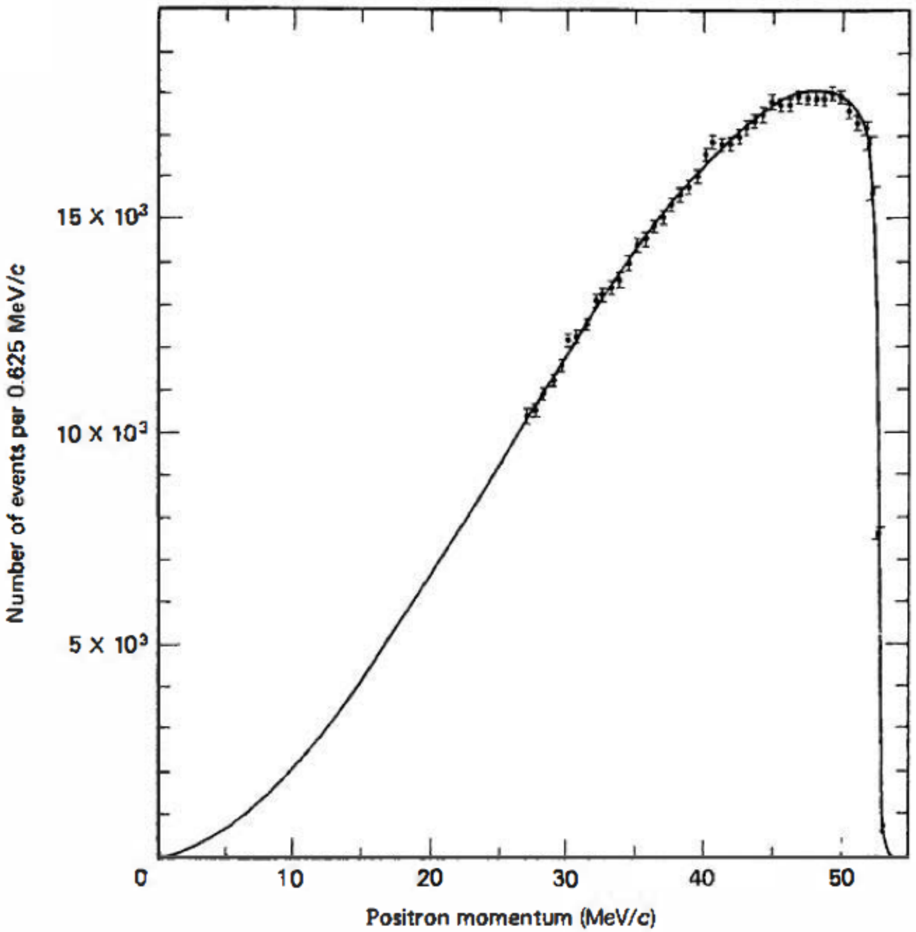


Fig. 9.1 Experimental spectrum of positrons in $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$. The solid line is the theoretically predicted spectrum based on Equation (9.33), corrected for electromagnetic effects. (*Source*: Bardon, M. *et al.* (1965) *Physical Review Letters*, **14**, 449. For the latest high-precision data on muon decay go to the TWIST collaboration web site at TRIUMF, Vancouver, BC.)

Note: In the SM, there are no muon decays without neutrinos. But that might happen in BSM theories! What would we see if muons could occasionally decay directly to electrons? mu2e!

Total decay width

$$\Gamma = \frac{g_W^4 m_\mu}{64\pi^3 M_W^4} \int_{E=0}^{E=\frac{m_\mu}{2}} \left(\frac{m_\mu E^2}{2} - \frac{2}{3} E^3 \right) dE$$

$$\Gamma = \frac{g_W^4 m_\mu}{64\pi^3 M_W^4} \frac{m_\mu^4}{96}$$

$$\tau = \frac{1}{\Gamma} = \frac{6144\pi^3 M_W^4}{g_W^4 m_\mu^5}$$

Plugging in numbers... muon lifetime is 2.2 microseconds, but that is in muon rest frame. BIG time dilation factors

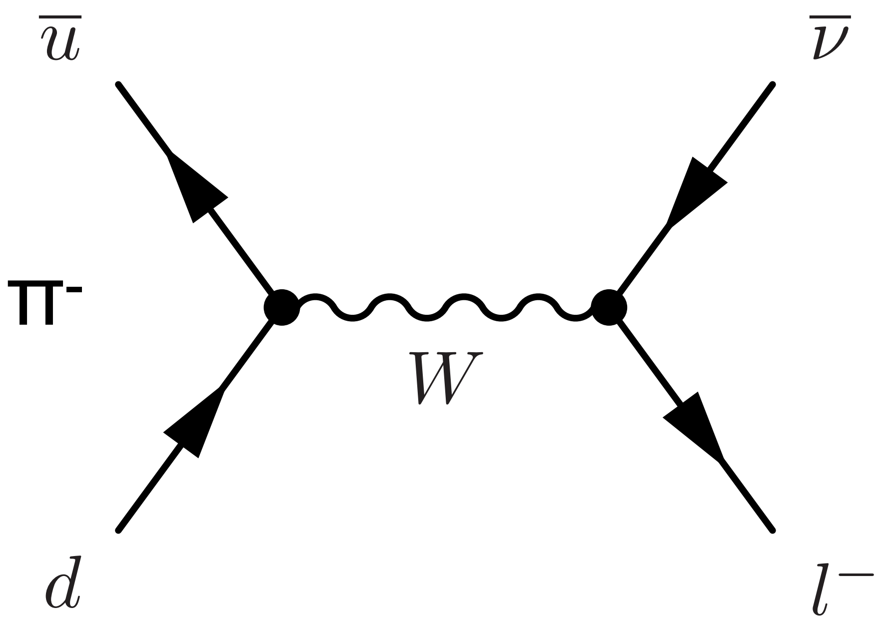
As Griffiths points out

Weak fine structure constant/weak coupling is LARGER than EM coupling by ~x5! It is weak due to the massive nature of W/Z bosons

Decay of the neutron

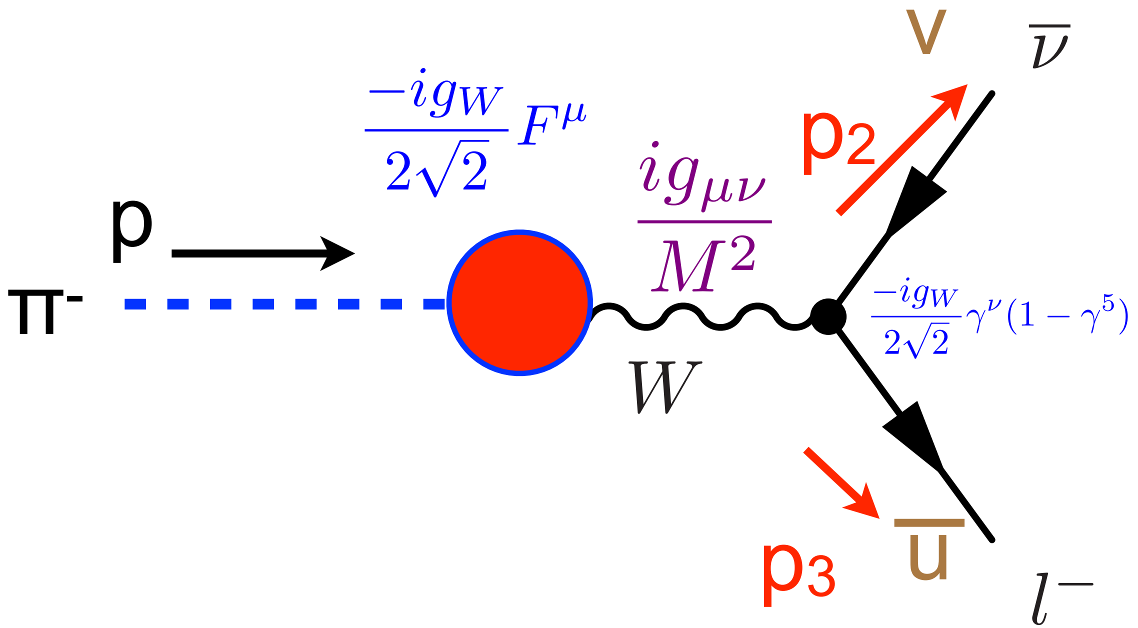
In principle (as Griffiths does) we could use the same formalism to calculate the lifetime of the neutron. We know of course that this will not be the full story, since the decay involves quarks, and not protons. Reminder that neutron lifetime (15 minutes!) is so long because of phase space considerations: masses of neutron and proton are so close together

Decay of the pion



Decay of charged pion to leptons. Note that we skipped the discussion of form factors earlier. General problem: We don’t know the overlap between the wave functions of the quark (if they are far apart in the pion bound state, then decay via this mode is less likely)

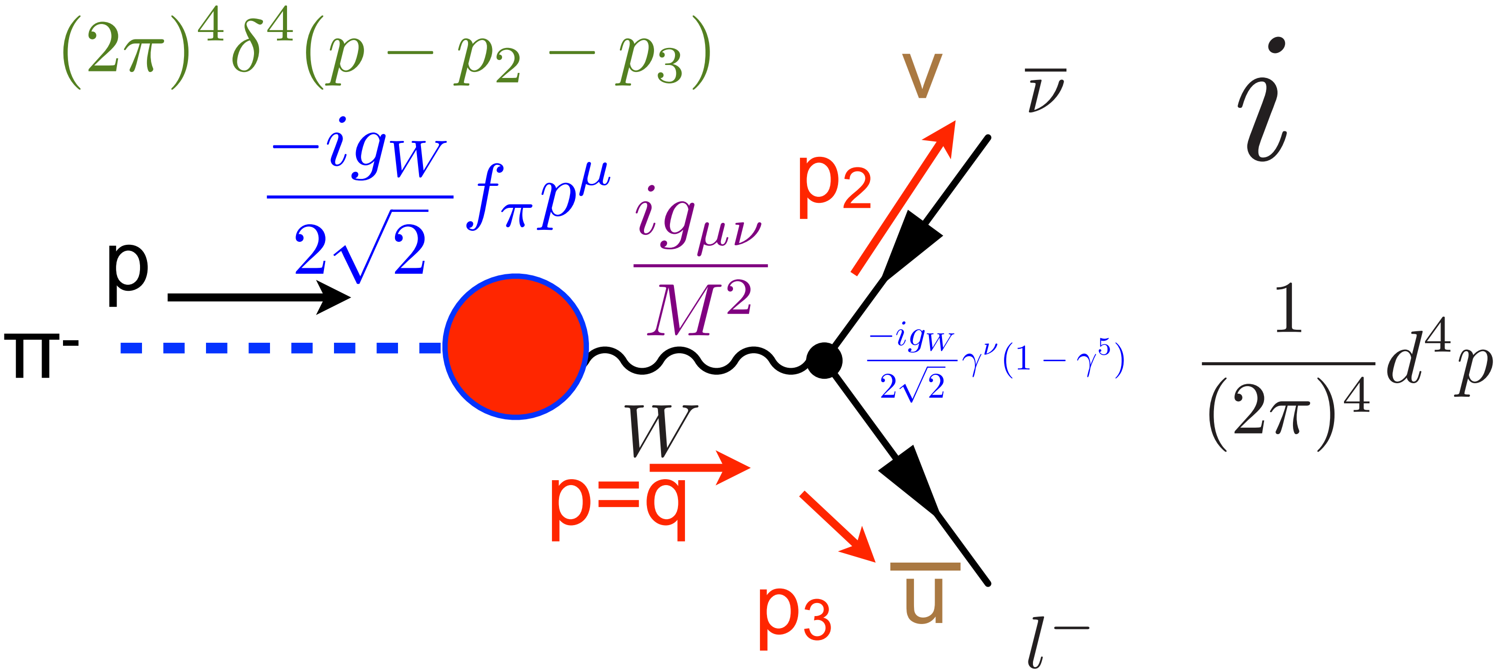
What can we do instead?



We know the coupling of the W to the leptons. Let's try and be as general as possible on the coupling of the pion to the W

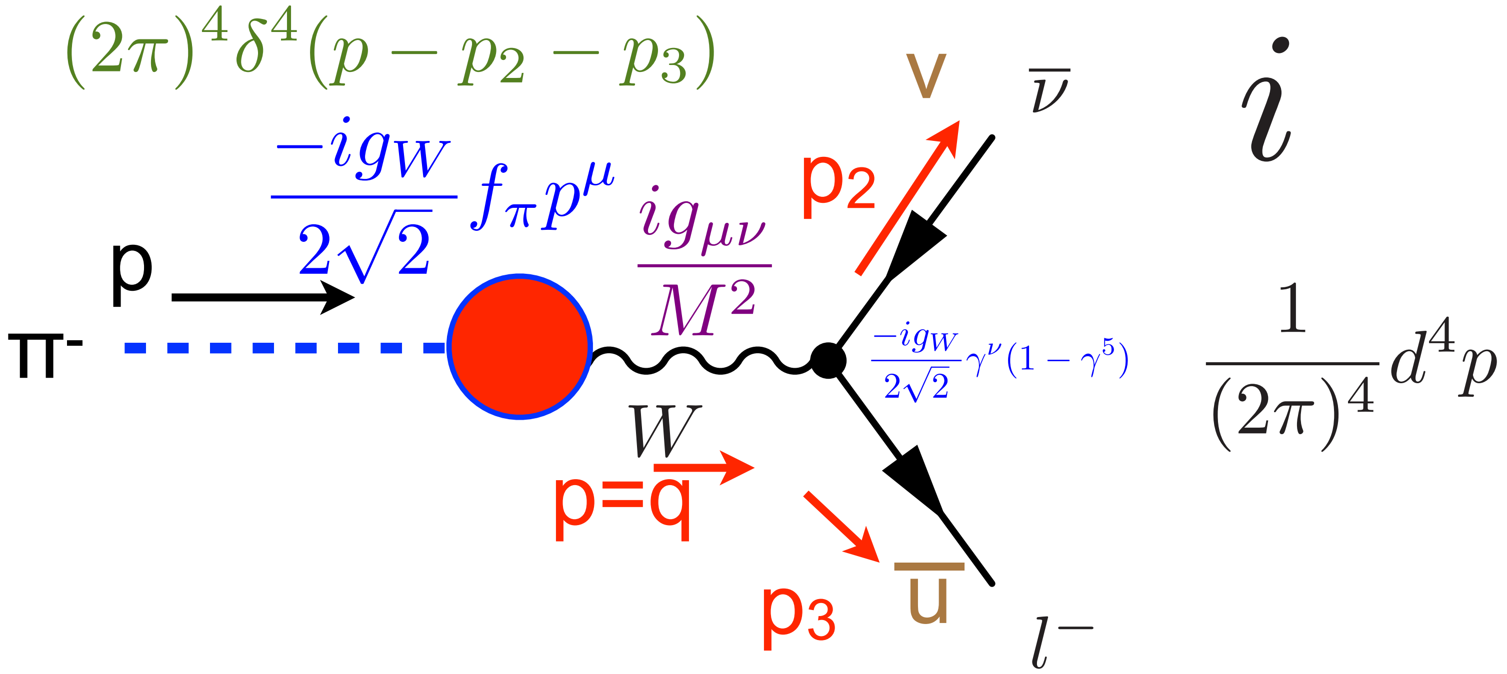
"Form factor" describing the blob must be a 4-vector (has to contract with γ_μ)

What form can F take?



F must be be of form (F^μ), but what must it be? Only quantity it can depend on is p, so it must be of form fp^μ, where f is the pion decay constant (this is the unknown that we get instead of the wave function overlap)

The matrix element



$$\mathcal{M} = \int i[\overline{u}(3)][-i\frac{g_W}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)][v(2)]\frac{ig_{\mu\nu}}{M^2}(-i\frac{g_W}{2\sqrt{2}}f_\pi p^\mu)(2\pi)^4(p - p_2 - p_3)\frac{d^4p}{(2\pi)^4}$$

The matrix element

$$\mathcal{M} = \int i[\overline{u}(3)][-i\frac{g_W}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)][v(2)]\frac{ig_{\mu\nu}}{M^2}(-i\frac{g_W}{2\sqrt{2}}f_\pi p^\mu)(2\pi)^4(p-p_2-p_3)\frac{d^4p}{(2\pi)^4}$$

$$\mathcal{M} = \frac{f_\pi g_W^2}{8M_W^2}[\overline{u}(3)][\gamma^\nu(1-\gamma^5)][v(2)]p_\nu$$

Sum over outgoing spins
and apply Casimir tricks.
Also, we know that $m_2 = 0$

$$< |\mathcal{M}|^2 >= \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 [\overline{u}(3)][\gamma^\nu(1-\gamma^5)][v(2)]p_\nu \times$$

$$\left[[\overline{u}(3)][\gamma^\mu(1-\gamma^5)][v(2)]p_\mu\right]^*$$

The matrix element

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 [\overline{u}(3)][\gamma^\nu(1-\gamma^5)][v(2)]p_\nu \times$$

$$\left[[\overline{u}(3)][\gamma^\mu(1-\gamma^5)][v(2)]p_\mu\right]^*$$

$$\sum_{spins} [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^* = \text{Trace}[\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2(\not{p}_a + m_a)]$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 p_\mu p_\nu Tr[\gamma^\nu(1-\gamma^5)(\not{p}_2 - m_2)\overline{\gamma^\mu(1-\gamma^5)}(\not{p}_3 + m_3)]$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 p_\mu p_\nu Tr[\gamma^\nu(1-\gamma^5)(\not{p}_2)\gamma^\mu(1-\gamma^5)(\not{p}_3 + m_l)]$$

$$8(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} p_1 \cdot p_3 - i\epsilon^{\mu\nu\lambda\sigma} p_{1\lambda} p_{3\sigma})$$

The matrix element

$$< |\mathcal{M}|^2 >= \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 p_\mu p_\nu Tr[\gamma^\nu(1-\gamma^5)(\not{p}_2)\gamma^\mu(1-\gamma^5)(\not{p}_3+m_l)]$$

Thankfully, we calculated that trace already...

$$< |\mathcal{M}|^2 >= \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 p_\mu p_\nu Tr[\gamma^\nu(1-\gamma^5)(\not{p}_2)\gamma^\mu(1-\gamma^5)(\not{p}_3+m_l)]$$

$$< |\mathcal{M}|^2 >= \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 8p_\mu p_\nu \left(p_2^\mu p_3^\nu + p_3^\mu p_2^\nu - g^{\mu\nu}(p_2 \cdot p_3) - i\epsilon^{\mu\nu\lambda\sigma} p_{2\lambda} p_{3\sigma}\right)$$

$$p = p_2 + p_3 \rightarrow$$

$$(p \cdot p_2) = p_2^2 + (p_2 \cdot p_3) = (p_2 \cdot p_3)$$

$$(p \cdot p_3) = p_3^2 + (p_2 \cdot p_3) = (p_2 \cdot p_3) + m_l^2$$

The matrix element

$$< |\mathcal{M}|^2 >= \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 8p_\mu p_\nu \left(p_2^\mu p_3^\nu + p_3^\mu p_2^\nu - g^{\mu\nu}(p_2 \cdot p_3) - i\epsilon^{\mu\nu\lambda\sigma} p_{2\lambda} p_{3\sigma}\right)$$

$$(p \cdot p_2) = p_2^2 + (p_2 \cdot p_3) = (p_2 \cdot p_3)$$

$$(p \cdot p_3) = p_3^2 + (p_2 \cdot p_3) = (p_2 \cdot p_3) + m_l^2$$

$$p^2 = (p_2 + p_3)^2 = m_\pi^2 = p_2^2 + p_3^2 + 2(p_2 \cdot p_3) = m_l^2 + 2(p_2 \cdot p_3)$$

$$(p_2 \cdot p_3) = \frac{m_\pi^2 - m_l^2}{2} = (p \cdot p_2)$$

$$(p \cdot p_3) = \frac{m_\pi^2 + m_l^2}{2}$$

$$< |\mathcal{M}|^2 >= \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 8\left((p \cdot p_2)(p \cdot p_3) + (p \cdot p_2)(p \cdot p_3) - p^2(p_2 \cdot p_3) - i\epsilon^{\mu\nu\lambda\sigma} p_\mu p_\nu p_{2\lambda} p_{3\sigma}\right)$$

Using earlier trick, epsilon terms will cancel...

$$\epsilon^{\mu\nu\lambda\sigma} p_\mu p_\nu p_{2\lambda} p_{3\sigma} = \epsilon^{01\lambda\sigma} p_0 p_1 p_{2\lambda} p_{3\sigma} + \epsilon^{10\lambda\sigma} p_0 p_1 p_{2\lambda} p_{3\sigma} + \ldots$$

The matrix element

$$< |\mathcal{M}|^2 >= \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 8\left(\left((p\cdot p_2)(p\cdot p_3) + (p\cdot p_2)(p\cdot p_3) - p^2(p_2\cdot p_3)\right)\right)$$

$$< |\mathcal{M}|^2 >= \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 8\left(\frac{m_\pi^2 - m_l^2}{2}\frac{m_\pi^2 + m_l^2}{2} + \frac{m_\pi^2 - m_l^2}{2}\frac{m_\pi^2 + m_l^2}{2} - m_\pi^2\frac{m_\pi^2 - m_l^2}{2}\right)$$

$$< |\mathcal{M}|^2 >= \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 8\left(\frac{m_\pi^4 - m_l^4}{2} - \frac{m_\pi^4}{2} + \frac{m_\pi^2 m_l^2}{2}\right)$$

$$< |\mathcal{M}|^2 >= \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 8\left(\frac{m_\pi^2 m_l^2 - m_l^4}{2}\right)$$

$$< |\mathcal{M}|^2 >= \left(\frac{f_\pi g_W^2}{4M_W^2}\right)^2 \left(m_\pi^2 m_l^2 - m_l^4\right)$$

What is pion decay rate?

