On to Feynman calculus

Recall that most of the particles we play with are unstable. Want to calculate their decay rate (**Γ**), the probability per unit time of decay

$$dN = -\Gamma N dt$$

$$N(t) = N(0)e^{-\Gamma t}$$

Number of particles that make it to N(t) and decay at N(t+dt)

$$N(t+dt) = N_0 e^{-\Gamma(t+dt)} = N_0 e^{-\Gamma t} e^{-\Gamma dt}$$

$$e^x \sim 1 + x \rightarrow e^{-\Gamma dt} \sim 1 - \Gamma dt$$

$$N(t+dt) = N_0 e^{-\Gamma t}(1 - \Gamma dt)$$

$$N(t) - N(t+dt) \sim N_0 e^{-\Gamma t} - N_0 e^{-\Gamma t}(1 - \Gamma dt) = \boxed{N_0 e^{-\Gamma t} \Gamma dt}$$

What is the average lifetime?

Average lifetime

$$\tau = \int_0^\infty t\left(e^{-\Gamma t}\Gamma dt\right)$$

Number of particles that make it to N(t) and decay at N(t+dt):

$$\boxed{N_0 e^{-\Gamma t}\Gamma dt}$$

$$\tau = \Gamma \int_0^\infty t e^{-\Gamma t} dt$$

$$u = t, dv = e^{-\Gamma t}dt, du = dt, v = \frac{-1}{\Gamma}e^{-\Gamma t}$$

$$\tau = \Gamma\left[\frac{-t}{\Gamma}e^{-\Gamma t}\right]_0^\infty - \Gamma\int_0^\infty \frac{-1}{\Gamma}e^{-\Gamma t}dt$$

$$\tau = -\Gamma\left[\frac{1}{\Gamma^2}e^{-\Gamma t}\right]_0^\infty = \frac{1}{\Gamma}$$

So, fraction that decay at N(t+dt) is then

$$\boxed{e^{-\Gamma t}\Gamma dt}$$

So average lifetime = 1/Γ

Decays

Typically define partial widths Γ_i which are defined as the rates for specific decays. The total decay rate is the sum of the partial widths, and the lifetime is given by τ:

$$\Gamma = \sum_{i=1}^{n} \Gamma_i$$

$$BR(i) = \frac{\Gamma_i}{\Gamma}$$

$$\tau = \frac{1}{\Gamma}$$

The branching ratio (BR) is the fraction of all decays to that go to a specific final state

Scattering

We’re also very often interested in the collision between two objects. Collisions can be:

A+B→A+B (elastic, no energy lost)
A+B→Other (inelastic, energy “lost” in the form of conversion to other particles)

Typically we refer to the cross section (σ) for a collision process. A natural way to think of a collision that relates to classical scattering theory. Units of area

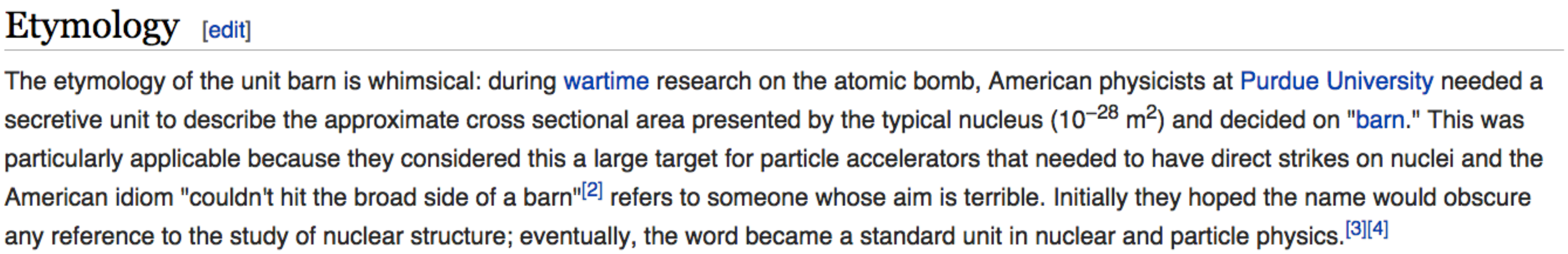
Cross sections

Sometimes in particle physics we think of differential cross sections (dσ/dX), which refer to how often a process occurs per unit of X

X can be energy (ex: cross section for collision to produce a particle with a certain energy)
X can be number of objects (ex: how often does a collision produce a process with a certain number of jets)
X can be angle (ex: cross section where collision decay products travel in a certain direction)

Cross section units

In particle physics, we typically use “barns” (b).
1 barn =10^{-28} m^2 (typically ~area of Uranium nucleus)
If you believe Wikipedia and its references ...