Recall for a matrix element that factorizes:

$$\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2$$

$$<|\mathcal{M}|^2>= \left(\frac{f_\pi g_W^2}{4M_W^2}\right)^2 (m_\pi^2 m_l^2 - m_l^4)$$

And we even calculated the momentum way back then:

$$r = |\mathbf{p_2}| = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2}$$

$$r = |\mathbf{p_2}| = \frac{1}{2m_\pi}\sqrt{m_\pi^4 + m_l^4 - 2m_\pi^2m_l^2}$$

Here:

$$r = |\mathbf{p_2}| = \frac{1}{2m_\pi}(m_\pi^2 - m_l^2)$$

What is pion decay rate?

$$\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2} |\mathcal{M}|^2 \qquad r = |\mathbf{p_2}| = \frac{1}{2m_\pi}(m_\pi^2 - m_l^2)$$

$$< |\mathcal{M}|^2 >= \left(\frac{f_\pi g_W^2}{4M_W^2} \right)^2 (m_\pi^2 m_l^2 - m_l^4)$$

$$\Gamma = \frac{1}{8\pi m_\pi^2} \frac{1}{2m_\pi} (m_\pi^2 - m_l^2) \left(\frac{f_\pi g_W^2}{4M_W^2} \right)^2 (m_\pi^2 m_l^2 - m_l^4)$$

$$\Gamma = \frac{1}{\pi m_\pi^3} \left(\frac{f_\pi g_W^2}{16M_W^2} \right)^2 m_l^2 (m_\pi^2 - m_l^2)^2$$

We can't calculate absolute rate, but we can do ratios

$$\frac{\Gamma(\pi^- \rightarrow e^- + \overline{\nu}_e)}{\Gamma(\pi^- \rightarrow \mu^- + \overline{\nu}_\mu)} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} = 10^{-4}$$

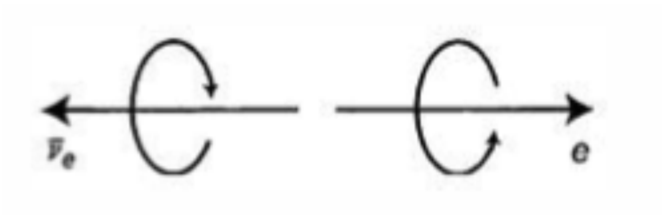
Agrees with experiment, but that is odd, because phase space arguments would predict the opposite! What is going on?

Note that if lepton mass is zero, decay rate is predicted to be zero! Why is that? Must have something to do with V-A coupling...

On V-A coupling and pion decay

Pion is spin zero, so in its rest frame its decay products must have opposite spin

But anti-neutrino is **always** right-handed, which means the electron must be right-handed too

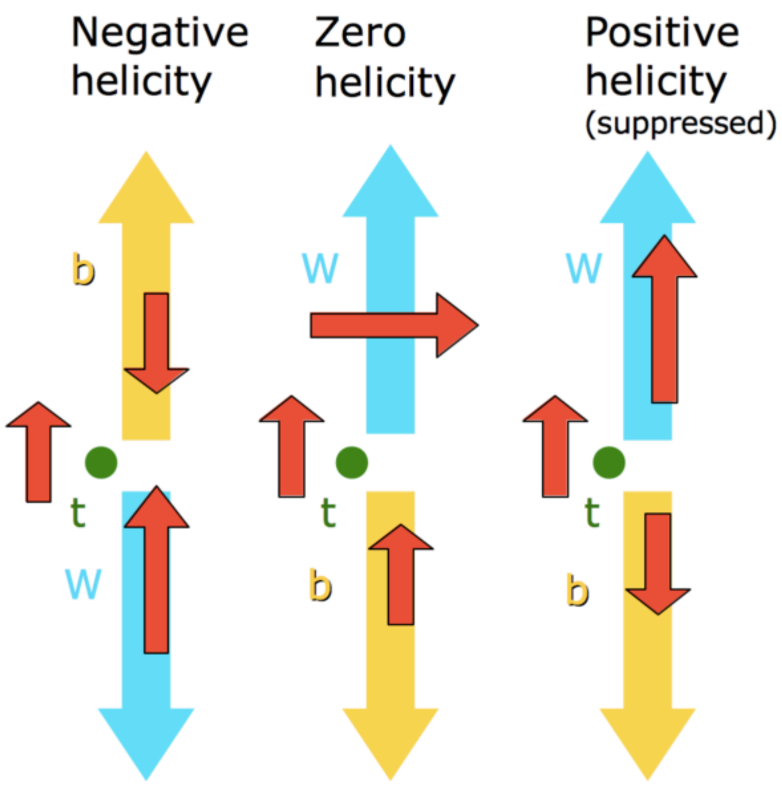


If electron had zero mass, it would only be left-handed. **Helicity suppression**

(Aside: What about decays to taus?)

W bosons in top quark decays

Top quarks provide a source of polarized W bosons! In the limit of $m_b = 0$, there are no positively polarized W bosons produced in top quark decays (at leading order).

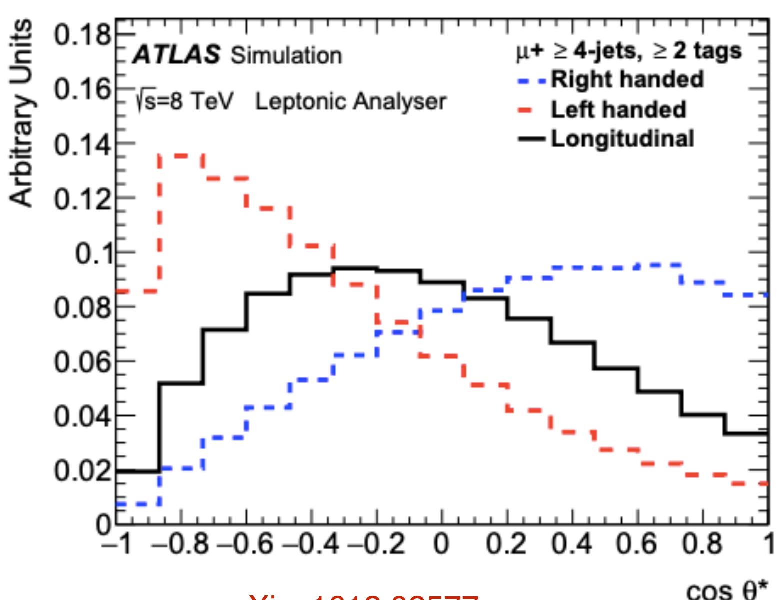


http://www.scholarpedia.org/article/File:Whelicity.png

W bosons in top quark decays

Top quarks provide a source of polarized W bosons! In the limit of $m_b = 0$, there are no positively polarized W bosons produced in top quark decays (at leading order).

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\theta^*} = \frac{3}{4}\left(1-\cos^2\theta^*\right) F_0 + \frac{3}{8}(1-\cos\theta^*)^2 \; F_L + \frac{3}{8}(1+\cos\theta^*)^2 \; F_R$$

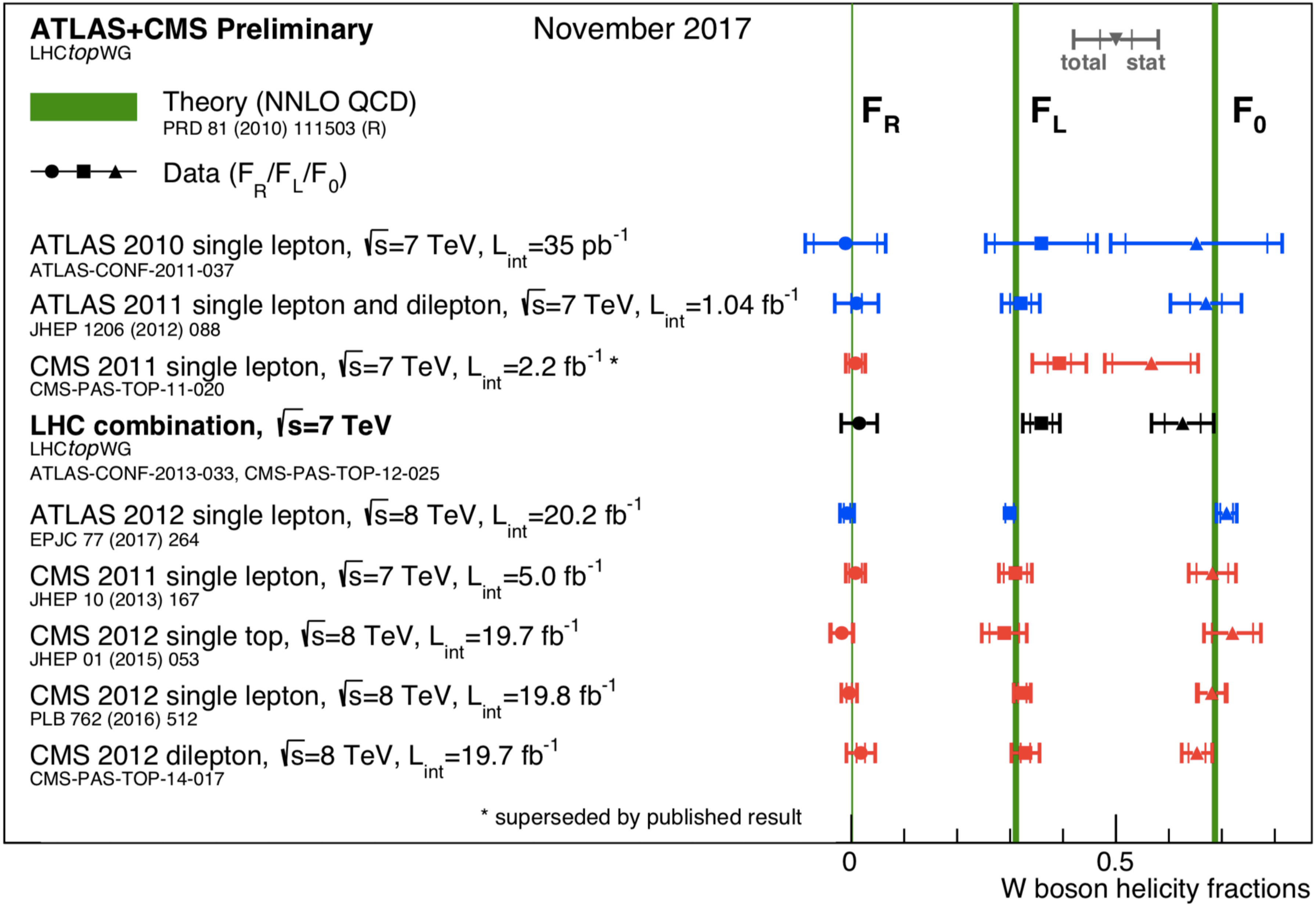


arXiv: 1612.02577

θ^* is the angle between the W boson decay product and the reversed flight direction of the b quark from the top quark decay, in the W rest frame (not that we must pick which b quark is the right one, then!)

W bosons in top quark decays

https://twiki.cern.ch/twiki/bin/view/LHCPhysics/LHCTopWGSummaryPlots#W_boson_helicity



On to charged weak interaction of quarks

In lepton sector, W couples only within a lepton generation (this is why electron number, muon number and tau number are conserved).

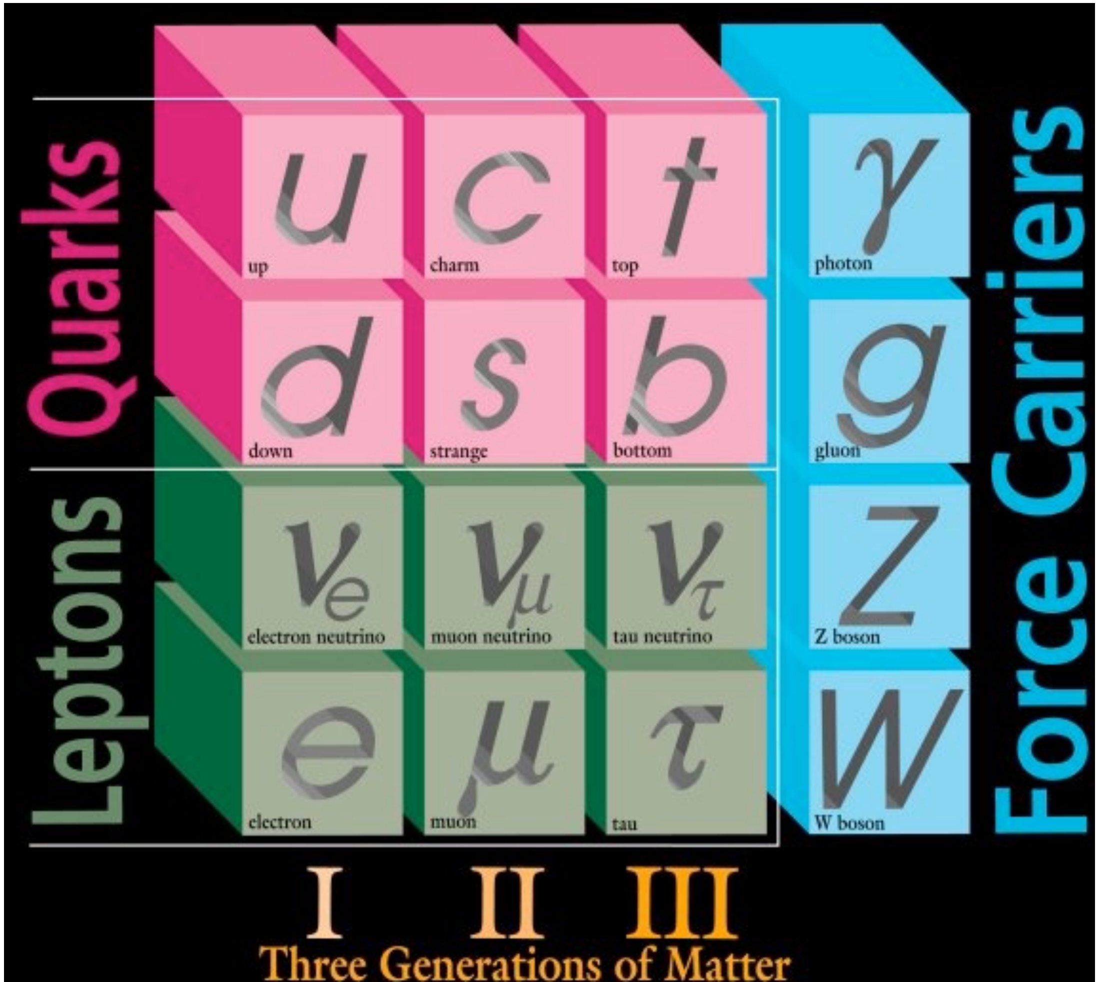
$$W^+ \rightarrow e^+ + \nu_e$$

$$W^+ \rightarrow \mu^+ + \nu_e$$

$$W^+ \rightarrow \tau^+ + \nu_e$$

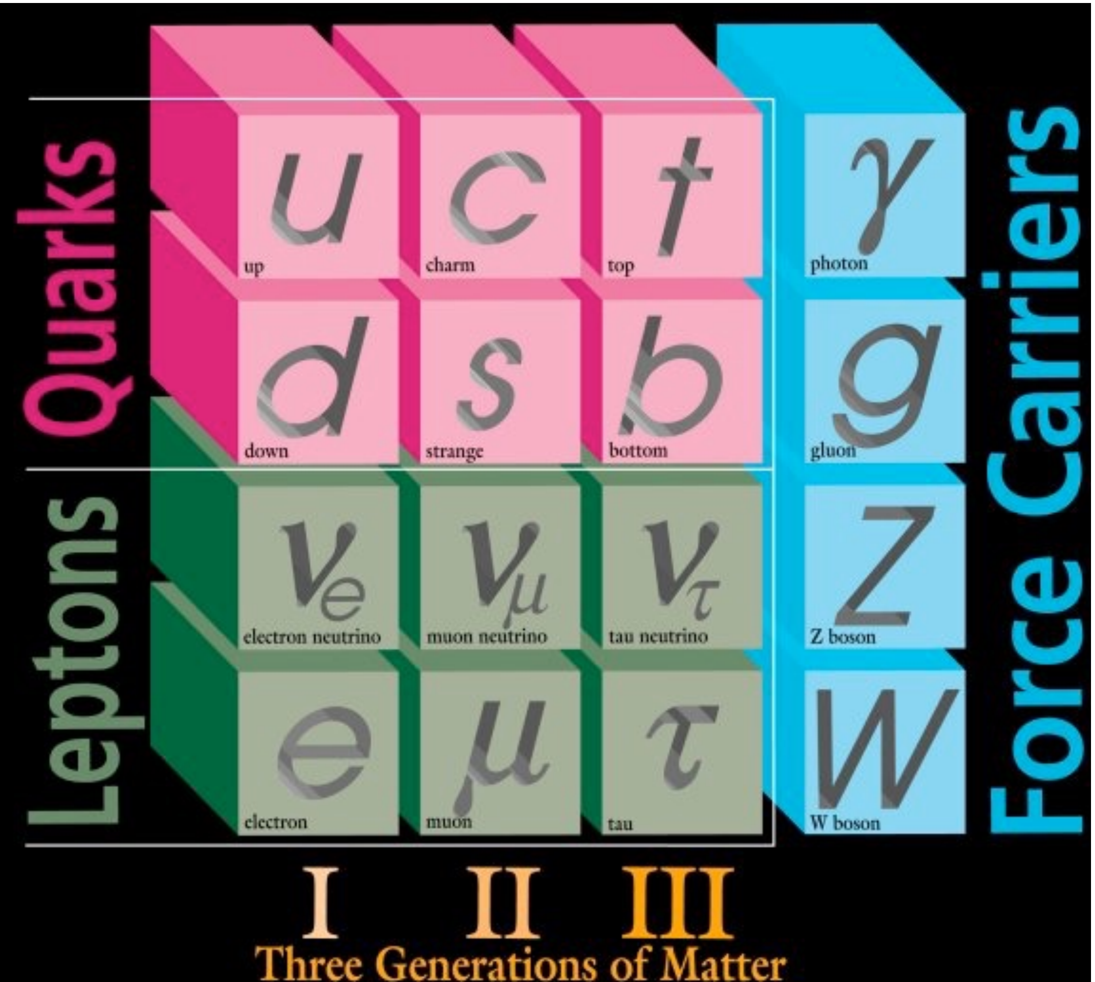
On to charged weak interaction of quarks