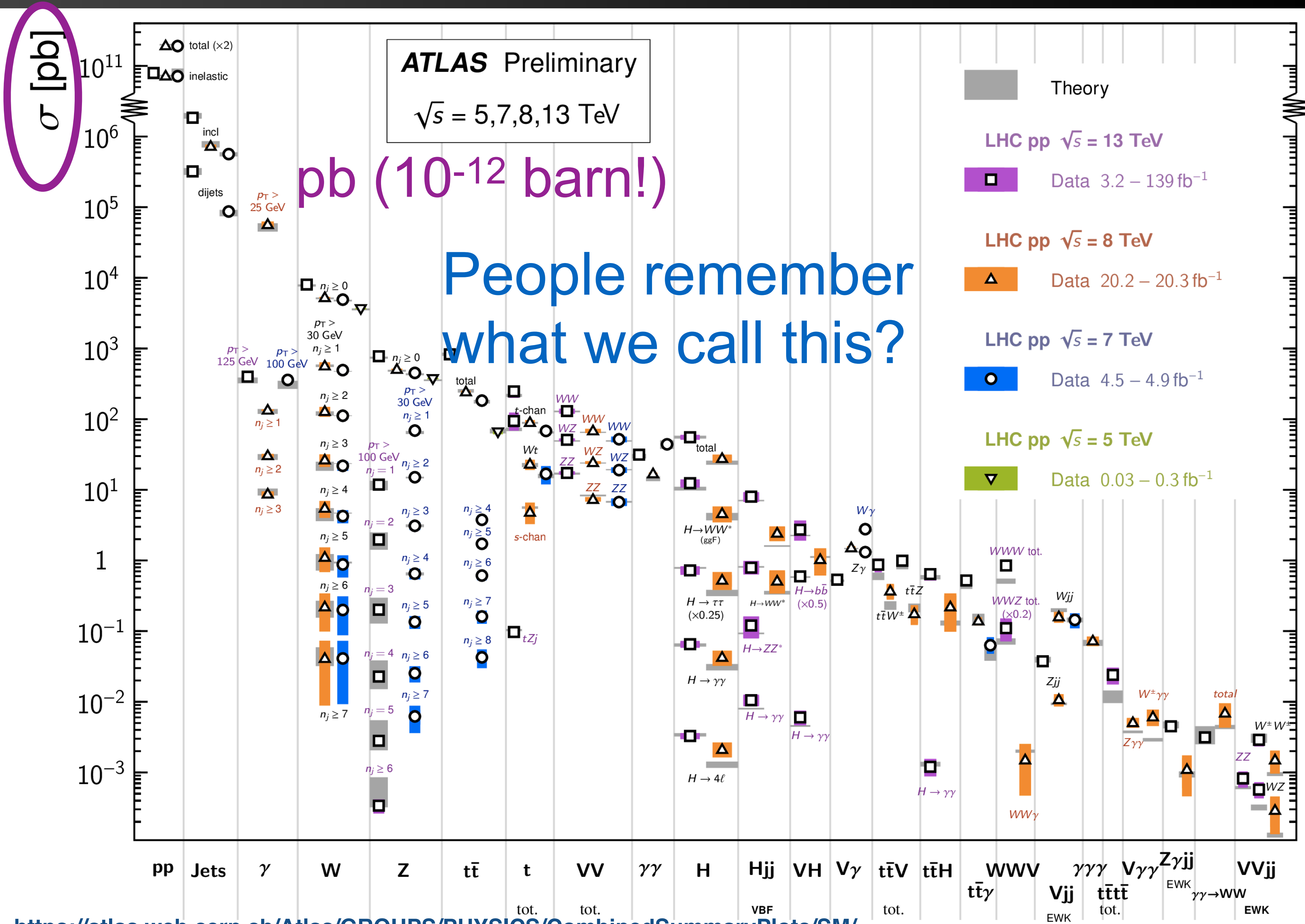


Etymology [edit]

The etymology of the unit barn is whimsical: during wartime research on the atomic bomb, American physicists at Purdue University needed a secretive unit to describe the approximate cross sectional area presented by the typical nucleus (10^{-28} m^2) and decided on "barn." This was particularly applicable because they considered this a large target for particle accelerators that needed to have direct strikes on nuclei and the American idiom "couldn't hit the broad side of a barn"[2] refers to someone whose aim is terrible. Initially they hoped the name would obscure any reference to the study of nuclear structure; eventually, the word became a standard unit in nuclear and particle physics.[3][4]

1 barn is a huge number in particle physics!

Cross sections at the LHC

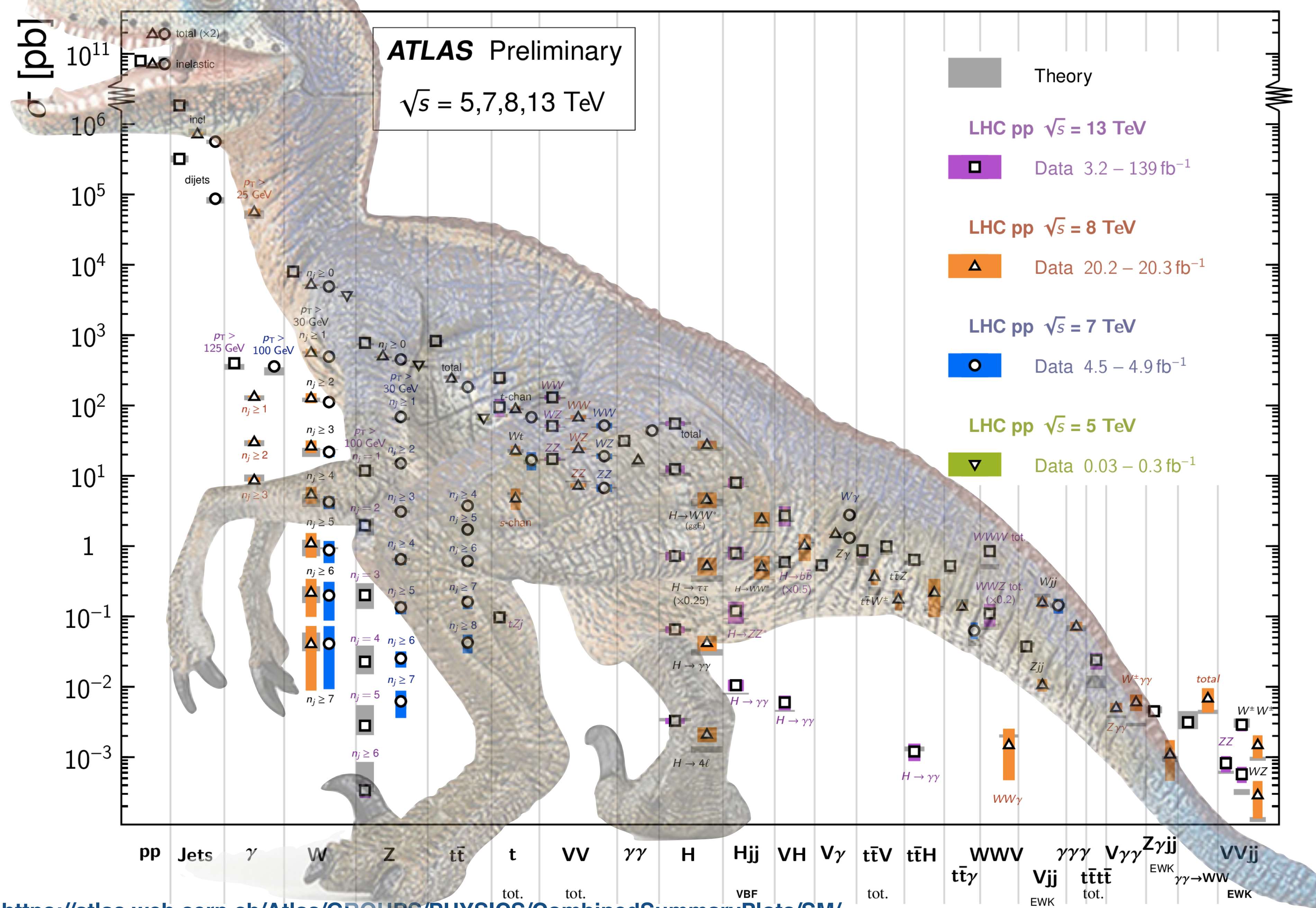


pb (10^{-12} barn!)