In quark sector, might expect something analogous to hold



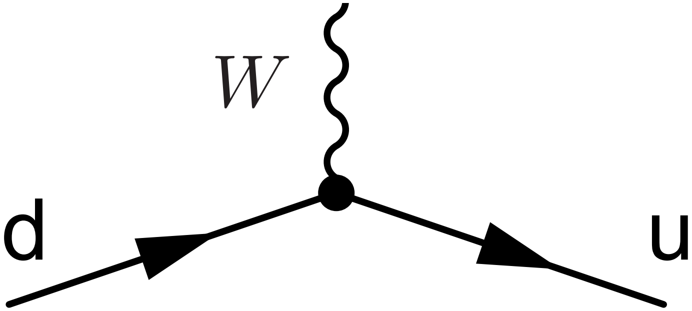
Coupling is predominantly but not entirely within a generation (there is no “conservation of 1st generation of quarks”)

This is how 2nd and 3rd generations of quarks can decay to the lighter 1st generation quarks

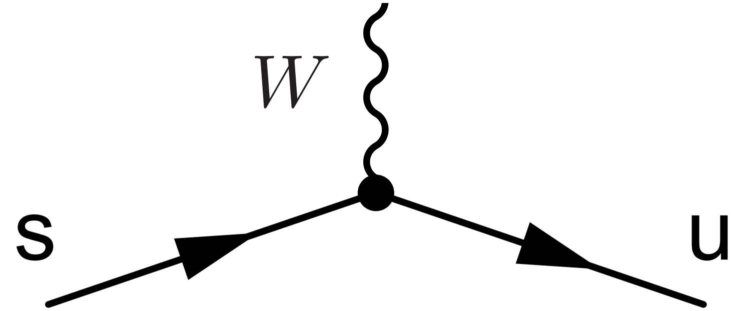


As we have discussed, the mass eigenstates that we know of are not the same eigenstates of the charged weak interaction, but instead are slightly rotated

Charged weak interaction vertices



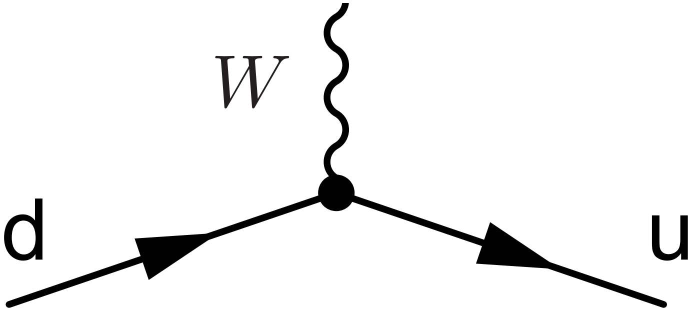
$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_c$$

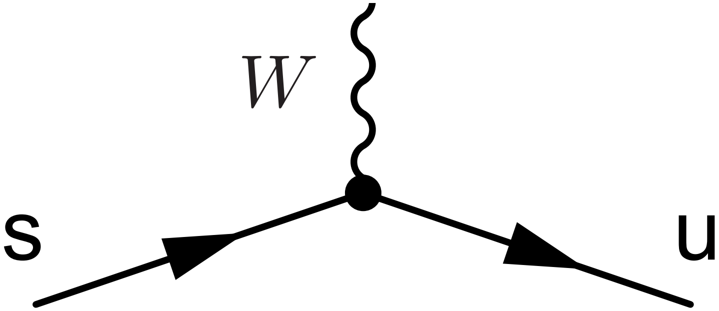


$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\sin\theta_c$$

θ_c (Cabbibo angle, after Nicola Cabibbo, who proposed it), is small (13°), so that strangeness-violating decay is weak

Example usage





$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_c \qquad \frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\sin\theta_c$$

$$\frac{\Gamma(K^- \to l - + \overline{\nu}_l)}{\Gamma(\pi^- \to l^- + \overline{\nu}_l)} = \frac{f_K^2}{f_\pi^2}\tan^2\theta_c \left(\frac{m_\pi}{m_K}\right)^3 \frac{m_l^2(m_K^2 - m_l^2)^2}{m_l^2(m_\pi^2 - m_l^2)^2}$$

Let's assume form factors for kaon and pion are ~the same

$$\frac{\Gamma(K^- \to l - + \overline{\nu}_l)}{\Gamma(\pi^- \to l^- + \overline{\nu}_l)} = \tan^2\theta_c \left(\frac{m_\pi}{m_K}\right)^3 \frac{m_l^2(m_K^2 - m_l^2)^2}{m_l^2(m_\pi^2 - m_l^2)^2}$$

Pion vs Kaon leptonic decays

Charged kaon mass ~0.494 GeV
Charged pion mass ~0.140 GeV
Electron mass ~ 0.0005 GeV
Muon mass ~0.106 GeV

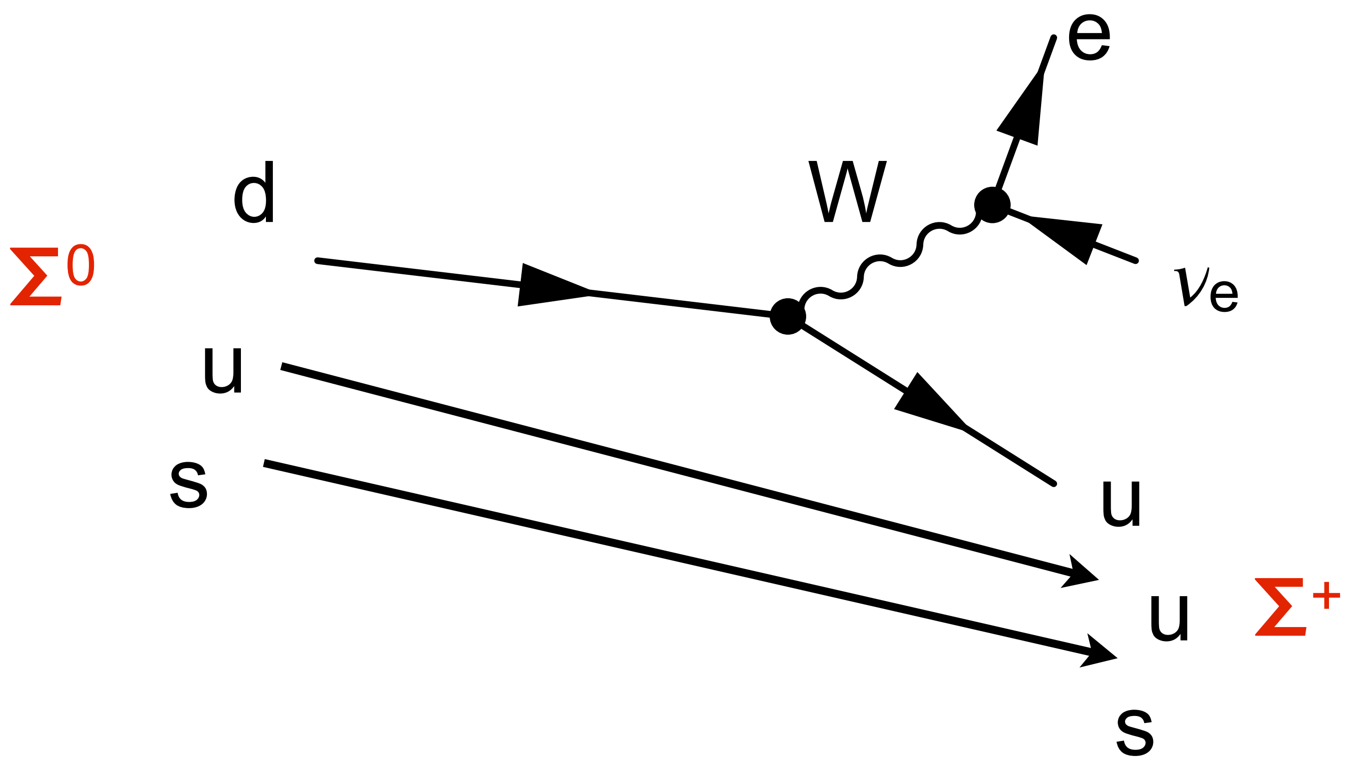
$$\frac{\Gamma(K^- \rightarrow l - + \overline{\nu}_l)}{\Gamma(\pi^- \rightarrow l^- + \overline{\nu}_l)} = \tan^2\theta_c \left(\frac{m_\pi}{m_K}\right)^3 \frac{(m_K^2 - m_l^2)^2}{(m_\pi^2 - m_l^2)^2}$$

Without Cabbibo angle:
electron ratio: 3.5
muon ratio: 17.6

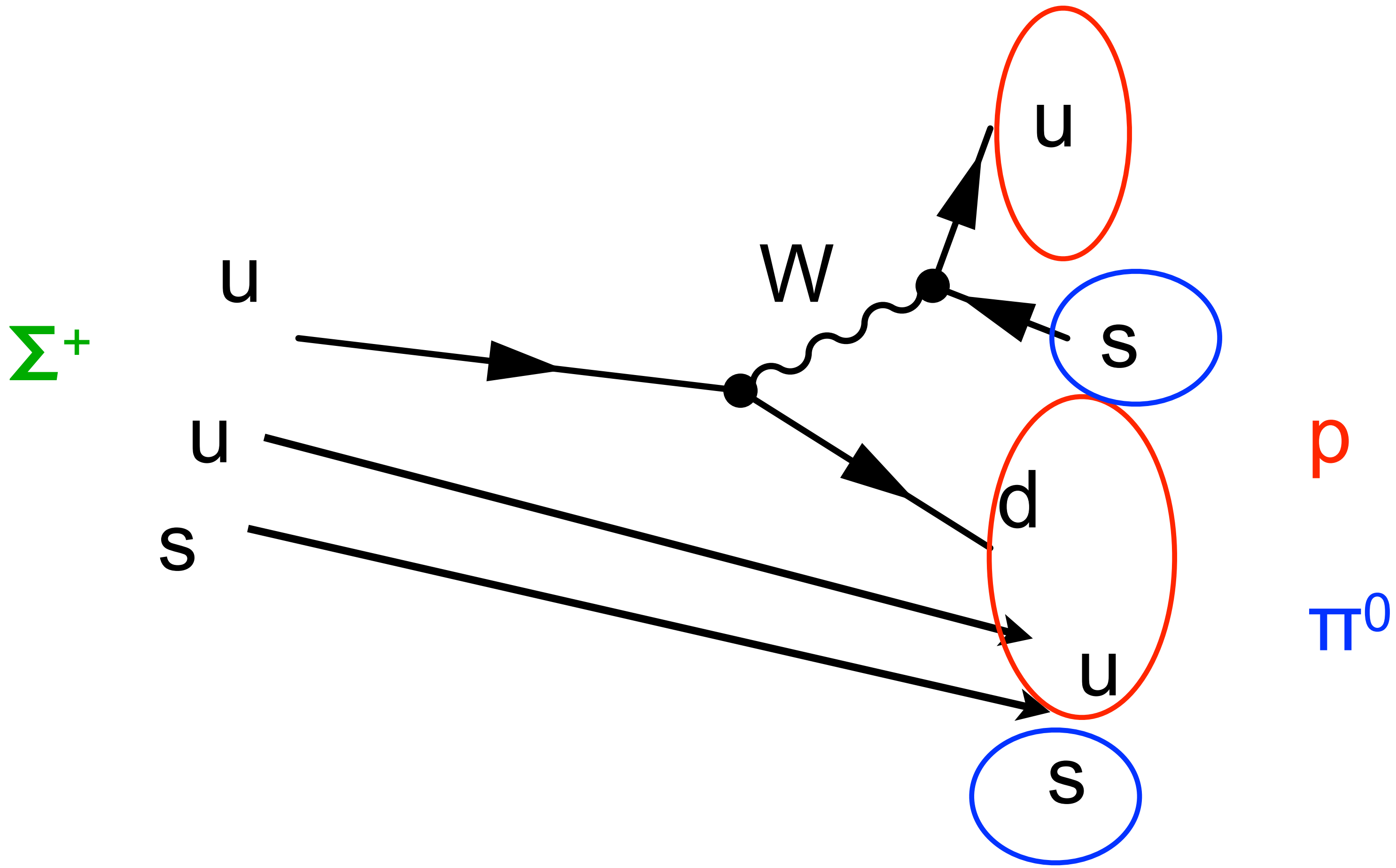
With Cabbibo angle:
electron ratio: 0.19
muon ratio: 0.96

Observed (0.26/1.34)

More complicated decays

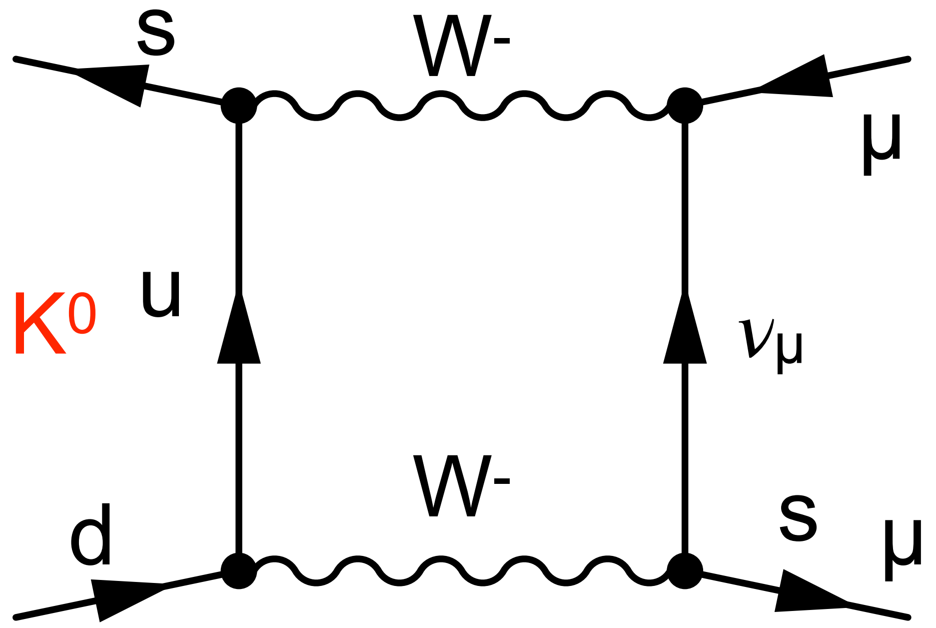


More complicated decays

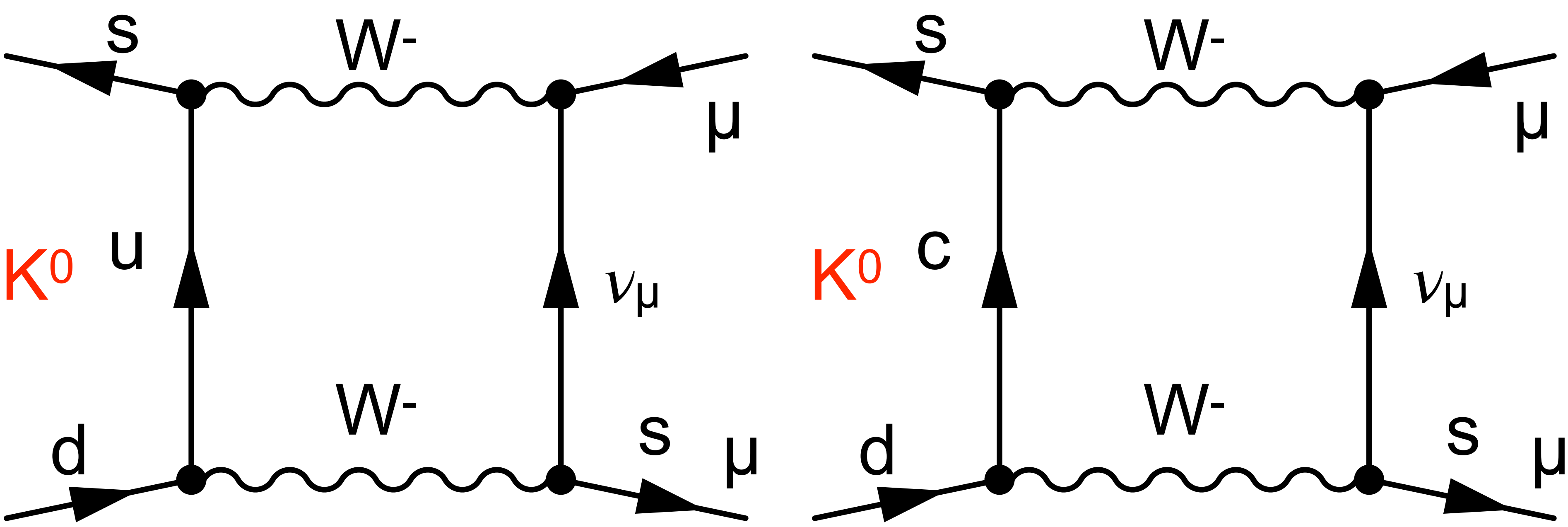


Kaon decays

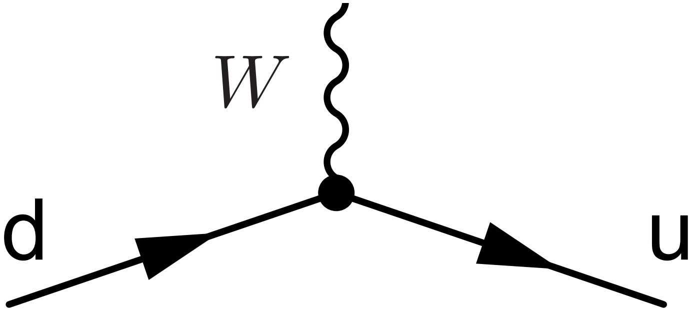
Kaon decays to muons observed to occur at a much much lower rate than observed. Why might that be?



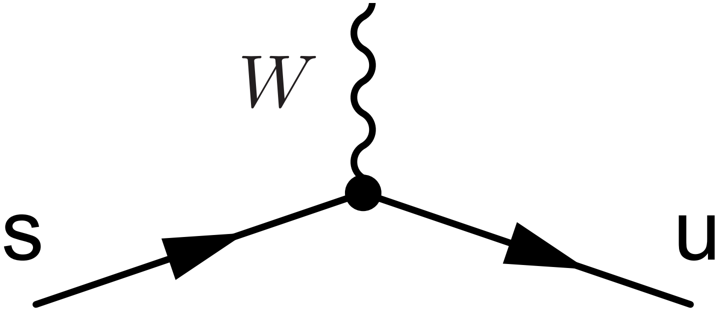
Glashow, Iliopoulous and Maiani proposed the "GIM" mechanism - a new 4th quark comes in with opposite sign and cancels the diagram!



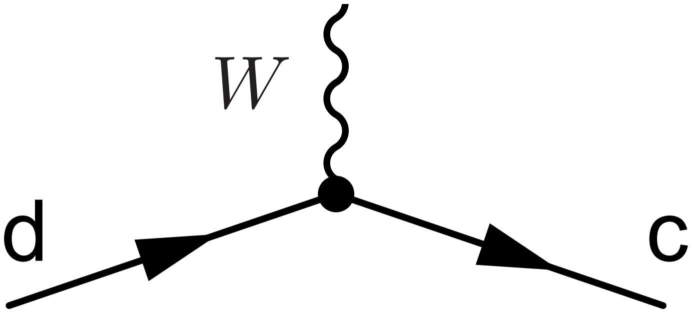
W couplings to new charm quark



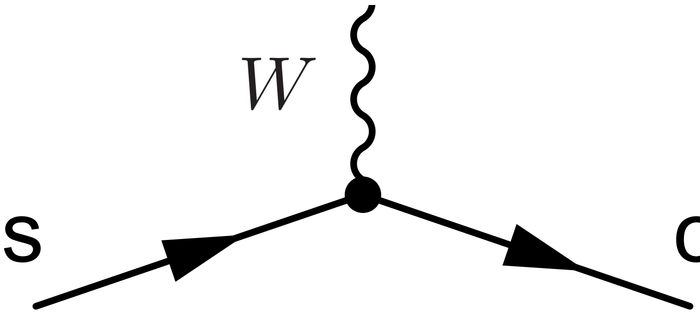
$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_c$$



$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\sin\theta_c$$



$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)(-\sin\theta_c)$$

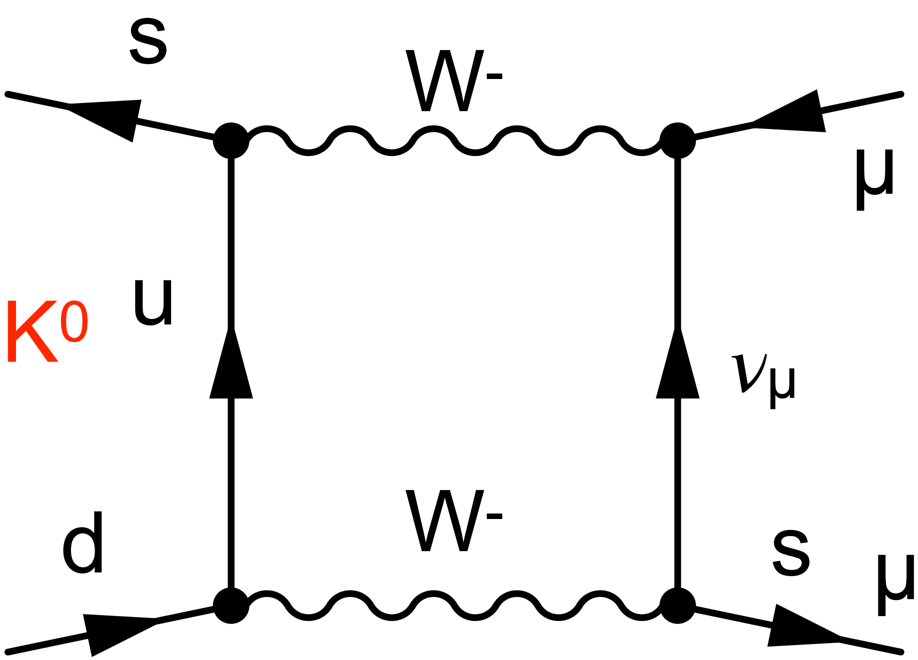


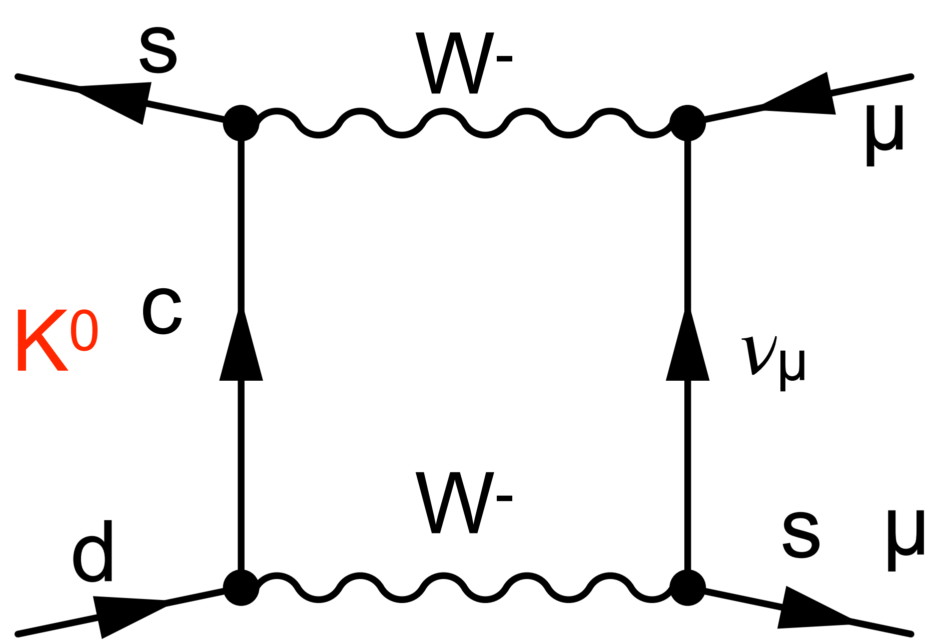
$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_c$$

Terms cancel!
(Up until charm mass effects, so not identically)

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\sin\theta_c$$

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_c$$





$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_c$$

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)(-\sin\theta_c)$$

Interpretation

$$d' = d\cos\theta_c + s\sin\theta_c$$
$$s' = -d\sin\theta_c + s\cos\theta_c$$

W bosons couple to u and c quarks, but to linear combinations d' and s', not d and s

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

$$\begin{pmatrix} u \\ d' \end{pmatrix} \begin{pmatrix} c \\ s' \end{pmatrix}$$

OR

$$\begin{pmatrix} u \\ d\cos\theta_c + s\sin\theta_c \end{pmatrix} \begin{pmatrix} c \\ -d\sin\theta_c + s\cos\theta_c \end{pmatrix}$$

Expanding to the CKM matrix (we saw this already!)

Kobayashi

Maskawa

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

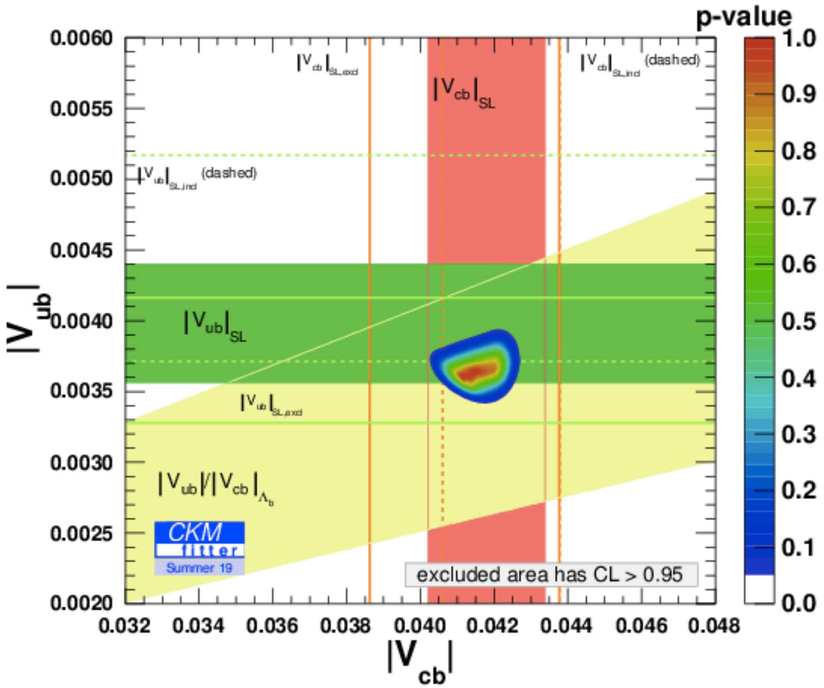
$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} 0.97434 & 0.22506 & 0.00357 \\ 0.22492 & 0.97351 & 0.0411 \\ 0.00875 & 0.0403 & 0.99915 \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

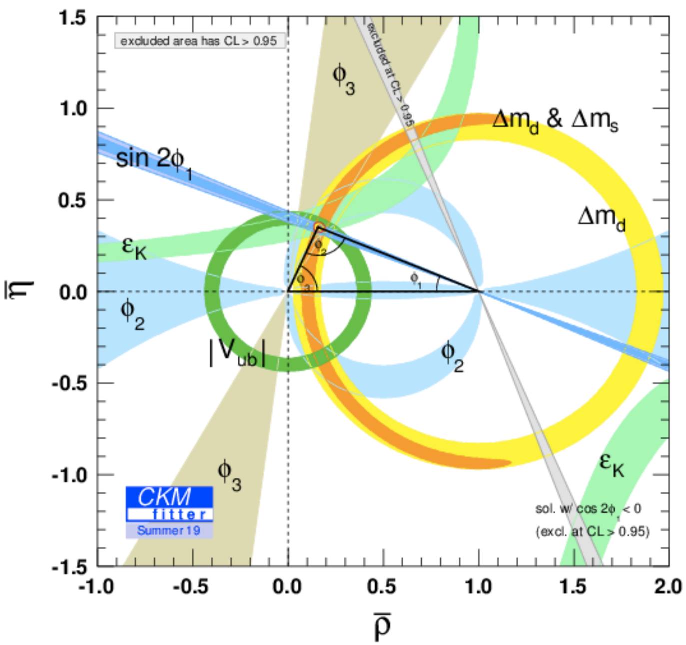
Expanding to the CKM matrix

CKM matrix V has in general 3 “angles” and one phase (which allows for CP violating effects!) Lots of efforts to probe and test the CKM matrix and look for new physics (for example, is it unitary?)

CKM Fitter collaboration

http://ckmfitter.in2p3.fr/www/results/plots_summer19/ckm_res_summer19.html





Expanding to the CKM matrix

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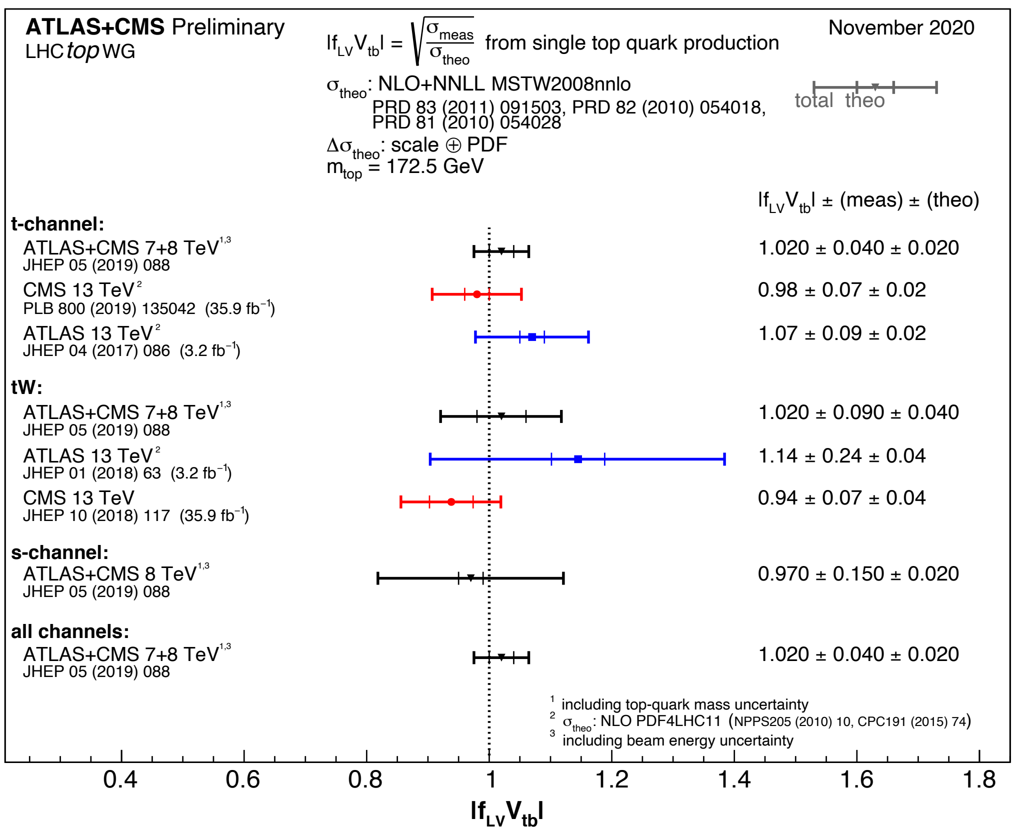
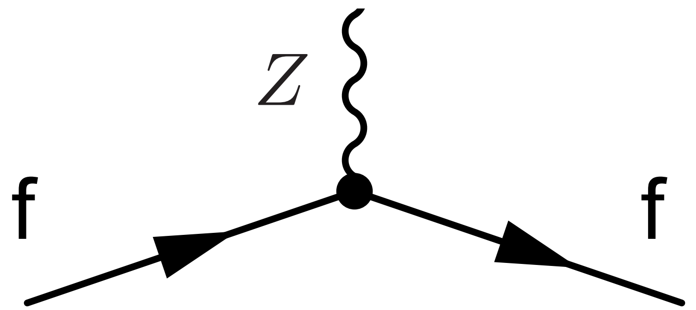


ATLAS+CMS V_{tb} measurements from single top quark production

https://twiki.cern.ch/twiki/bin/view/LHCPhysics/LHCTopWGSummaryPlots#Single_Top_Quark_Production

Electroweak unification (as we'll see) requires the existence of neutral weak processes



$$\frac{-ig_Z}{2}\gamma^\mu(c_V^f - c_A^f\gamma^5)$$

$$g_w = \frac{g_e}{\sin\theta_w}$$

Careful: 2 vs 2*sqrt(2)!

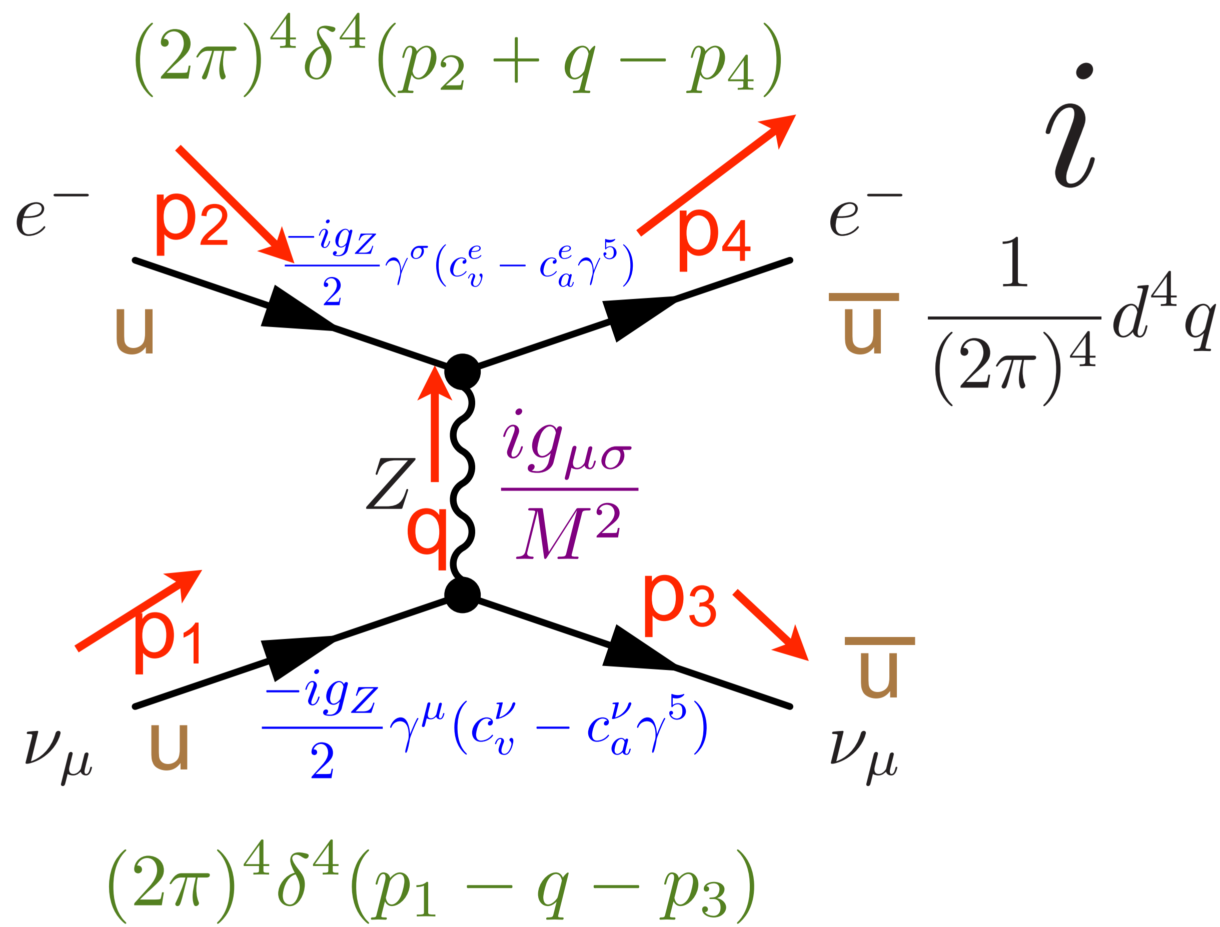
$$g_z = \frac{g_e}{\sin\theta_w \cos\theta_w}$$

$$\theta_w = 28.75^o$$

$$\sin^2\theta_w = 0.2314$$

f	c_V	c_A
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	$\frac{1}{2}$
e, μ, τ	$\frac{-1}{2} + 2\sin^2\theta_w$	$\frac{-1}{2}$
u, c, t	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_w$	$\frac{1}{2}$
d, s, b	$\frac{-1}{2} + \frac{2}{3}\sin^2\theta_w$	$\frac{-1}{2}$

Neutrino-electron scattering



Careful about “nu” in numerator! Not an index

Assume small momentum transfer

$$\mathcal{M} = \int i[\overline{u}(3)][-i\frac{g_Z}{2}\gamma^\mu(c_V^\nu - c_A^\nu\gamma^5)][u(1)]\frac{ig_{\mu\sigma}}{M^2}[\overline{u}(4)][-i\frac{g_Z}{2}\gamma^\sigma(c_V^e - c_A^e\gamma^5)][u(2)]$$

$$(2\pi)^4\delta^4(p_1-q-p_3)(2\pi)^4\delta^4(p_2+q-p_4)\frac{d^4q}{(2\pi)^4}$$

Neutrino-electron scattering matrix element

$$\mathcal{M} = \int i[\overline{u}(3)][-i\frac{g_Z}{2}\gamma^\mu(c_V^\nu - c_A^\nu\gamma^5)][u(1)]\frac{ig_{\mu\sigma}}{M^2}[\overline{u}(4)][-i\frac{g_Z}{2}\gamma^\sigma(c_V^e - c_A^e\gamma^5)][u(2)]$$

$$(2\pi)^4\delta^4(p_1 - q - p_3)(2\pi)^4\delta^4(p_2 + q - p_4)\frac{d^4q}{(2\pi)^4}$$

Know that we can use earlier result (with appropriate substitution, and being careful about sqrt(2) difference in coupling) and ignoring electron mass

$$|\mathcal{M}|^2 = \frac{g_Z^4}{16M_Z^4}Tr[\gamma^\mu(C_V^\nu - C_A^\nu\gamma^5)(\not{p}_1)(\gamma^\sigma((C_V^\nu - C_A^\nu\gamma^5)(\not{p}_3)]\times$$

$$Tr[\gamma_\mu(C_V^e - C_A^e\gamma^5)(\not{p}_2)(\gamma_\sigma((C_V^e - C_A^e\gamma^5)(\not{p}_4)]$$

Neutrino-electron scattering matrix element

$$|\mathcal{M}|^2 = \frac{g_Z^4}{16M_Z^4} Tr[\gamma^\mu (C_V^\nu - C_A^\nu \gamma^5)(\not{p}_1)(\gamma^\sigma ((C_V^\nu - C_A^\nu \gamma^5)(\not{p}_3)] \times$$
$$Tr[\gamma_\mu (C_V^e - C_A^e \gamma^5)(\not{p}_2)(\gamma_\sigma ((C_V^e - C_A^e \gamma^5)(\not{p}_4)]$$