People remember what we call this?

https://atlas.web.cern.ch/Atlas/GROUPS/PHYSICS/CombinedSummaryPlots/SM/

Cross sections at the LHC



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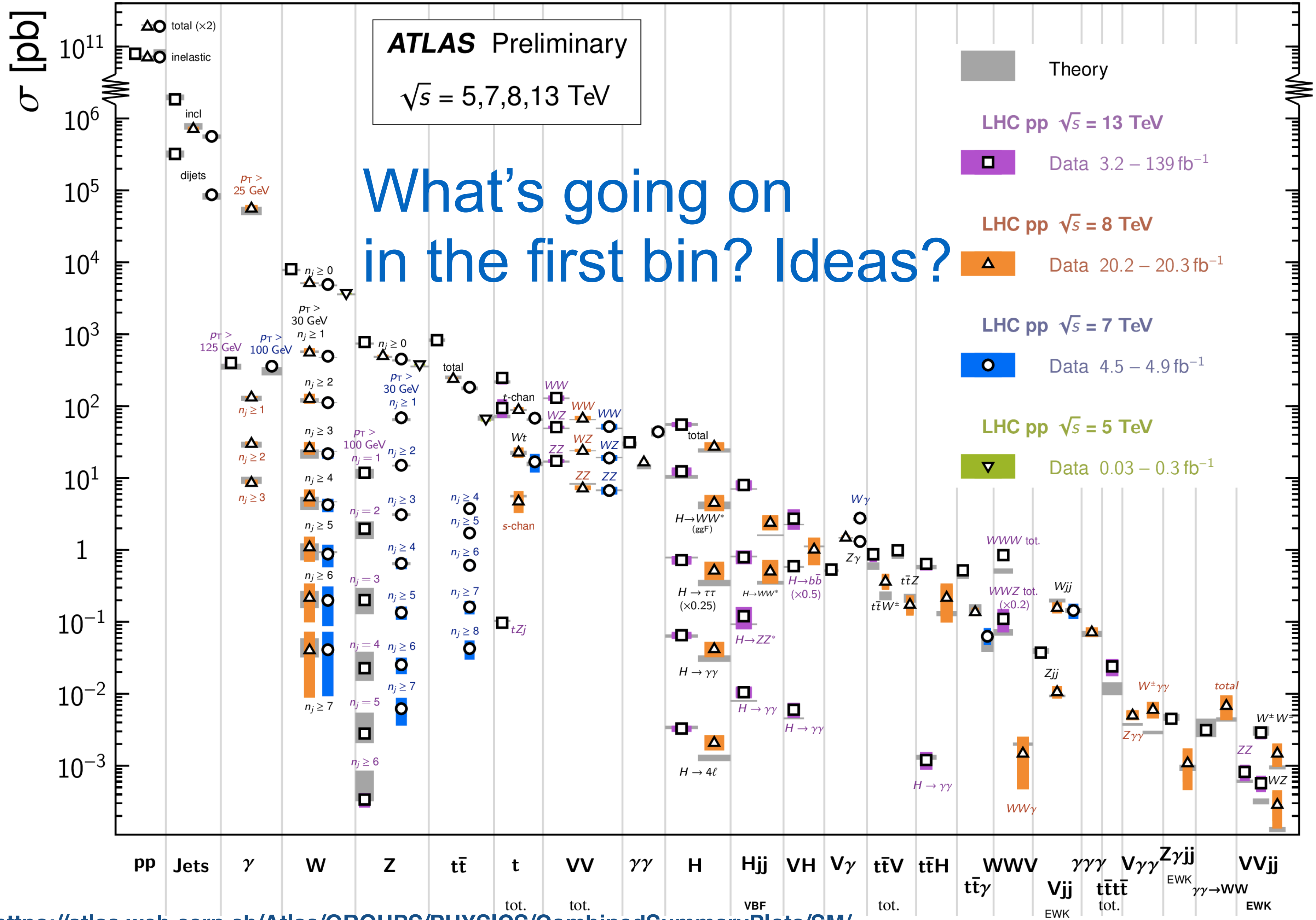
Cross sections at the LHC



What's going on in the first bin? Ideas?

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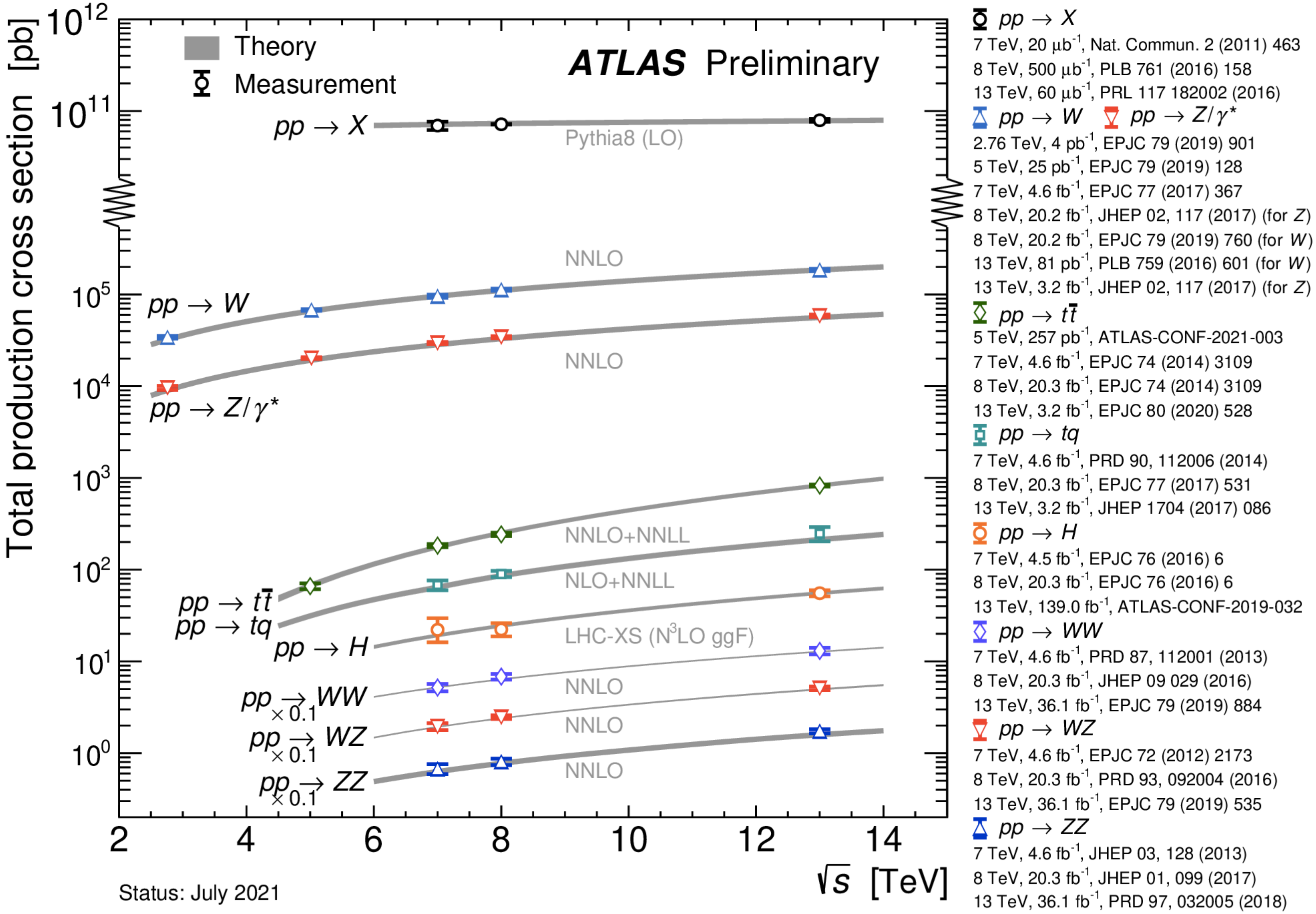
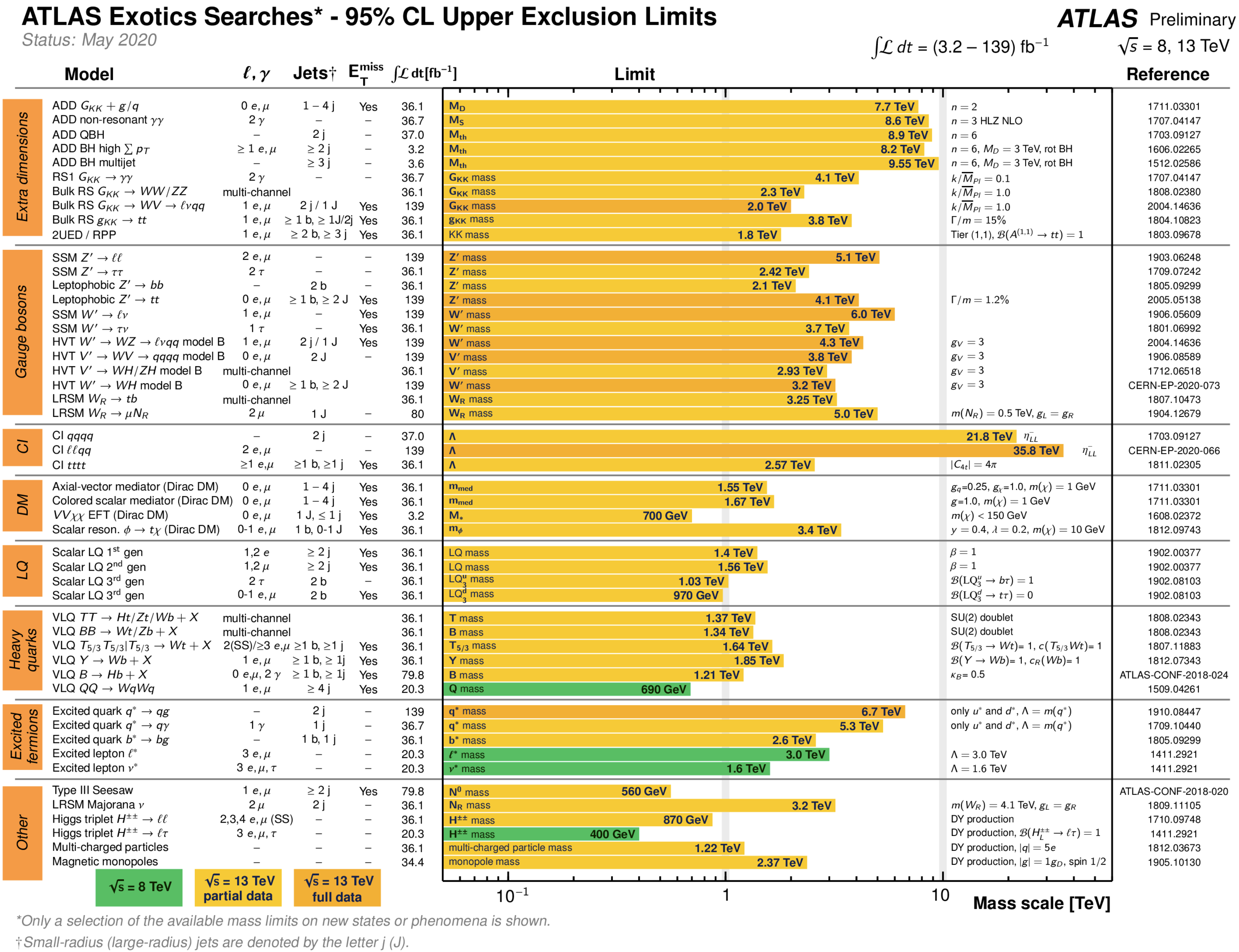
Cross sections at the LHC