Let's focus on second trace

$$Tr[\gamma_\mu (C_V^e - C_A^e \gamma^5)(\not{p}_2)(\gamma_\sigma (C_V^e - C_A^e \gamma^5)(\not{p}_4)] =$$
$$Tr[\gamma_\mu C_V^e \gamma_\nu p_2^\nu \gamma_\sigma C_V^e \gamma_\alpha p_4^\alpha - \gamma_\mu C_V^e \gamma_\nu p_2^\nu \gamma_\sigma C_A^e \gamma^5 \gamma_\alpha p_4^\alpha -$$
$$\gamma_\mu C_A^e \gamma^5 \gamma_\nu p_2^\nu \gamma_\sigma C_V^e \gamma_\alpha p_4^\alpha + \gamma_\mu C_A^e \gamma^5 \gamma_\nu p_2^\nu \gamma_\sigma C_A^e \gamma^5 \gamma_\alpha p_4^\alpha]$$

$$Tr[\gamma_\mu (C_V^e - C_A^e \gamma^5)(\not{p}_2)(\gamma_\sigma (C_V^e - C_A^e \gamma^5)(\not{p}_4)] =$$
$$Tr[\gamma_\mu C_V^e \gamma_\nu p_2^\nu \gamma_\sigma C_V^e \gamma_\alpha p_4^\alpha + \gamma^5 \gamma_\mu C_V^e \gamma_\nu p_2^\nu \gamma_\sigma C_A^e \gamma_\alpha p_4^\alpha +$$
$$\gamma^5 \gamma_\mu C_A^e \gamma_\nu p_2^\nu \gamma_\sigma C_V^e \gamma_\alpha p_4^\alpha + \gamma_\mu C_A^e \gamma_\nu p_2^\nu \gamma_\sigma C_A^e \gamma_\alpha p_4^\alpha]$$

Neutrino-electron scattering matrix element

$$Tr[\gamma_\mu(C_V^e - C_A^e\gamma^5)(\not{p}_2)(\gamma_\sigma(C_V^e - C_A^e\gamma^5)(\not{p}_4)] =$$
$$Tr[\gamma_\mu C_V^e \gamma_\nu p_2^\nu \gamma_\sigma C_V^e \gamma_\alpha p_4^\alpha + \gamma^5\gamma_\mu C_V^e \gamma_\nu p_2^\nu \gamma_\sigma C_A^e \gamma_\alpha p_4^\alpha +$$
$$\gamma^5\gamma_\mu C_A^e \gamma_\nu p_2^\nu \gamma_\sigma C_V^e \gamma_\alpha p_4^\alpha + \gamma_\mu C_A^e \gamma_\nu p_2^\nu \gamma_\sigma C_A^e \gamma_\alpha p_4^\alpha]$$

$$Tr[\gamma_\mu(C_V^e - C_A^e\gamma^5)(\not{p}_2)(\gamma_\sigma(C_V^e - C_A^e\gamma^5)(\not{p}_4)] =$$
$$4(C_V^e)^2 p_2^\nu p_4^\alpha \left(g_{\mu\nu}g_{\sigma\alpha} - g_{\mu\sigma}g_{\nu\alpha} + g_{\mu\alpha}g_{\nu\sigma}\right) + 4iC_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha} p_2^\nu p_4^\alpha +$$
$$4iC_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha} p_2^\nu p_4^\alpha + 4(C_A^e)^2 p_2^\nu p_4^\alpha \left((g_{\mu\nu}g_{\sigma\alpha} - g_{\mu\sigma}g_{nu\alpha} + g_{\mu\alpha}g_{\nu\sigma})\right]$$

$$Tr[\gamma_\mu(C_A^e - C_V^e\gamma^5)(\not{p}_2)(\gamma_\sigma(C_A^e - C_V^e\gamma^5)(\not{p}_4)] =$$
$$4((C_V^e)^2 + (C_A^e)^2) p_2^\nu p_4^\alpha \left(g_{\mu\nu}g_{\sigma\alpha} - g_{\mu\sigma}g_{\nu\alpha} + g_{\mu\alpha}g_{\nu\sigma}\right) + 8iC_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha} p_2^\nu p_4^\alpha$$

Neutrino-electron scattering matrix element

$$Tr[\gamma_\mu(C_A^e - C_V^e\gamma^5)(\not{p}_2)(\gamma_\sigma(C_A^e - C_V^e\gamma^5)(\not{p}_4)] =$$

$$4((C_V^e)^2 + (C_A^e)^2)p_2^\nu p_4^\alpha\,(g_{\mu\nu}g_{\sigma\alpha} - g_{\mu\sigma}g_{\nu\alpha} + g_{\mu\alpha}g_{\nu\sigma}) + 8iC_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha$$

$$= 4((C_V^e)^2 + (C_A^e)^2)(p_{2\mu}p_{4\sigma} - g_{\mu\sigma}(p_2\cdot p_4) + p_{2\sigma}p_{4\mu}) + 8iC_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha$$

And then first trace is

$$= 4((C_V^\nu)^2 + (C_A^\nu)^2)(p_1^\mu p_3^\sigma - g^{\mu\sigma}(p_1\cdot p_3) + p_1^\sigma p_3^\mu) + 8iC_V^\nu C_A^\nu \epsilon^{\mu\beta\sigma\lambda}p_{1\beta}p_{3\lambda}$$

Multiplying them and contracting gives

$$= 16((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2)\times$$

$$((p_1\cdot p_2)(p_3\cdot p_4) - (p_2\cdot p_4)(p_1\cdot p_3) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)(p_2\cdot p_4) + 4(p_1\cdot p_3)(p_2\cdot p_4) -$$

$$(p_1\cdot p_3)(p_2\cdot p_4) + (p_2\cdot p_3)(p_1\cdot p_4) - (p_2\cdot p_4)(p_1\cdot p_3) + (p_1\cdot p_2)(p_3\cdot p_4))$$

$$-64C_V^e C_A^\nu C_V^\nu C_A^e \epsilon^{\mu\beta\sigma\lambda}p_{1\beta}p_{3\lambda}\epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha$$

$$+32iC_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha(p_1^\mu p_3^\sigma - g^{\mu\sigma}(p_1\cdot p_3) + p_1^\sigma p_3^\mu)$$

$$+32iC_V^\nu C_A^\nu \epsilon^{\mu\beta\sigma\lambda}p_{1\beta}p_{3\lambda}(p_{2\mu}p_{4\sigma} - g_{\mu\sigma}(p_2\cdot p_4) + p_{2\sigma}p_{4\mu})$$

Neutrino-electron scattering matrix element

$$= 16((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2)\times$$

$$((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_2 \cdot p_4)(p_1 \cdot p_3) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4) + 4(p_1 \cdot p_3)(p_2 \cdot p_4) -$$
$$(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_2 \cdot p_3)(p_1 \cdot p_4) - (p_2 \cdot p_4)(p_1 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4))$$

$$-64 C_V^e C_A^\nu C_V^\nu C_A^e \epsilon^{\mu\beta\sigma\lambda} p_{1\beta} p_{3\lambda} \epsilon_{\mu\nu\sigma\alpha} p_2^\nu p_4^\alpha$$
$$+32 i C_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha} p_2^\nu p_4^\alpha (p_1^\mu p_3^\sigma - g^{\mu\sigma}(p_1 \cdot p_3) + p_1^\sigma p_3^\mu)$$
$$+32 i C_V^\nu C_A^\nu \epsilon^{\mu\beta\sigma\lambda} p_{1\beta} p_{3\lambda} (p_{2\mu} p_{4\sigma} - g_{\mu\sigma}(p_2 \cdot p_4) + p_{2\sigma} p_{4\mu})$$

Pairs of 1st and 3rd terms cancel as we saw earlier (anti-symmetry)

g terms require μ=σ but then ε is zero

$$= 16((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times (2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3))$$
$$-64 C_V^e C_A^\nu C_V^\nu C_A^e \epsilon^{\mu\beta\sigma\lambda} p_{1\beta} p_{3\lambda} \epsilon_{\mu\nu\sigma\alpha} p_2^\nu p_4^\alpha$$

Recall:

$$\epsilon^{\mu\beta\sigma\lambda}\epsilon_{\mu\nu\sigma\alpha} = (-\epsilon^{\mu\sigma\beta\lambda} - \epsilon_{\mu\sigma\nu\alpha}) = \epsilon^{\mu\sigma\beta\lambda}\epsilon_{\mu\sigma\nu\alpha} = -2(\delta_\nu^\beta \delta_\alpha^\lambda - \delta_\alpha^\beta \delta_\nu^\lambda)$$

Neutrino-electron scattering matrix element

$$= 16((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times (2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3))$$
$$-64 C_V^e C_A^\nu C_V^\nu C_A^e \epsilon^{\mu\beta\sigma\lambda} p_{1\beta} p_{3\lambda} \epsilon_{\mu\nu\sigma\alpha} p_2^\nu p_4^\alpha$$
$$\epsilon^{\mu\beta\sigma\lambda}\epsilon_{\mu\nu\sigma\alpha} = (-\epsilon^{\mu\sigma\beta\lambda} - \epsilon_{\mu\sigma\nu\alpha}) = \epsilon^{\mu\sigma\beta\lambda}\epsilon_{\mu\sigma\nu\alpha} = -2(\delta_\nu^\beta \delta_\alpha^\lambda - \delta_\alpha^\beta \delta_\nu^\lambda)$$

$$= 16((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times (2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3))$$
$$-128 C_V^e C_A^\nu C_V^\nu C_A^e p_{1\beta} p_{3\lambda} (p_4^\beta p_2^\lambda - p_2^\beta p_4^\lambda)$$

$$= 32((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$
$$+128 C_V^e C_A^\nu C_V^\nu C_A^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

Almost there

$$= 32((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$
$$+128 C_V^e C_A^\nu C_V^\nu C_A^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

Divide by two to average over electron spins

$$= 16((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$
$$+64 C_V^e C_A^\nu C_V^\nu C_A^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

Add in missing terms from matrix element

$$= \frac{g_Z^4}{M_Z^4}((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$
$$+\frac{4g_Z^4}{M_Z^4} C_V^e C_A^\nu C_V^\nu C_A^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

Let's plug in C terms for neutrino (both 1/2)

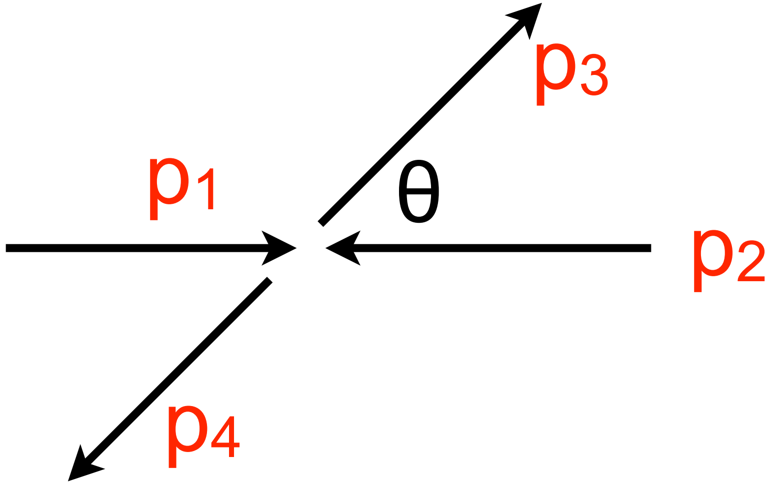
$$= \frac{g_Z^4}{2M_Z^4}((C_V^e)^2 + (C_A^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$
$$+\frac{g_Z^4}{M_Z^4} C_V^e C_A^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

Let's get some kinematics in CoM frame

$$= \frac{g_Z^4}{2M_Z^4}((C_V^e)^2 + (C_A^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$+ \frac{g_Z^4}{M_Z^4} C_V^e C_A^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

Assume massless electron. Pick center of mass frame where $E_1 = E_2 = E_3 = E_4 = E$ and $|\mathbf{p}_i| = E$



$$(p_1 \cdot p_2) = E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 = E^2 + E^2 = 2E^2$$

$$(p_3 \cdot p_4) = E_3 E_4 - \mathbf{p}_3 \cdot \mathbf{p}_4 = E^2 + E^2 = 2E^2$$

$$(p_1 \cdot p_4) = E_1 E_4 - \mathbf{p}_1 \cdot \mathbf{p}_4 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

$$(p_2 \cdot p_3) = E_2 E_3 - \mathbf{p}_2 \cdot \mathbf{p}_3 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

Let's get some kinematics in CoM frame

$$= \frac{g_Z^4}{2M_Z^4}((C_A^e)^2 + (C_V^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$+ \frac{g_Z^4}{M_Z^4} C_A^e C_V^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$(p_1 \cdot p_2) = E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 = E^2 + E^2 = 2E^2$$

$$(p_3 \cdot p_4) = E_3 E_4 - \mathbf{p}_3 \cdot \mathbf{p}_4 = E^2 + E^2 = 2E^2$$

$$(p_1 \cdot p_4) = E_1 E_4 - \mathbf{p}_1 \cdot \mathbf{p}_4 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

$$(p_2 \cdot p_3) = E_2 E_3 - \mathbf{p}_2 \cdot \mathbf{p}_3 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

$$= \frac{g_Z^4}{2M_Z^4}((C_A^e)^2 + (C_V^e)^2) \times (4E^4 + E^4(1 + \cos\theta)^2)$$

$$+ \frac{g_Z^4}{M_Z^4} C_A^e C_V^e (4E^4 - E^4(1 + \cos\theta)^2)$$

Let's get some kinematics in CoM frame

$$= \frac{g_Z^4}{2M_Z^4}((C_A^e)^2 + (C_V^e)^2) \times (4E^4 + E^4(1+\cos\theta)^2)$$

$$+ \frac{g_Z^4}{M_Z^4} C_A^e C_V^e (4E^4 - E^4(1+\cos\theta)^2)$$

Use half angle formula $\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$

$$= \frac{g_Z^4 E^4}{2M_Z^4}((C_A^e)^2 + (C_V^e)^2) \times (4 + 4\cos^4\frac{\theta}{2}) + \frac{g_Z^4 E^4}{M_Z^4} C_A^e C_V^e (4 - 4\cos^4\frac{\theta}{2})$$

Simplifying further

$$< |\mathcal{M}|^2 >= \frac{g_Z^4 E^4}{M_Z^4}((C_A^e)^2 + (C_V^e)^2)(2 + 2\cos^4\frac{\theta}{2})$$

$$+ \frac{g_Z^4 E^4}{M_Z^4} C_A^e C_V^e (4 - 4\cos^4\frac{\theta}{2})$$

$$< |\mathcal{M}|^2 >= \frac{2g_Z^4 E^4}{M_Z^4}((C_A^e)^2 + (C_V^e)^2 + 2C_A^e C_V^e) + \cos^4\frac{\theta}{2}((C_A^e)^2 + (C_V^e)^2 - 2C_A^e C_V^e)$$

$$< |\mathcal{M}|^2 >= \frac{2g_Z^4 E^4}{M_Z^4}((C_A^e + C_V^e)^2) + \cos^4\frac{\theta}{2}(C_A^e - C_V^e)^2)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{|M|^2}{(E_1 + E_2)^2}\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{1}{4E^2}\frac{2g_Z^4 E^4}{M_Z^4}((C_A^e + C_V^e)^2) + \cos^4\frac{\theta}{2}(C_A^e - C_V^e)^2)$$

Differential xsection to xsection

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{1}{4E^2}\frac{2g_Z^4 E^4}{M_Z^4}(((C_A^e + C_V^e)^2) + \cos^4\frac{\theta}{2}(C_A^e - C_V^e)^2)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{128\pi^2}\frac{g_Z^4 E^2}{M_Z^4}(((C_A^e + C_V^e)^2) + \cos^4\frac{\theta}{2}(C_A^e - C_V^e)^2)$$

Φ →

$$\sigma = \frac{2\pi}{128\pi^2}\frac{g_Z^4 E^2}{M_Z^4}\int_0^\pi [(((C_A^e + C_V^e)^2) + \cos^4\frac{\theta}{2}(C_A^e - C_V^e)^2)]\sin\theta d\theta$$

First piece of integral (sin theta) gives 2

$$\int_0^\pi \cos^4\frac{\theta}{2}\sin\theta d\theta = \frac{1}{4}\int_0^\pi (1+\cos\theta)^2\sin\theta d\theta$$

$$= \frac{1}{4}\int_0^\pi [\underbrace{\sin\theta}_{2} + 2\cos\theta\sin\theta + \cos^2\theta\sin\theta]d\theta$$

Differential xsection to xsection

$$\frac{1}{4}\int_0^{\pi}[\sin\theta + 2\cos\theta\sin\theta + \cos^2\theta\sin\theta]d\theta = \frac{1}{4}[2 + \int_0^{\pi} 2\cos\theta\sin\theta d\theta + \cos^2\theta\sin\theta d\theta]$$

$$u = \cos\theta, du = -\sin\theta d\theta$$

$$\frac{1}{4}[2 + \int_0^{\pi}(-2u - u^2)du]$$

$$\frac{1}{2} + \frac{1}{4}[-\cos^2\theta]_{\theta=0}^{\theta=\pi} + \frac{1}{4}[-\cos^3\theta/3]_{\theta=0}^{\theta=\pi} = \frac{1}{2} + \frac{1}{4}\frac{1}{3}(2) = \frac{2}{3}$$

$$\sigma = \frac{2\pi}{128\pi^2}\frac{g_Z^4 E^2}{M_Z^4}(2(C_A^e + C_V^e)^2) + \frac{2}{3}(C_A^e - C_V^e)^2)$$

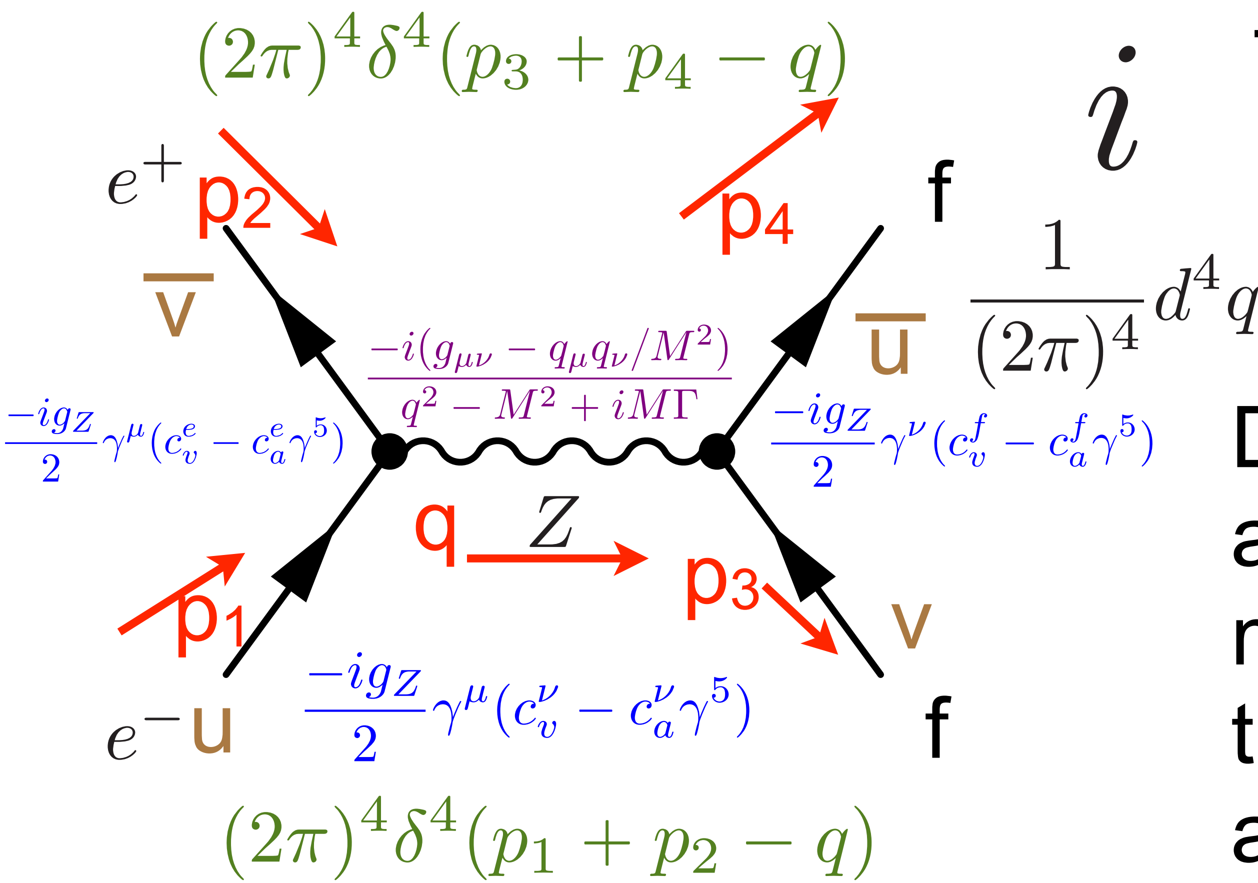
$$\sigma = \frac{1}{64\pi}\frac{g_Z^4 E^2}{M_Z^4}\frac{8}{3}[(C_A^e)^2 + (C_V^e)^2 + (C_V^e)(C_A^e)]$$

$$\sigma = \frac{1}{24\pi}\frac{g_Z^4 E^2}{M_Z^4}[(C_A^e)^2 + (C_V^e)^2 + (C_V^e)(C_A^e)]$$

Moving on to the Z boson

Z boson not discovered until 1983! Why? Most of the interesting processes are masked by electromagnetism. Need lots of energy to reach masses where you are near Z pole (near Z resonance)...