Small but non-zero dependence

https://atlas.web.cern.ch/Atlas/GROUPS/PHYSICS/CombinedSummaryPlots/SM/

Limits on processes instead



ATLAS Exotics Searches* - 95% CL Upper Exclusion Limits

Status: May 2020

ATLAS Preliminary

$\int\mathcal{L}\,dt = (3.2 - 139)\ \mathrm{fb}^{-1}$

$\sqrt{s}$ = 8, 13 TeV

	Model	ℓ,γ	Jets†	E_T^{miss}	$\int\mathcal{L}\,dt[\mathrm{fb}^{-1}]$	Limit		Reference
Extra dimensions	ADD $G_{KK}+g/q$	$0\ e,\mu$	1 – 4 j	Yes	36.1	M_D 7.7 TeV	$n=2$	1711.03301
	ADD non-resonant $\gamma\gamma$	$2\ \gamma$	–	–	36.7	M_S 8.6 TeV	$n=3$ HLZ NLO	1707.04147
	ADD QBH	–	2 j	–	37.0	M_{th} 8.9 TeV	$n=6$	1703.09127
	ADD BH high $\sum p_T$	$\geq 1\ e,\mu$	≥ 2 j	–	3.2	M_{th} 8.2 TeV	$n=6$, $M_D=3$ TeV, rot BH	1606.02265
	ADD BH multijet	–	≥ 3 j	–	3.6	M_{th} 9.55 TeV	$n=6$, $M_D=3$ TeV, rot BH	1512.02586
	RS1 $G_{KK}\to\gamma\gamma$	$2\ \gamma$	–	–	36.7	G_{KK} mass 4.1 TeV	$k/\overline{M}_{Pl}=0.1$	1707.04147
	Bulk RS $G_{KK}\to WW/ZZ$	multi-channel			36.1	G_{KK} mass 2.3 TeV	$k/\overline{M}_{Pl}=1.0$	1808.02380
	Bulk RS $G_{KK}\to WV\to\ell\nu qq$	$1\ e,\mu$	2 j / 1 J	Yes	139	G_{KK} mass 2.0 TeV	$k/\overline{M}_{Pl}=1.0$	2004.14636
	Bulk RS $g_{KK}\to tt$	$1\ e,\mu$	≥ 1 b, ≥ 1J/2j	Yes	36.1	g_{KK} mass 3.8 TeV	$\Gamma/m=15\%$	1804.10823
	2UED / RPP	$1\ e,\mu$	≥ 2 b, ≥ 3 j	Yes	36.1	KK mass 1.8 TeV	Tier (1,1), $\mathcal{B}(A^{(1,1)}\to tt)=1$	1803.09678
Gauge bosons	SSM $Z'\to\ell\ell$	$2\ e,\mu$	–	–	139	Z' mass 5.1 TeV		1903.06248
	SSM $Z'\to\tau\tau$	$2\ \tau$	–	–	36.1	Z' mass 2.42 TeV		1709.07242
	Leptophobic $Z'\to bb$	–	2 b	–	36.1	Z' mass 2.1 TeV		1805.09299
	Leptophobic $Z'\to tt$	$0\ e,\mu$	≥ 1 b, ≥ 2 J	Yes	139	Z' mass 4.1 TeV	$\Gamma/m=1.2\%$	2005.05138
	SSM $W'\to\ell\nu$	$1\ e,\mu$	–	Yes	139	W' mass 6.0 TeV		1906.05609
	SSM $W'\to\tau\nu$	$1\ \tau$	–	Yes	36.1	W' mass 3.7 TeV		1801.06992
	HVT $W'\to WZ\to\ell\nu qq$ model B	$1\ e,\mu$	2 j / 1 J	Yes	139	W' mass 4.3 TeV	$g_V=3$	2004.14636
	HVT $V'\to WV\to qqqq$ model B	$0\ e,\mu$	2 J	–	139	V' mass 3.8 TeV	$g_V=3$	1906.08589
	HVT $V'\to WH/ZH$ model B	multi-channel			36.1	V' mass 2.93 TeV	$g_V=3$	1712.06518
	HVT $W'\to WH$ model B	$0\ e,\mu$	≥ 1 b, ≥ 2 J		139	W' mass 3.2 TeV	$g_V=3$	CERN-EP-2020-073
	LRSM $W_R\to tb$	multi-channel			36.1	W_R mass 3.25 TeV		1807.10473
	LRSM $W_R\to\mu N_R$	$2\ \mu$	1 J	–	80	W_R mass 5.0 TeV	$m(N_R)=0.5$ TeV, $g_L=g_R$	1904.12679
CI	CI $qqqq$	–	2 j	–	37.0	Λ 21.8 TeV η_{LL}^-		1703.09127
	CI $\ell\ell qq$	$2\ e,\mu$	–	–	139	Λ 35.8 TeV	η_{LL}^-	CERN-EP-2020-066
	CI $tttt$	$\geq 1\ e,\mu$	≥1 b, ≥1 j	Yes	36.1	Λ 2.57 TeV	$\lvert C_{4t}\rvert=4\pi$	1811.02305
DM	Axial-vector mediator (Dirac DM)	$0\ e,\mu$	1 – 4 j	Yes	36.1	m_{med} 1.55 TeV	g_q=0.25, g_χ=1.0, $m(\chi)=1$ GeV	1711.03301
	Colored scalar mediator (Dirac DM)	$0\ e,\mu$	1 – 4 j	Yes	36.1	m_{med} 1.67 TeV	g=1.0, $m(\chi)=1$ GeV	1711.03301
	$VV\chi\chi$ EFT (Dirac DM)	$0\ e,\mu$	1 J, ≤ 1 j	Yes	3.2	M_* 700 GeV	$m(\chi)<150$ GeV	1608.02372
	Scalar reson. $\phi\to t\chi$ (Dirac DM)	0-1 e,μ	1 b, 0-1 J	Yes	36.1	m_ϕ 3.4 TeV	$y=0.4$, $\lambda=0.2$, $m(\chi)=10$ GeV	1812.09743
LQ	Scalar LQ 1st gen	$1{,}2\ e$	≥ 2 j	Yes	36.1	LQ mass 1.4 TeV	$\beta=1$	1902.00377
	Scalar LQ 2nd gen	$1{,}2\ \mu$	≥ 2 j	Yes	36.1	LQ mass 1.56 TeV	$\beta=1$	1902.00377
	Scalar LQ 3rd gen	$2\ \tau$	2 b	–	36.1	LQ_3^u mass 1.03 TeV	$\mathcal{B}(\mathrm{LQ}_3^u\to b\tau)=1$	1902.08103
	Scalar LQ 3rd gen	0-1 e,μ	2 b	Yes	36.1	LQ_3^d mass 970 GeV	$\mathcal{B}(\mathrm{LQ}_3^d\to t\tau)=0$	1902.08103
Heavy quarks	VLQ $TT\to Ht/Zt/Wb+X$	multi-channel			36.1	T mass 1.37 TeV	SU(2) doublet	1808.02343
	VLQ $BB\to Wt/Zb+X$	multi-channel			36.1	B mass 1.34 TeV	SU(2) doublet	1808.02343
	VLQ $T_{5/3}T_{5/3}\vert T_{5/3}\to Wt+X$	2(SS)/≥3 e,μ	≥1 b, ≥1 j	Yes	36.1	$T_{5/3}$ mass 1.64 TeV	$\mathcal{B}(T_{5/3}\to Wt)$= 1, $c(T_{5/3}Wt)$= 1	1807.11883
	VLQ $Y\to Wb+X$	$1\ e,\mu$	≥ 1 b, ≥ 1j	Yes	36.1	Y mass 1.85 TeV	$\mathcal{B}(Y\to Wb)$= 1, $c_R(Wb)$= 1	1812.07343
	VLQ $B\to Hb+X$	$0\ e,\mu$, $2\ \gamma$	≥ 1 b, ≥ 1j	Yes	79.8	B mass 1.21 TeV	κ_B= 0.5	ATLAS-CONF-2018-024
	VLQ $QQ\to WqWq$	$1\ e,\mu$	≥ 4 j	Yes	20.3	Q mass 690 GeV		1509.04261
Excited fermions	Excited quark $q^*\to qg$	–	2 j	–	139	q^* mass 6.7 TeV	only u^* and d^*, $\Lambda=m(q^*)$	1910.08447
	Excited quark $q^*\to q\gamma$	$1\ \gamma$	1 j	–	36.7	q^* mass 5.3 TeV	only u^* and d^*, $\Lambda=m(q^*)$	1709.10440
	Excited quark $b^*\to bg$	–	1 b, 1 j	–	36.1	b^* mass 2.6 TeV		1805.09299
	Excited lepton ℓ^*	$3\ e,\mu$	–	–	20.3	ℓ^* mass 3.0 TeV	$\Lambda=3.0$ TeV	1411.2921
	Excited lepton ν^*	$3\ e,\mu,\tau$	–	–	20.3	ν^* mass 1.6 TeV	$\Lambda=1.6$ TeV	1411.2921
Other	Type III Seesaw	$1\ e,\mu$	≥ 2 j	Yes	79.8	N^0 mass 560 GeV		ATLAS-CONF-2018-020
	LRSM Majorana ν	$2\ \mu$	2 j	–	36.1	N_R mass 3.2 TeV	$m(W_R)=4.1$ TeV, $g_L=g_R$	1809.11105
	Higgs triplet $H^{\pm\pm}\to\ell\ell$	2,3,4 e,μ (SS)	–	–	36.1	$H^{\pm\pm}$ mass 870 GeV	DY production	1710.09748
	Higgs triplet $H^{\pm\pm}\to\ell\tau$	$3\ e,\mu,\tau$	–	–	20.3	$H^{\pm\pm}$ mass 400 GeV	DY production, $\mathcal{B}(H_L^{\pm\pm}\to\ell\tau)=1$	1411.2921
	Multi-charged particles	–	–	–	36.1	multi-charged particle mass 1.22 TeV	DY production, $\lvert q\rvert=5e$	1812.03673
	Magnetic monopoles	–	–	–	34.4	monopole mass 2.37 TeV	DY production, $\lvert g\rvert=1g_D$, spin 1/2	1905.10130

$\sqrt{s}$ = 8 TeV

$\sqrt{s}$ = 13 TeV partial data

$\sqrt{s}$ = 13 TeV full data

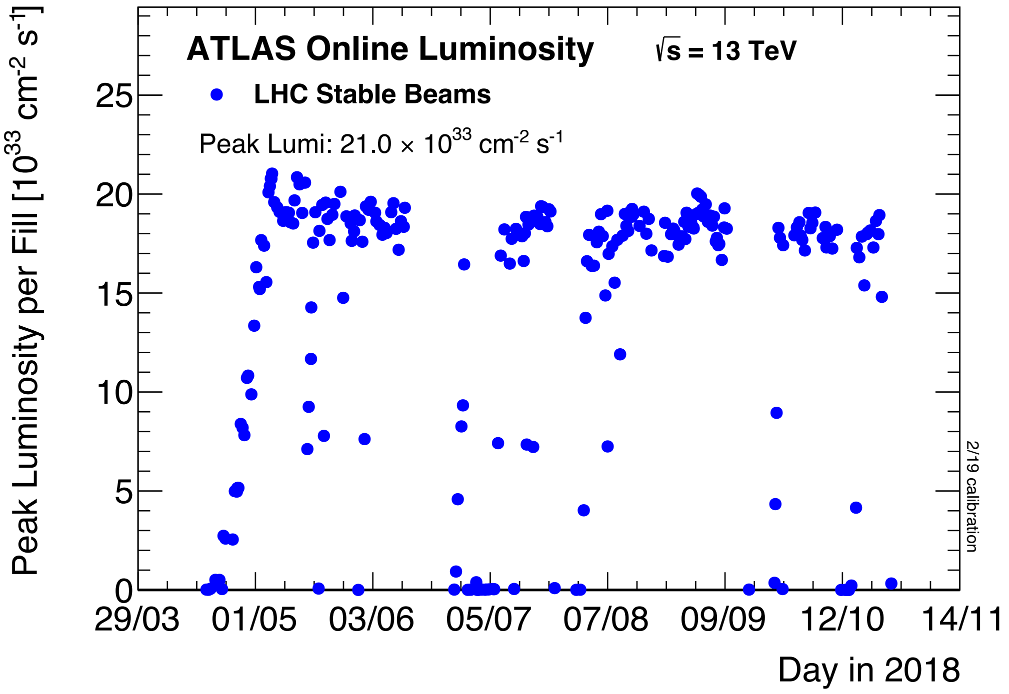
*Only a selection of the available mass limits on new states or phenomena is shown.