Studying the Z pole (Z mass at resonance)



The LEP collider! electron-positron collisions

Do NOT assume small momentum transfer! But assume fermion mass = 0

$$\mathcal{M} = \int i[\overline{u}(4)][-i\frac{g_Z}{2}\gamma^\nu(c_V^f - c_A^f\gamma^5)][v(3)]\left(\frac{-ig_{\mu\nu} - q_\mu q_\nu/M^2}{q^2 - M^2 + iM\Gamma}\right)[\overline{v}(2)][-i\frac{g_Z}{2}\gamma^\mu(c_V^e - c_A^e\gamma^5)][u(1)]$$
$$(2\pi)^4\delta^4(p_1 + p_2 - q)(2\pi)^4\delta^4(p_3 + p_4 - q)\frac{d^4q}{(2\pi)^4}$$

New term in propagator

$$\mathcal{M} = \int i[\overline{u}(4)][-i\frac{g_Z}{2}\gamma^\nu(c_V^f - c_A^f\gamma^5)][v(3)]\left(\frac{-ig_{\mu\nu} - q_\mu q_\nu/M^2}{q^2 - M^2 + iM\Gamma}\right)[\overline{v}(2)][-i\frac{g_Z}{2}\gamma^\mu(c_V^e - c_A^e\gamma^5)][u(1)]$$
$$(2\pi)^4\delta^4(p_1 + p_2 - q)(2\pi)^4\delta^4(p_3 + p_4 - q)\frac{d^4q}{(2\pi)^4}$$

$$[\overline{u}(4)][\gamma^\nu(c_A^f - c_V^f\gamma^5)][v(3)]q_\mu q_\nu = [\overline{u}(4)][\not{q}(c_A^f - c_V^f\gamma^5)][v(3)]q_\mu =$$
$$[\overline{u}(4)][(\not{p}_3 + \not{p}_4)(c_A^f - c_V^f\gamma^5)][v(3)]q_\mu$$

BUT $[\overline{u}(4)]\not{p}_4 = 0$ for m = 0

Why? Recall Dirac equation…

$$\overline{u}\left(\gamma^\mu k_\mu - m\right) = 0$$
$$\overline{v}\left(\gamma^\mu k_\mu + m\right) = 0$$

Plug in m=0

New term in propagator

$$\mathcal{M} = \int i[\overline{u}(4)][-i\frac{g_Z}{2}\gamma^\nu(c_V^f - c_A^f\gamma^5)][v(3)]\left(\frac{-ig_{\mu\nu} - q_\mu q_\nu/M^2}{q^2 - M^2 + iM\Gamma}\right)[\overline{v}(2)][-i\frac{g_Z}{2}\gamma^\mu(c_V^e - c_A^e\gamma^5)][u(1)]$$
$$(2\pi)^4\delta^4(p_1 + p_2 - q)(2\pi)^4\delta^4(p_3 + p_4 - q)\frac{d^4q}{(2\pi)^4}$$

$$[\overline{u}(4)][\gamma^\nu(c_A^f - c_V^f\gamma^5)][v(3)]q_\mu q_\nu = [\overline{u}(4)][\not{q}(c_A^f - c_V^f\gamma^5)][v(3)]q_\mu =$$
$$[\overline{u}(4)][(\not{p}_4 + \not{p}_3)(c_A^f - c_V^f\gamma^5)][v(3)]q_\mu = [\overline{u}(4)][(c_A^f + c_V^f\gamma^5)]\not{p}_3[v(3)]q_\mu$$
$$\not{p}_3[v(3)] = \overline{u}(4)\not{p}_4 = 0$$ for m = 0

$$\mathcal{M} = \int i[\overline{u}(4)][-i\frac{g_Z}{2}\gamma^\nu(c_V^f - c_A^f\gamma^5)][v(3)]\left(\frac{-ig_{\mu\nu}}{q^2 - M^2 + iM\Gamma}\right)[\overline{v}(2)][-i\frac{g_Z}{2}\gamma^\mu(c_V^e - c_A^e\gamma^5)][u(1)]$$
$$(2\pi)^4\delta^4(p_1 + p_2 - q)(2\pi)^4\delta^4(p_3 + p_4 - q)\frac{d^4q}{(2\pi)^4}$$

$$\mathcal{M} = -\frac{g_z^2}{4(q^2 - M^2 + iM\Gamma)}[\overline{u}(4)][\gamma_\mu(c_V^f - c_A^f\gamma^5)][v(3)][\overline{v}(2)][\gamma^\mu(c_V^e - c_A^e\gamma^5)][u(1)]$$

Turning the crank

$$\mathcal{M} = -\frac{g_z^2}{4(q^2 - M^2 + iM\Gamma)}[\overline{u}(4)][\gamma_\mu(c_V^f - c_A^f\gamma^5)][v(3)][\overline{v}(2)][\gamma^\mu(c_V^e - c_A^e\gamma^5)][u(1)]$$

After averaging over spins

$$< |\mathcal{M}|^2 >= \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2}\times$$

$$Tr[\gamma_\mu(C_V^f - C_A^f\gamma^5)\not{p}_3\gamma_\nu(c_V^f - c_A^f\gamma^5)\not{p}_4]\times$$

$$Tr[\gamma^\mu(C_V^e - C_e^f\gamma^5)\not{p}_1\gamma^\nu(c_V^e - c_A^e\gamma^5)\not{p}_2]$$

Thankfully, we calculated this before...

$$Tr[\gamma_\mu(C_A^e - C_V^e\gamma^5)(\not{p}_2)(\gamma_\sigma(C_A^e - C_V^e\gamma^5)(\not{p}_4)] =$$

$$4((C_V^e)^2 + (C_A^e)^2)(p_{2\mu}p_{4\sigma} - g_{\mu\sigma}(p_2 \cdot p_4) + p_{2\sigma}p_{4\mu}) + 8iC_A^eC_V^e\epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha$$

Turning the crank some more (lots of cranks in this course)

$$< |\mathcal{M}|^2 >= \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times$$

$$Tr[\gamma_\mu (C_V^f - C_A^f \gamma^5) \not{p}_3 \gamma_\nu (c_V^f - c_A^f \gamma^5) \not{p}_4] \times$$

$$Tr[\gamma^\mu (C_V^e - C_e^f \gamma^5) \not{p}_1 \gamma^\nu (c_V^e - c_A^e \gamma^5) \not{p}_2]$$

$$< |\mathcal{M}|^2 >= \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times$$

$$[4((C_V^f)^2 + (C_A^f)^2)(p_{3\mu} p_{4\nu} + p_{4\mu} p_{3\nu} - g_{\mu\nu}(p_3 \cdot p_4))$$

$$+8iC_V^f C_A^f \epsilon_{\mu\sigma\nu\alpha} p_3^\sigma p_4 \alpha] \times$$

$$[4((C_V^e)^2 + (C_A^e)^2)(p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu} - g_{\mu\nu}(p_1 \cdot p_2))$$

$$+8iC_V^e C_A^e \epsilon^{\mu\lambda\nu\beta} p_{1\lambda} p_{2\beta}]$$

Turning the crank some more (lots of cranks in this course)

$$< |\mathcal{M}|^2 > = \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times$$

$$[4((C_V^f)^2 + (C_A^f)^2)(p_{3\mu}p_{4\nu} + p_{4\mu}p_{3\nu} - g_{\mu\nu}(p_3 \cdot p_4))$$

$$+8iC_V^f C_A^f \epsilon_{\mu\sigma\nu\alpha} p_3^\sigma p_4 \alpha] \times$$

$$[4((C_V^e)^2 + (C_A^e)^2)(p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - g_{\mu\nu}(p_1 \cdot p_2))$$

$$+8iC_V^e C_A^e \epsilon^{\mu\lambda\nu\beta} p_{1\lambda}p_{2\beta}]$$

$$< |\mathcal{M}|^2 > = \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times [$$

$$16((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_2 \cdot p_3)(p_1 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4)$$

$$+(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_2 \cdot p_4)(p_1 \cdot p_3) - (p_1 \cdot p_2)(p_3 \cdot p_4)$$

$$-(p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) + 4(p_1 \cdot p_2)(p_3 \cdot p_4)]$$

$$+4((C_V^f)^2 + (C_A^f)^2)(8iC_V^f C_A^f \epsilon_{\mu\sigma\nu\alpha} p_3^\sigma p_4^\alpha)(p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - g_{\mu\nu}(p_1 \cdot p_2))$$

$$+4((C_V^e)^2 + (C_A^e)^2)(8iC_V^e C_A^e \epsilon^{\mu\lambda\nu\beta} p_{1\lambda}p_{2\beta}(p_{3\mu}p_{4\nu} + p_{4\mu}p_{3\nu} - g_{\mu\nu}(p_3 \cdot p_4))$$

$$-64C_V^e C_A C_V^f C_A^f \epsilon_{\mu\sigma\nu\alpha} p_3^\sigma p_4^\alpha \epsilon^{\mu\lambda\nu\beta} p_{1\lambda}p_{2\beta}]$$

As before

Turning the crank some more (lots of cranks in this course)

$$< |\mathcal{M}|^2 >= \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times [$$

$$16((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2(p_2 \cdot p_3)(p_1 \cdot p_4)]$$

$$-64 C_V^e C_A C_V^f C_A^f \epsilon_{\mu\sigma\nu\alpha} p_3^\sigma p_4^\alpha \epsilon^{\mu\lambda\nu\beta} p_{1\lambda} p_{2\beta}]$$

Recall:

$$\epsilon^{\mu\beta\sigma\lambda}\epsilon_{\mu\nu\sigma\alpha} = (-\epsilon^{\mu\sigma\beta\lambda} - \epsilon_{\mu\sigma\nu\alpha}) = \epsilon^{\mu\sigma\beta\lambda}\epsilon_{\mu\sigma\nu\alpha} = -2(\delta_\nu^\beta \delta_\alpha^\lambda - \delta_\alpha^\beta \delta_\nu^\lambda)$$

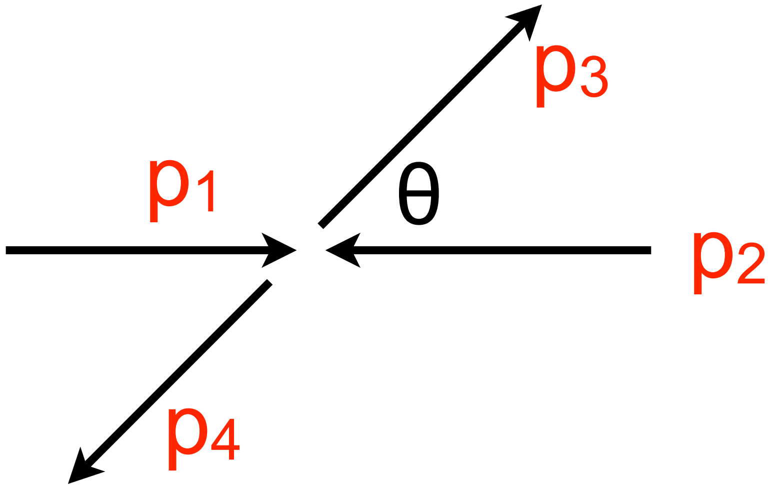
$$< |\mathcal{M}|^2 >= \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times [$$

$$16((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2(p_2 \cdot p_3)(p_1 \cdot p_4)]$$

$$+128 C_V^e C_A C_V^f C_A^f ((p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3)]$$

Almost there

Assume massless fermions and electrons. Pick center of mass frame where $E_1 = E_2 = E_3 = E_4 = E$ and $|\mathbf{p}_i| = E$



$$(p_1 \cdot p_3) = E_1 E_3 - \mathbf{p}_1 \cdot \mathbf{p}_3 = E^2 - E^2 \cos\theta = E^2(1 - \cos\theta)$$

$$(p_2 \cdot p_4) = E_2 E_4 - \mathbf{p}_2 \cdot \mathbf{p}_4 = E^2 - E^2 \cos\theta = E^2(1 - \cos\theta)$$

$$(p_2 \cdot p_3) = E_2 E_3 - \mathbf{p}_2 \cdot \mathbf{p}_3 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

$$(p_1 \cdot p_4) = E_1 E_4 - \mathbf{p}_1 \cdot \mathbf{p}_4 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

Almost there

$q=p_1+p_2$
and need to be a bit more careful about the magnitude of the imaginary number!

$$<|\mathcal{M}|^2>=\frac{g_z^4}{64(q^2-M^2+iM\Gamma)^2}\times[$$

$$16((C_V^f)^2+(C_A^f)^2)((C_V^e)^2+(C_A^e)^2)[2(p_1\cdot p_3)(p_2\cdot p_4)+2(p_2\cdot p_3)(p_1\cdot p_4)]$$

$$+128C_V^e C_A C_V^f C_A^f((p_1\cdot p_3)(p_2\cdot p_4)-(p_1\cdot p_4)(p_2\cdot p_3)]$$

$$(p_1\cdot p_3)=E_1E_3-\mathbf{p}_1\cdot\mathbf{p}_3=E^2-E^2\cos\theta=E^2(1-\cos\theta)$$

$$(p_2\cdot p_4)=E_2E_4-\mathbf{p}_2\cdot\mathbf{p}_4=E^2-E^2\cos\theta=E^2(1-\cos\theta)$$

$$(p_2\cdot p_3)=E_2E_3-\mathbf{p}_2\cdot\mathbf{p}_3=E^2+E^2\cos\theta=E^2(1+\cos\theta)$$

$$(p_1\cdot p_4)=E_1E_4-\mathbf{p}_1\cdot\mathbf{p}_4=E^2+E^2\cos\theta=E^2(1+\cos\theta)$$

$$q^2=(p_1+p_2)^2=p_1^2+p_2^2+2(p_1\cdot p_2)=4E^2$$

Almost there

$$< |\mathcal{M}|^2 >= \frac{g_z^4}{64(4E^2 - M^2)^2 + 64M^2\Gamma^2} \times [$$

$$16((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[2E^4(1-\cos\theta)^2 + 2E^4(1+\cos\theta)^2]$$

$$+128 C_V^e C_A C_V^f C_A^f (E^4(1-\cos\theta)^2 - E^4(1+\cos\theta)^2)]$$

$$< |\mathcal{M}|^2 >= \frac{g_z^4 E^4}{2(4E^2 - M^2)^2 + 2M^2\Gamma^2} \times [$$

$$((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[(1-\cos\theta)^2 + (1+\cos\theta)^2]$$

$$+4 C_V^e C_A C_V^f C_A^f ((1-\cos\theta)^2 - (1+\cos\theta)^2)]$$

$$< |\mathcal{M}|^2 >= \frac{g_z^4 E^4}{2(4E^2 - M^2)^2 + 2M^2\Gamma^2} \times [$$

$$((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[2 + 2\cos^2\theta]$$

$$+4 C_V^e C_A C_V^f C_A^f (-4\cos\theta)$$

A bit of simplification

$$< |\mathcal{M}|^2 >= \frac{g_z^4 E^4}{2(4E^2 - M^2)^2 + 2M^2\Gamma^2} \times [$$

$$((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[2 + 2\cos^2\theta]$$

$$+4C_V^e C_A C_V^f C_A^f(-4\cos\theta)$$

$$< |\mathcal{M}|^2 >= \frac{g_z^4 E^4}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta\right]$$

Differential cross section

$$< |\mathcal{M}|^2 >= \frac{g_z^4 E^4}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta\right]$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|M|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

Equal

$$\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2 E^2} \frac{g_z^4 E^4}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta\right]$$

Differential cross section to total cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2 E^2} \frac{g_z^4 E^4}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta\right]$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2} \frac{g_z^4 E^2}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta\right]$$

Φ →

$$\sigma = \frac{2\pi}{256\pi^2} \frac{g_z^4 E^2}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\int_{\theta=0}^{\theta=\pi} \left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta\right] \sin\theta \, d\theta$$

Total cross section

$$\sigma = \frac{1}{128\pi}\frac{g_z^4 E^2}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\int_{\theta=0}^{\theta=\pi} \Big[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1+\cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta\Big] \sin\theta d\theta$$

$$\int_{\theta=0}^{\theta=\pi} \sin\theta d\theta = 2$$

$$\int_{\theta=0}^{\theta=\pi} \cos^2\theta \sin\theta d\theta = \frac{2}{3}$$

$$\int_{\theta=0}^{\theta=\pi} \cos\theta \sin\theta d\theta = 0$$

We just evaluated these a handful of slides ago

$$\sigma = \frac{1}{48\pi}\frac{g_z^4 E^2}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times \Big[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)\Big]$$

On Z boson width

Predictions for branching ratios to specific final states!