†Small-radius (large-radius) jets are denoted by the letter j (J).

https://atlas.web.cern.ch/Atlas/GROUPS/PHYSICS/CombinedSummaryPlots/EXOTICS/

Luminosity

Luminosity defines how many particles we have to collide. More specifically, the number of particles per unit time per unit area. We often think of the instantaneous luminosity, which is the luminosity at any one given time



https://twiki.cern.ch/twiki/bin/view/AtlasPublic/LuminosityPublicResultsRun2#Multiple_Year_Collision_Plots

Luminosity

We also think of the integrated luminosity over time, which when multiplied by a cross section, tells us how many events of a certain process we expected to produce.

Example: ATLAS collected ~140 fb^{-1} of data at energy of 13 TeV. The cross section for Higgs bosons at 13 TeV is ~50 pb = 50,000 fb, so ~7.5 million Higgs bosons were produced in that data set at ATLAS

The branching ratio for Higgs bosons to pairs of photons is 0.0023, so ~17,000 Higgs bosons were produced in the diphoton final state

Fermi’s Golden Rule

Have you seen this before in Quantum Mechanics? We’ll need the relativistic version of it. If not, suggest you look it up

Fermi was a smart man (hard to think of someone with more things named after him). He told us that the rate for a process to occur is equal to the square of the quantum mechanical amplitude (aka the matrix element), multiplied by the density of states

But first, an aside on Dirac Delta functions

Dirac was also a smart man (maybe fewer things named after him than Fermi, but not by that much)

$$\delta(x) = \begin{matrix} 0, & x \neq 0 \\ \infty, & x = 0 \end{matrix} \qquad \delta(x-a) = \begin{matrix} 0, & x \neq a \\ \infty, & x = a \end{matrix}$$

$$f(x)\delta(x) = f(0)\delta(x)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(0)\delta(x)dx = f(0)\int_{-\infty}^{\infty} \delta(x)dx = f(0)$$

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = \int_{-\infty}^{\infty} f(a)\delta(x-a)dx = f(a)\int_{-\infty}^{\infty} \delta(x-a)dx = f(a)$$

Aside on Dirac Delta functions

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = ?$$

$$z = kx, dx = dz/k$$

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \frac{1}{k}\int_{z=-\infty}^{z=\infty} f(z/k)\delta(z)dz =$$

$$\frac{1}{k}\int_{z=-\infty}^{z=\infty} f(z/k)\delta(z)dz = \frac{f(0)}{k}\int_{-\infty}^{\infty} \delta(z)z = \frac{f(0)}{k}$$

Note that here, limits of integration go from -infinity to +infinity only if k is positive

What if k is negative?

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = ?$$

$$z = kx, dx = dz/k \text{ (k negative)}$$

$$\int_{x=-\infty}^{x=\infty} f(x)\delta(kx)dx = \frac{1}{k}\int_{z=+\infty}^{z=-\infty} f(z/k)\delta(z)dz = -\frac{1}{k}\int_{z=-\infty}^{z=+\infty} f(z/k)\delta(z)dz$$

$$-\frac{1}{k}\int_{-\infty}^{\infty} f(z/k)\delta(z)dz = \frac{f(0)}{|k|}\int_{-\infty}^{\infty} \delta(z)z = \frac{f(0)}{|k|}$$

$$\boxed{\delta(kx) = \frac{1}{|k|}\delta(x)}$$

What about any arbitrary function?

Any arbitrary function with potentially any number of zeros.

$$\delta(g(x)), g(x_i) = 0, i = 1, 2, 3...$$

$$g(x) = g(x_i) + (x - x_i)g'(x_i) + \frac{1}{2}(x - x_i)^2 g''(x_i) + ...$$

if x_i something other than zero, this just shifts the delta function

$g'(x_i)$ is “k” in the previous slide

What about any arbitrary function?

For one zero:

$$\delta(g(x)) = \frac{1}{|g'(x_i)|}\delta(x - x_i)$$

In total:

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x - x_i)$$

Heaviside step function

$$\theta(x) = 0(x < 0)$$

$$\theta(x) = 1(x > 0)$$

$$\delta(x) = \frac{d\theta}{dx}$$

What does he look like to you?

Wikipedia image of Oliver Heaviside

Maybe it's just me?

Let’s do

Griffiths problems A1 and A3 together

$$A.1a)\int_0^3 (2x^2+7x+3)\delta(x-1)dx$$

$$A.1b)\int_0^3 \ln(1+x)\delta(\pi-x)dx$$

$$A.3)\text{Simplify }\ \delta(\sin x)$$

Back to Fermi

Fermi’s Golden Rule: nothing to do with how you should treat others (that’s a different Golden Rule). It tells us that the rate for a process (a given collision or decay) is a product of the square of the matrix element (dynamics specific to the theory of the forces at play) and the phase space (recall that things like to happen the more phase space there is for it to happen)

On the phase space

For example, let’s begin by considering
$1 \rightarrow 2+3+4+....n$

In other words, object 1 decaying to objects 2, 3, 4... (n-1 total particles)

Up to some overall normalization, consider the phase space of the jth object as $d^4p_j=d(p^0_j)d^3(\mathbf{p}_j)$

Hopefully that makes some intuitive sense as a definition of phase space?

But of course, the jth object can’t just have any arbitrary value of energy and momentum

Constraints on the phase space

$$1\to 2+3+4+....n$$

The decay products have a definite mass. In other words, $p_j^2 = m_j^2$. Can enforce this in an integral with a delta function, $\delta(p_j^2 - m_j^2)$

Don't allow negative energy states of decay productions, so $p_j^0 > 0$. Can enforce this with Heaviside function, $\theta(p_j^0)$

Conserve energy and momentum. Can enforce this with $\delta(p_1-p_2-p_3-...p_n)$

Golden rule for decays

$1 \rightarrow 2+3+4+\ldots.n$

Normalization

Matrix element squared (to be worked on later)

Decay rate

Momentum/energy conservation

$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n)\times$$

$$\prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0)\frac{d^4 p_j}{(2\pi)^4}$$

On-shell final products

E>0

Phase space

Where did that come from, though?