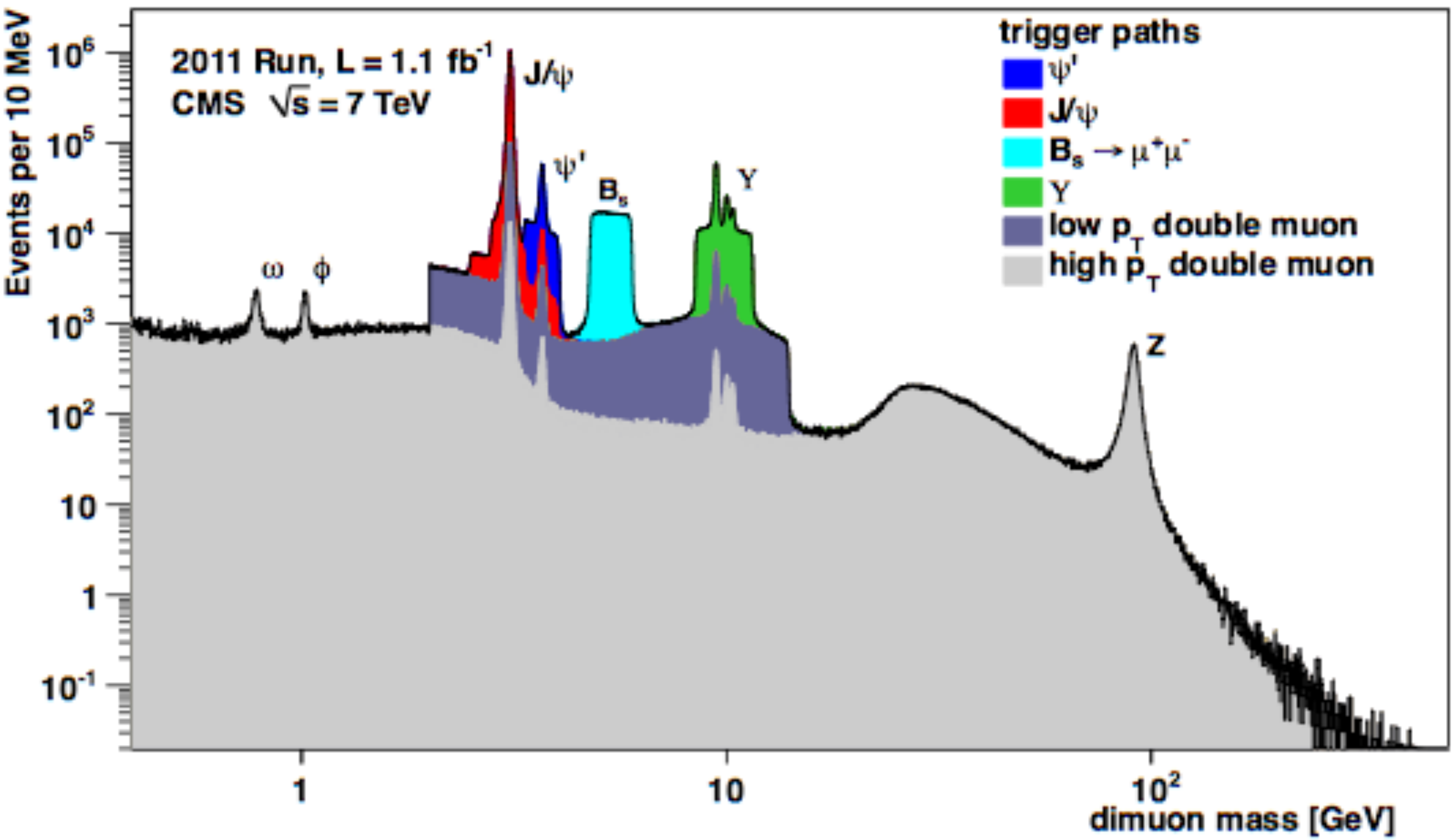
$$\sigma = \frac{1}{48\pi}\frac{g_z^4 E^2}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times \left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)\right]$$

Z boson mass = 91.2 GeV

Z boson width = 2.5 GeV

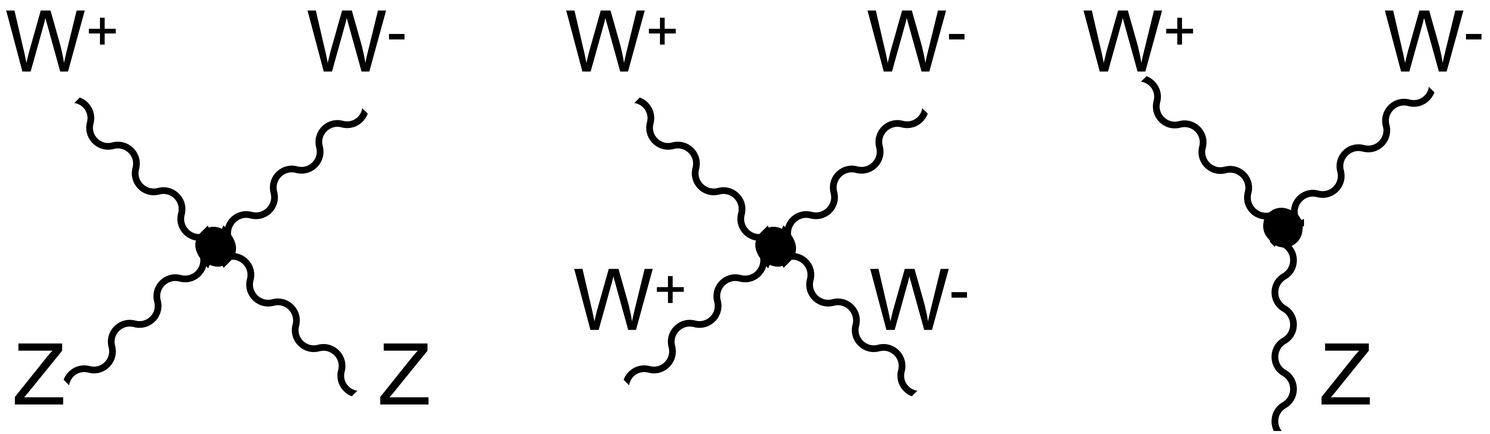
If E not near M_Z, width term is negligible (at Z pole, it's needed - otherwise we end up with infinite cross section!) Reason width is there is that Z has a finite lifetime (so it's not a stable particle). EM (photon-mediated) contribution to the process is much bigger than neutral current process, except at Z pole

CMS dimuon data

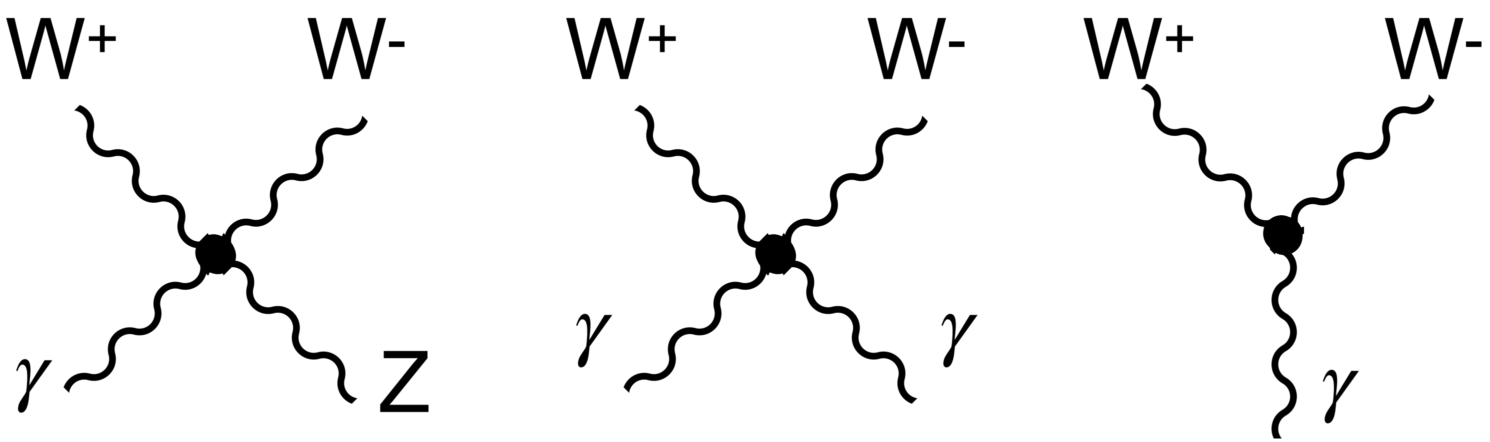


https://twiki.cern.ch/twiki/bin/view/CMSPublic/PhysicsResultsMUO#Invariant_mass_spectra_of_opposi

Last comment on weak vector bosons



We won't study these vertices, but they exist, too!



On to electroweak unification

Long history of unifying forces in physics. Most famous example is electricity and magnetism (Maxwell). Einstein (SR) then showed that they are one and the same thing, differing only in reference frame. Wouldn't it be great to unify the weak force and E&M?

First problem - their strength is very different! Solution (Glashow): Give masses to W/Z bosons and not to photon

New problem - why are the masses so different? Solution: next chapter

Next problem with EW unification

Weak forces contain both axial and vector components, E&M contains a pure vector component

New solution: redefine our spinors

Remember the Dirac equation solutions:

Electrons

Spin up

$$\psi_1 = e^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

Spin down

$$\psi_2 = e^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Positrons

Spin down

$$\psi_3 = e^{+imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Spin up

$$\psi_4 = e^{+imt}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

What are the eigenvectors of helicity?

Reminder that helicity h of a particle is the normalized dot product of its spin and direction of flight (h=+1 if spin is in same direction as motion and h=-1 if in opposite direction)

Also remember - helicity is NOT Lorentz invariant (can always change direction of motion in different reference frame unless m = 0)

Spin matrix → $\hat{\Sigma}$

$$\hat{h} = \frac{\hat{\Sigma}\cdot\vec{p}}{|\vec{p}|} = \frac{1}{|\vec{p}|}\begin{pmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix}$$

What are the eigenvectors of helicity?

Let's find the eigenvalues of the helicity operator

$$\frac{1}{p}\begin{pmatrix} \sigma\cdot\vec{p} & 0 \\ 0 & \sigma\cdot\vec{p} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \lambda \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$(\sigma\cdot\hat{\mathbf{p}})u_A = p\lambda u_A$$

Way back when ...

$$(\sigma\cdot\hat{\mathbf{p}})u_B = p\lambda u_B$$

$$\vec{p}\cdot\vec{\sigma} = p_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$(\vec{p}\cdot\vec{\sigma})^2 = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$(\vec{p}\cdot\vec{\sigma})^2 = \begin{pmatrix} p_z^2 + (p_x - ip_y)(p_x + ip_y) & p_z(p_x - ip_y) - p_z(p_x - ip_y) \\ p_z(p_x + ip_y) - p_z(p_x + ip_y) & (p_x + ip_y)(p_x - ip_y) + p_z^2 \end{pmatrix}$$

$$(\vec{p}\cdot\vec{\sigma})^2 = \mathbf{p}^2 1$$

What are the eigenvectors of helicity?

$$(\sigma \cdot \hat{\mathbf{p}})u_A = p\lambda u_A$$

$$(\sigma \cdot \hat{\mathbf{p}})u_B = p\lambda u_B$$

$$(\sigma \cdot \hat{\mathbf{p}})(\sigma \cdot \hat{\mathbf{p}})u_A = p\lambda(\sigma \cdot \hat{\mathbf{p}})u_A$$

$$(\sigma \cdot \hat{\mathbf{p}})(\sigma \cdot \hat{\mathbf{p}})u_B = p\lambda(\sigma \cdot \hat{\mathbf{p}})u_B$$

$$p^2 u_A = p^2\lambda^2 u_A$$

$$p^2 u_B = p^2\lambda^2 u_B$$

Helicity eigenvalue = +/- 1

What else can we say?

Also way back when we had for spinor components

$$u_B = \frac{\vec{p}\cdot\vec{\sigma}}{E+m}u_A \qquad \gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^5 u = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_A \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}u_A \end{pmatrix}$$

u_A is a 2-component spinor

Back to our favorite matrix

$$\gamma^5 u = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_A \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} u_A \end{pmatrix}$$

$$\gamma^5 u = \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m} u_A \\ u_A \end{pmatrix}$$

Try this:

$$\begin{pmatrix} \frac{(\vec{p}\cdot\vec{\sigma})}{E+m} & 0 \\ 0 & \frac{(\vec{p}\cdot\vec{\sigma})}{E-m} \end{pmatrix} u = \begin{pmatrix} \frac{(\vec{p}\cdot\vec{\sigma})}{E+m} & 0 \\ 0 & \frac{(\vec{p}\cdot\vec{\sigma})}{E-m} \end{pmatrix} \begin{pmatrix} u_A \\ \frac{(\vec{p}\cdot\vec{\sigma})}{E+m} u_A \end{pmatrix} = \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m} u_A \\ \frac{(\vec{p}\cdot\vec{\sigma})^2}{E^2-m^2} u_A \end{pmatrix} = \begin{pmatrix} \frac{(\vec{p}\cdot\vec{\sigma})}{E+m} u_A \\ \frac{p^2}{p^2} u_A \end{pmatrix}$$

$$\begin{pmatrix} \frac{(\vec{p}\cdot\vec{\sigma})}{E+m} & 0 \\ 0 & \frac{(\vec{p}\cdot\vec{\sigma})}{E-m} \end{pmatrix} u = \begin{pmatrix} \frac{(\vec{p}\cdot\vec{\sigma})}{E+m} u_A \\ u_A \end{pmatrix} = \gamma^5 u$$

What if the particle is massless?

In certain limits

$$\gamma^5 u = \begin{pmatrix} \frac{(\vec{p}\cdot\vec{\sigma})}{E} & 0 \\ 0 & \frac{(\vec{p}\cdot\vec{\sigma})}{E} \end{pmatrix}$$

If m small (or E>>m)

But in that case, we get back helicity operator!

$$E = |\vec{p}| \rightarrow \gamma^5 u = \begin{pmatrix} (\hat{p}\cdot\vec{\sigma}) & 0 \\ 0 & (\hat{p}\cdot\vec{\sigma}) \end{pmatrix}$$

Left-/right-handed projection operators

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5)$$

What is $(\Sigma \cdot \hat{p})(P_\pm u)$ if E>>m?

$$(\Sigma \cdot \hat{p})(P_\pm u) = \frac{1}{2}\begin{pmatrix} (\sigma \cdot \hat{p}) & 0 \\ 0 & (\sigma \cdot \hat{p}) \end{pmatrix}(1 \pm \gamma^5)u$$

$$= \frac{1}{2}\left[\begin{pmatrix} (\sigma \cdot \hat{p}) & 0 \\ 0 & (\sigma \cdot \hat{p}) \end{pmatrix} u \pm \begin{pmatrix} (\sigma \cdot \hat{p}) & 0 \\ 0 & (\sigma \cdot \hat{p}) \end{pmatrix} \gamma^5 u\right]$$

$$= \frac{1}{2}\left[\gamma^5 u \pm (\gamma^5)^2 u\right] = \frac{1}{2}\left[\gamma^5 \pm 1\right] u = \pm\frac{1}{2}(1 \pm \gamma^5)u = \pm P_\pm u$$

Algebra of projection operators

$$(P_+P_+)u = \frac{1}{2}(1+\gamma^5)\frac{1}{2}(1+\gamma^5)u = \frac{1}{4}(1+2\gamma^5+(\gamma^5)^2)u$$

$$(P_+P_+)u = \frac{1}{2}(1+\gamma^5)u = P_+u$$

$$(P_-P_-)u = \frac{1}{2}(1-\gamma^5)\frac{1}{2}(1-\gamma^5)u = \frac{1}{4}(1-2\gamma^5+(\gamma^5)^2)u$$

$$(P_-P_-)u = \frac{1}{2}(1-\gamma^5)u = P_-u$$

$$(P_+P_-)u = \frac{1}{2}(1+\gamma^5)\frac{1}{2}(1-\gamma^5)u = \frac{1}{4}(1-\gamma^5+\gamma^5-(\gamma^5)^2)u = 0$$

$$(P_-P_+)u = \frac{1}{2}(1-\gamma^5)\frac{1}{2}(1+\gamma^5)u = \frac{1}{4}(1+\gamma^5-\gamma^5-(\gamma^5)^2)u = 0$$

$$P_- + P_+ = \frac{1}{2}(1-\gamma^5) + \frac{1}{2}(1+\gamma^5) = 1$$

Algebra of projection operators

In massless limit, P_- picks out the left-handed component of helicity and P_+ picks out the right-handed component of helicity

If we're not in the massless limit, can still define these operators and refer to them as the **chiral** components, even if helicity is not unique. Only in massless limit are the two equivalent

Using these operators in electroweak unification

$$u_L(p) = \frac{1}{2}(1-\gamma^5)u(p)$$

$$u_R(p) = \frac{1}{2}(1+\gamma^5)u(p)$$

$$v_L(p) = \frac{1}{2}(1+\gamma^5)v(p)$$

$$v_R(p) = \frac{1}{2}(1-\gamma^5)v(p)$$

What about the adjoints?

$$\overline{u}_L(p) = \left[\frac{1}{2}(1-\gamma^5)u(p)\right]^\dagger \gamma^0 = u^\dagger(p)\frac{1}{2}(1-\gamma^{5\dagger})\gamma^0$$

$$\overline{u}_L(p) = u^\dagger(p)\frac{1}{2}(1-\gamma^5)\gamma^0 = u^\dagger(p)\frac{1}{2}(\gamma^0-\gamma^5\gamma^0)$$

$$\overline{u}_L(p) = u^\dagger(p)\frac{1}{2}(\gamma^0+\gamma^0\gamma^5) = u^\dagger(p)\gamma^0\frac{1}{2}(1+\gamma^5)$$

$$\overline{u}_L(p) = \overline{u}(p)\frac{1}{2}(1+\gamma^5)$$

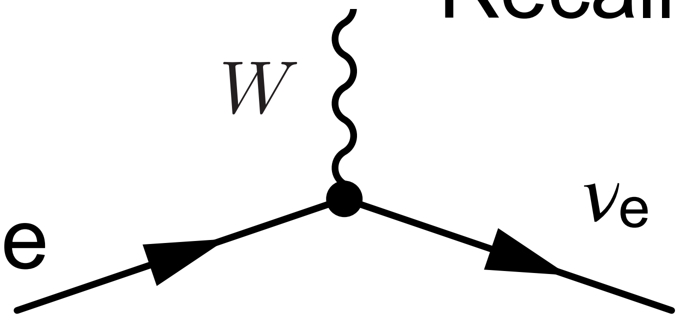
Similarly...

$$\overline{u}_R(p) = \overline{u}(p)\frac{1}{2}(1-\gamma^5)$$

$$\overline{v}_L(p) = \overline{v}(p)\frac{1}{2}(1-\gamma^5)$$

$$\overline{v}_R(p) = \overline{v}(p)\frac{1}{2}(1+\gamma^5)$$

How does this help us?



Recall that we had this interaction....

$$\propto \quad \overline{\nu}\gamma_\mu \frac{1-\gamma^5}{2} e$$

Call this the **weak current** j^+_μ

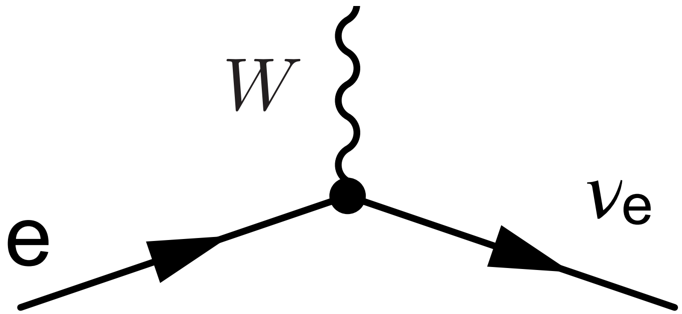
"Bar" is not anti-particle,
but adjoint!

$$\left(\frac{1-\gamma^5}{2}\right)^2 = \left(\frac{1-\gamma^5}{2}\right)$$

$$\gamma_\mu \left(\frac{1-\gamma^5}{2}\right) = \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu$$

$$\rightarrow \gamma_\mu \left(\frac{1-\gamma^5}{2}\right) = \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu \left(\frac{1-\gamma^5}{2}\right)$$

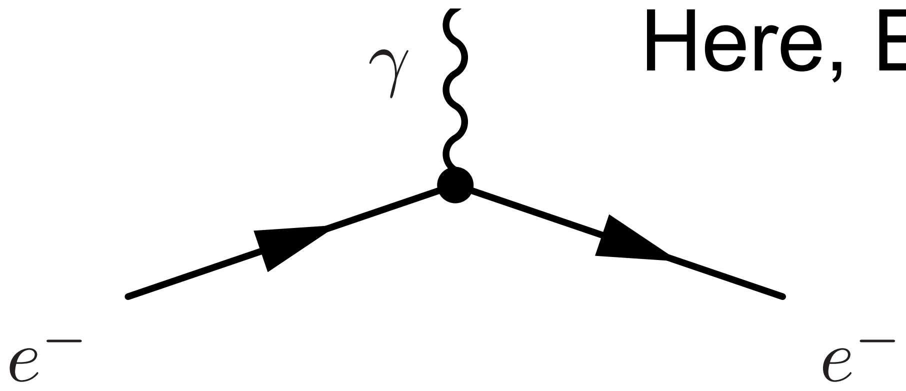
Weak current



$$j_\mu^+ = \overline{\nu}\gamma_\mu \left(\frac{1-\gamma^5}{2}\right) e = \overline{\nu} \left(\frac{1+\gamma^5}{2}\right) \gamma_\mu \left(\frac{1-\gamma^5}{2}\right) e = \overline{\nu}_L \gamma_\mu e_L$$

So vertex couples left-handed electrons to left-handed neutrinos. This is now **purely** a vector coupling, just like in E&M

What about electricity+magnetism?



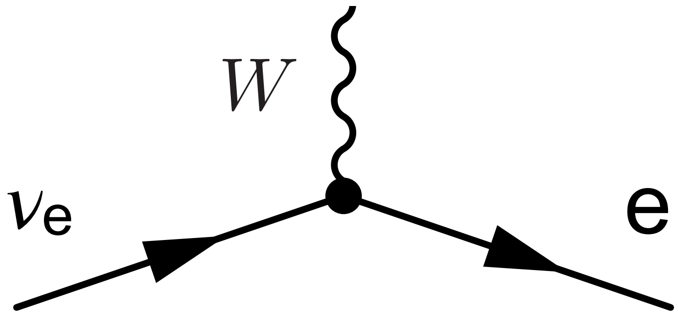
Here, EM current is

$$j_\mu^{EM} = -\bar{e}\gamma_\mu e = -(\bar{e}_L + \bar{e}_R)\gamma_\mu(e_L + e_R)$$

Since, as we saw: $\bar{e}_R\gamma_\mu e_L = \bar{e}_L\gamma_\mu e_R = 0$

Is it clear why that is?

Other weak current process



$$j_\mu^- = \overline{e}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)\nu = \overline{e}_L\gamma_\mu\nu_L$$

Define: $\chi_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$

Conveniently then write:

$$j_\mu^\pm = \overline{\chi}_L\gamma_\mu\tau^\pm\chi_L$$

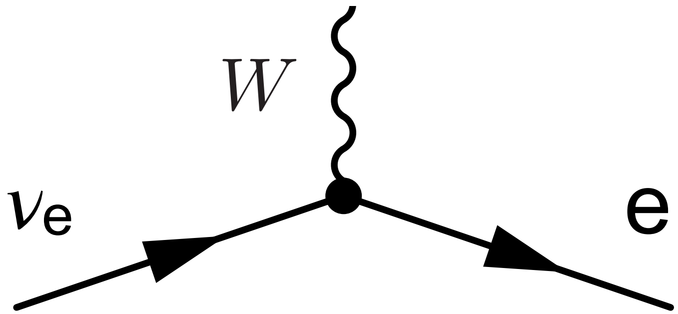
$$\tau^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Just like Pauli spin matrices!

$$\tau^\pm = \frac{1}{2}(\tau^1 \pm i\tau^2)$$

What about neutral current?



$$j_\mu^- = \bar{e}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)\nu = \bar{e}_L\gamma_\mu\nu_L$$

$$\chi_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$$

Conveniently then write:

$$j_\mu^\pm = \overline{\chi}_L\gamma_\mu\tau^\pm\chi_L$$

$$\tau^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Reminds us perhaps of isospin? Maybe current from Z boson is then...

$$j_\mu^3 = \overline{\chi}_L\gamma_\mu\frac{1}{2}\tau^3\chi_L = \frac{1}{2}\overline{\nu}_L\gamma_\mu\nu_L - \frac{1}{2}\overline{e}_L\gamma_\mu e_L$$

Not quite

NOPE! Need neutral current to couple to right-handed components, too...

“Weak hypercharge” current, instead

Symmetry of our new, combined theory!

$$SU(2)_L \otimes U(1)$$

Let’s check this definition of Y (weak hypercharge):

$$j_\mu^Y = 2j_\mu^{EM} - 2j_\mu^3 = -2\overline{e}_R\gamma_\mu e_R - \overline{e}_L\gamma_\mu e_L - \overline{\nu}_L\gamma_\mu\nu_L = -2\overline{e}_R\gamma_\mu e_R + \overline{\chi}_L\gamma_\mu\chi_L$$

In total we have three “weak isospin” currents

$$\mathbf{j}_\mu = \frac{1}{2}\overline{\chi}_L\gamma_\mu\tau\chi_L$$

And weak hypercharge current in addition

$$j_\mu^Y = -2\overline{e}_R\gamma_\mu\nu_R + \overline{\chi}_L\gamma_\mu\chi_L$$

Glashow/Weinberg/Salam theory

Have three weak isospin currents coupling with the same strength to a "weak isotriplet" of vector bosons **W** (coupling g_W), and an isosinglet vector boson B coupled to the weak hypercharge current (coupling g'/2). Write as:

$$-i\left[g_W \mathbf{j}_\mu \mathbf{W}^\mu + \frac{g'}{2} j_\mu^y B^\mu\right]$$

Expanding out

$$\mathbf{j}_\mu \mathbf{W}^\mu = j^1_\mu W^{\mu^1} + j^2_\mu W^{\mu^2} + j^3_\mu W^{\mu^3}$$

$$\mathbf{j}_\mu \mathbf{W}^\mu = \frac{1}{\sqrt{2}} j^+_\mu W^{\mu+} + \frac{1}{\sqrt{2}} j^-_\mu W^{\mu-} + j^3_\mu W^{\mu^3}$$

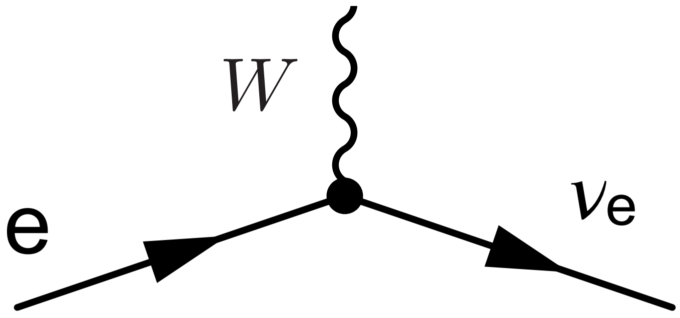
W bosons observed →

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \pm iW^2_\mu)$$

Plugging in to the interaction:

$$-i\left[\frac{g_W}{\sqrt{2}} j^+_\mu W^{\mu+} + \frac{g_W}{\sqrt{2}} j^-_\mu W^{\mu-} + g_W j^3_\mu W^{\mu^3} + \frac{g'}{2} j^y_\mu B^\mu\right]$$

What is this vertex then?



Back to when we first studied this, we get a term in matrix element from vertex of:

$$-ig_W \frac{1}{2\sqrt{2}} \gamma_\mu (1-\gamma^5)$$

Vertex factor agrees!

This is the term of interest (have W+)

$$j_\mu^+ = \overline{\nu}_L \gamma_\mu e_L = \overline{\nu} \gamma_\mu \frac{(1-\gamma^5)}{2} e$$

$$-i \left[\frac{g_W}{\sqrt{2}} j_\mu^+ W^{\mu +} + \frac{g_W}{\sqrt{2}} j_\mu^- W^{\mu -} + g_W j_\mu^3 W^{\mu^3} + \frac{g'}{2} j_\mu^y B^\mu \right]$$

Neutral states

They mix! One of them is massless (photon) and other is the massive Z boson

$$A_\mu = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W$$
$$Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W$$

Rewrite:

$$(A_\mu \sin\theta_W + Z_\mu \cos\theta_W) = W^3_\mu(\sin^2\theta_W + \cos^2\theta_W)$$
$$A_\mu \sin\theta_W + Z_\mu \cos\theta_W = W^3_\mu$$
$$(A_\mu \cos\theta_W - Z_\mu \sin\theta_W) = B_\mu(\cos^2\theta_W + \sin^2\theta_W)$$
$$A_\mu \cos\theta_W - Z_\mu \sin\theta_W = B_\mu$$

Neutral states

$$A_\mu \sin\theta_W + Z_\mu \cos\theta_W = W^3_\mu$$

$$A_\mu \cos\theta_W - Z_\mu \sin\theta_W = B_\mu$$

$$-i\left[\frac{g_W}{\sqrt{2}} j^+_\mu W^{\mu +} + \frac{g_W}{\sqrt{2}} j^-_\mu W^{\mu -} + g_W j^3_\mu {W^\mu}^3 + \frac{g'}{2} j^y_\mu B^\mu\right]$$

Neutral pieces:

$$-i\left[g_W j^3_\mu \left(A^\mu \sin\theta_W + Z^\mu \cos\theta_W\right) + \frac{g'}{2} j^y_\mu \left(A^\mu \cos\theta_W - Z^\mu \sin\theta_W\right)\right]$$

$$-i\left[A^\mu \left(g_W j^3_\mu \sin\theta_W + \frac{g'}{2} j^y_\mu \cos\theta_W\right) + Z^\mu \left(g_W j^3_\mu \cos\theta_W - \frac{g'}{2} j^y_\mu \sin\theta_W\right)\right]$$

Electromagnetic coupling

$$-i\left[A^\mu\left(g_W j^3_\mu \sin\theta_W + \frac{g'}{2} j^y_\mu \cos\theta_W\right) + Z^\mu\left(g_W j^3_\mu \cos\theta_W - \frac{g'}{2} j^y_\mu \sin\theta_W\right)\right]$$

$$j^{em}_\mu = j^3_\mu + \frac{1}{2} j^Y_\mu$$

$$g_W \sin\theta_W = g' \cos\theta_W = g_e$$

Relates the electromagnetic and weak couplings!

Neutral current coupling

$$-i\left[A^\mu\left(g_W j^3_\mu \sin\theta_W + \frac{g'}{2} j^y_\mu \cos\theta_W\right) + Z^\mu\left(g_W j^3_\mu \cos\theta_W - \frac{g'}{2} j^y_\mu \sin\theta_W\right)\right]$$

$$j^Y_\mu = 2j^{EM}_\mu - 2j^3_\mu$$

$$g_W \sin\theta_W = g' \cos\theta_W = g_e$$

$$g_W = g_e / \sin\theta_W$$

$$g' = g_e / \cos\theta_W$$

$$\left(g_W j^3_\mu \cos\theta_W - g' j^{EM}_\mu \sin\theta_W + g' j^3_\mu \sin\theta_W\right)$$

$$j^3_\mu \left(g_W \cos\theta_W + g' \sin\theta_W\right) - g' j^{EM}_\mu \sin\theta_W$$

$$j^3_\mu \left(g_e \frac{\cos\theta_W}{\sin\theta_W} + g_e \frac{\sin\theta_W}{\cos\theta_W}\right) - g_e j^{EM}_\mu \frac{\sin\theta_W}{\cos\theta_W}$$

Neutral current coupling

$$j_\mu^3 \left(g_e \frac{\cos\theta_W}{\sin\theta_W} + g_e \frac{\sin\theta_W}{\cos\theta_W} \right) - g_e j_\mu^{EM} \frac{\sin\theta_W}{\cos\theta_W}$$

Define: $$g_Z = \frac{g_e}{\sin\theta_W \cos\theta_W}$$

$$g_Z \left[j_\mu^3 \left(\cos^2\theta_W + \sin^2\theta_W \right) - j_\mu^{EM} \sin^2\theta_W \right]$$

$$g_Z \left[j_\mu^3 - j_\mu^{EM} \sin^2\theta_W \right]$$

Now we're getting somewhere...

Couplings to Z boson are of the form:

$$-ig_Z\left[j^3_\mu - j^{EM}_\mu \sin^2\theta_W\right]$$

Recall:

$$j^3_\mu = \overline{\chi}_L\gamma_\mu\frac{1}{2}\tau^3\chi_L = \frac{1}{2}\overline{\nu}_L\gamma_\mu\nu_L - \frac{1}{2}\overline{e}_L\gamma_\mu e_L$$

$$\overline{\nu}_L\gamma_\mu\nu_L = \overline{\nu}\frac{1}{2}(1+\gamma^5)\gamma_\mu\frac{1}{2}(1-\gamma^5)\nu$$

$$\overline{\nu}_L\gamma_\mu\nu_L = \frac{1}{4}\overline{\nu}(1+\gamma^5)\gamma_\mu(1-\gamma^5)\nu$$

$$\overline{\nu}_L\gamma_\mu\nu_L = \frac{1}{4}\overline{\nu}\gamma_\mu(1-\gamma^5)(1-\gamma^5)\nu$$

$$\overline{\nu}_L\gamma_\mu\nu_L = \frac{1}{4}\overline{\nu}\gamma_\mu(1-2\gamma^5+(\gamma^5)^2)\nu$$

$$\overline{\nu}_L\gamma_\mu\nu_L = \frac{1}{4}\overline{\nu}\gamma_\mu(1-2\gamma^5+1)\nu$$

$$\overline{\nu}_L\gamma_\mu\nu_L = \frac{1}{2}\overline{\nu}\gamma_\mu(1-\gamma^5)\nu$$

So:

$$j^3_\mu = \frac{1}{4}\left[\overline{\nu}\gamma_\mu(1-\gamma^5)\nu - \overline{e}\gamma_\mu(1-\gamma^5)e\right]$$

Now we're getting somewhere...

Couplings to Z boson are of the form:

$$-ig_Z\left[j^3_\mu - j^{EM}_\mu \sin^2\theta_W\right]$$

$$j^{EM}_\mu = -\bar{e}\gamma_\mu e \qquad j^3_\mu = \frac{1}{4}\left[\bar{\nu}\gamma_\mu(1-\gamma^5)\nu - \bar{e}\gamma_\mu(1-\gamma^5)e\right]$$

$$-i\frac{g_Z}{4}\left[\bar{\nu}\gamma_\mu(1-\gamma^5)\nu - \bar{e}\gamma_\mu(1-\gamma^5)e\right] + ig_Z\sin^2\theta_W\bar{e}\gamma_\mu e$$

Compare with Neutral weak vertex factor:

$$\frac{-ig_Z}{2}\gamma_\mu(c^f_V - c^f_A\gamma^5)$$

Now we're getting somewhere...

Couplings to Z boson are of the form:

$$\frac{-ig_Z}{4}\left[\overline{\nu}\gamma_\mu(1-\gamma^5)\nu - \overline{e}\gamma_\mu(1-\gamma^5)e\right] - i\sin^2\theta_W\overline{e}\gamma_\mu e$$

Compare with Neutral weak vertex factor:

$$\frac{-ig_Z}{2}\gamma_\mu(c_V^f - c_A^f\gamma^5)$$

For neutrinos:

$$\frac{-ig_Z}{2}\gamma_\mu(c_V^f - c_A^f\gamma^5) = -i\frac{g_Z}{4}\gamma_\mu(1-\gamma^5)$$

$c_V = 1/2$, $c_A = 1/2$

Now we're getting somewhere...

Couplings to Z boson are of the form:

$$\frac{-ig_Z}{4}\left[\overline{\nu}\gamma_\mu(1-\gamma^5)\nu - \overline{e}\gamma_\mu(1-\gamma^5)e\right] - i\sin^2\theta_W\overline{e}\gamma_\mu e$$

Compare with Neutral weak vertex factor:

$$\frac{-ig_Z}{2}\gamma_\mu(c_V^f - c_A^f\gamma^5)$$

For electrons:

$$\frac{-ig_Z}{2}\gamma_\mu(c_V^f - c_A^f\gamma^5) = +i\frac{g_Z}{4}\gamma_\mu(1-\gamma^5) - i\sin^2\theta_W\gamma_\mu$$

$c_V = -1/2+2\sin^2\theta_W$, $c_A = -1/2$

Let's try this out for quarks

Couplings to Z boson are of the form:

$$-ig_Z\left[j_\mu^3 - j_\mu^{EM}\sin^2\theta_W\right] \qquad \chi = \begin{pmatrix} \text{u} \\ \text{d} \end{pmatrix}$$

$$j_\mu^3 = \overline{\chi}_L \gamma_\mu \frac{1}{2}\tau^3 \chi_L = \frac{1}{2}(\overline{u}\ \overline{d})_L \gamma_\mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \text{u} \\ \text{d} \end{pmatrix}_L$$

$$j_\mu^3 = \frac{1}{2}(\overline{u}\ \overline{d})_L \gamma_\mu \begin{pmatrix} \text{u} \\ \text{-d} \end{pmatrix}_L$$

$$j_\mu^3 = \frac{1}{2}(\overline{u}_L \gamma_\mu u_L - \overline{d}_L \gamma_\mu d_L)$$

$$j_\mu^3 = \frac{1}{2}\left(\overline{u}\frac{(1+\gamma^5)}{2}\gamma_\mu \frac{(1-\gamma^5)}{2}u - \overline{d}\frac{(1+\gamma^5)}{2}\gamma_\mu \frac{(1-\gamma^5)}{2}d\right)$$

Let's try this out for quarks

Couplings to Z boson are of the form:

$$-ig_Z\left[j_\mu^3 - j_\mu^{EM}\sin^2\theta_W\right]$$

$$j_\mu^3 = \frac{1}{2}\left(\overline{u}\frac{(1+\gamma^5)}{2}\gamma_\mu\frac{(1+\gamma^5)}{2}u - \overline{d}\frac{(1-\gamma^5)}{2}\gamma_\mu\frac{(1-\gamma^5)}{2}d\right)$$

$$j_\mu^3 = \frac{1}{8}\left(\overline{u}(1+\gamma^5)\gamma_\mu(1-\gamma^5)u - \overline{d}(1+\gamma^5)\gamma_\mu(1-\gamma^5)d\right)$$

$$j_\mu^3 = \frac{1}{8}\left(\overline{u}\gamma_\mu(1-\gamma^5)(1-\gamma^5)u - \overline{d}\gamma_\mu(1-\gamma^5)(1-\gamma^5)d\right)$$

$$j_\mu^3 = \frac{1}{4}\left(\overline{u}\gamma_\mu(1-\gamma^5)u - \overline{d}\gamma_\mu(1-\gamma^5)d\right)$$

and

$$j_\mu^{EM} = \left(\frac{2}{3}\overline{u}\gamma_\mu u - \frac{1}{3}\overline{d}\gamma_\mu d\right)$$

Let's try this out for quarks

Couplings to Z boson are of the form:

$$-ig_Z\left[j^3_\mu - j^{EM}_\mu \sin^2\theta_W\right]$$

$$j^3_\mu = \frac{1}{4}\left(\overline{u}\gamma_\mu(1-\gamma^5)u - \overline{d}\gamma_\mu(1-\gamma^5)d\right)$$

$$j^{EM}_\mu = \left(\frac{2}{3}\overline{u}\gamma_\mu u - \frac{1}{3}\overline{d}\gamma_\mu d\right)$$

Compare with Neutral weak vertex factor:

$$\frac{-ig_Z}{2}\gamma_\mu(c^f_V - c^f_A\gamma^5)$$

For up quarks:

$c_V = 1/2-(4/3)\sin^2\theta_w$

$c_A = 1/2$

For down quarks:

$c_V = -1/2+(2/3)\sin^2\theta_w$

$c_A = -1/2$

That is what we wanted!

f	c_V	c_A
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	$\frac{1}{2}$
e, μ, τ	$\frac{-1}{2} + 2\sin^2\theta_w$	$\frac{-1}{2}$
u, c, t	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_w$	$\frac{1}{2}$
d, s, b	$\frac{-1}{2} + \frac{2}{3}\sin^2\theta_w$	$\frac{-1}{2}$

You might be asking

The symmetries we examined are clearly not intact in the final results (they are ‘broken’) - why is the photon massless but the W and Z bosons are not? Are we ready for your (presumably) first introduction to field theory? :)

But first some homework

9.5, 9.16, 9.29, 9.30, 9.31 (you may want to use the results from problem 9.1, which you do not need to prove)

And you should have a close to final presentation already by now!