Recall from basic QM: What is the transition rate from state |i> to state |f>, given some interaction Hamiltonian ($\hat{H}'$)?

Given by Γ_{fi}:

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_i)$$

Matrix element (“transition”)

$$|T_{fi}| = < f|\hat{H}'|i > + \sum_{j \neq i} \frac{< f|\hat{H}'|j >< j|\hat{H}'|i >}{E_i - E_j} + \ldots$$

$$\rho(E_i) = \left| \frac{dn}{dE} \right|_{E_i}$$

Density of states (phase space)

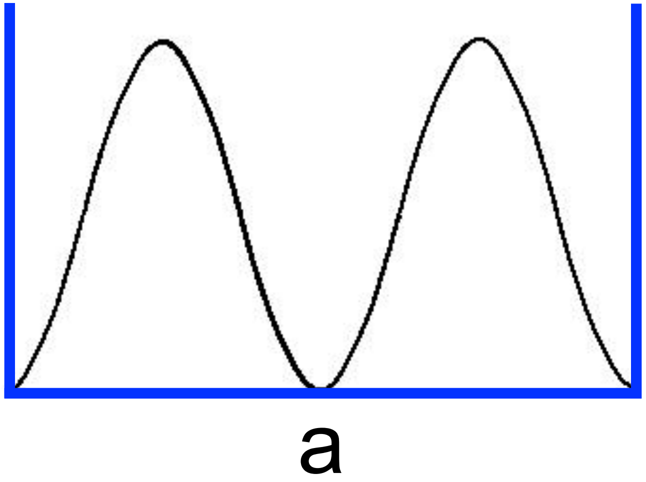
We can rewrite phase space as:

$$\rho(E_i) = \left|\frac{dn}{dE}\right|_{E_i}$$

$$\left|\frac{dn}{dE}\right|_{E_i} = \int \frac{dn}{dE}\delta(E_i - E)dE$$

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)dn$$

1d Boundary conditions in non-relativistic QM



Recall in a one dimensional box that boundary conditions force quantization of momentum

$$\psi(x+a) = \psi(x) \rightarrow p_x = n_x \frac{2\pi}{a}$$

3d Boundary conditions in non-relativistic QM

$$(p_x, p_y, p_z) = (n_x, n_y, n_z)\frac{2\pi}{a}$$

Similarly, in 3 dimensions (if in a 3D cube of length a on each side, total volume V)

What is the volume of a state in momentum space?

$$d^3\mathbf{p} = dp_x dp_y dp_z = \left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$

Normalizing things

Common to normalize to a particle per unit volume, so that the number of states for ith particle is

$$dn_i = \frac{d^3\mathbf{p}_i}{(2\pi)^3}$$

And then the total number of states is

$$dn = \prod_{i=1}^{N-1} dn_i = \prod_{i=1}^{N-1} \frac{d^3\mathbf{p}_i}{(2\pi)^3}$$

Note that we “lost” the last dn_i because it is not independent (fixed, due to momentum conservation)

Rewriting this a bit more nicely

We can add in the last missing $d^3\mathbf{p}_N$ by including a delta function, which forces momentum conservation (particle a is the one decaying), and accounting for the extra $(2\pi)^3$

$$\boxed{dn = (2\pi)^3 \prod_{i=1}^{N} \frac{d^3\mathbf{p}_i}{(2\pi)^3} \delta^3\left(\mathbf{p}_a - \sum_{i=1}^{N} \mathbf{p}_i\right)}$$

Putting it together

For particle a decaying
to particles 1 and 2...

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_a - E_1 - E_2) dn$$

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_a - E_1 - E_2) \frac{d^3\mathbf{p_1}}{(2\pi)^3} \frac{d^3\mathbf{p_2}}{(2\pi)^3} \delta^3 \left(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2\right)$$

What about relativistic mechanics?

Recall that we normalized our transition matrix element to one particle per unit volume.

What happens in another reference frame?
Perpendicular to direction of motion, nothing.

Parallel to direction of motion, we get a Lorentz contraction of $1/\gamma = m/E$, therefore to be Lorentz invariant our normalization must be proportional to $1/E$ (we choose $1/2E$ by Griffiths’ convention)

Why 2E and not E?

$$\prod_{i=1}^{N} \frac{d^3\mathbf{p_i}}{2E_i(2\pi)^3}$$

Our new Lorentz-invariant phase space

$$\int \delta(E_i^2 - \mathbf{p}_i^2 - m_i^2)dE_i = \frac{1}{2E_i}$$

Energy-momentum relation delta function. Is this clear?

So we can rewrite the phase space again

$$\prod_{i=1}^{N} \frac{d^3\mathbf{p_i}}{2E_i(2\pi)^3} = \prod_{i=1}^{N} \frac{d^3\mathbf{p_i}}{(2\pi)^3} \int \delta(E_i^2 - \mathbf{p}_i^2 - m_i^2)dE_i$$

$$\prod_{i=1}^{N} \frac{d^3\mathbf{p_i}}{2E_i(2\pi)^3} = \int \prod_{i=1}^{N} \frac{d^3\mathbf{p}_i dE_i}{(2\pi)^3} \delta(E_i^2 - \mathbf{p}_i^2 - m_i^2)$$

Now let's use 4-vector notation

$$\boxed{\text{phase space} = \int \prod_{i=1}^{N} \frac{d^4p_i}{(2\pi)^3} \delta(p_i^2 - m_i^2)}$$

More on phase space

Let’s take another look at our Lorentz-invariant phase space and check that it really is Lorentz-invariant

$$\frac{d^3\mathbf{p}}{(2\pi)^3 2E}$$

Let’s look at a transformation along the z axis. What is the phase space?

$$d^3\mathbf{p}' = dp'_x dp'_y dp'_z = dp_x dp_y \frac{dp_z^{'}}{dp_z} dp_z = d^3\mathbf{p} \frac{dp_z^{'}}{dp_z}$$

More on phase space

$$d^3\mathbf{p}' = dp'_x dp'_y dp'_z = dp_x dp_y \frac{dp'_z}{dp_z} dp_z = d^3\mathbf{p} \frac{dp'_z}{dp_z}$$

$$p'_z = \gamma(p_z - \beta E), E' = \gamma(E - \beta p_z), E^2 = p_x^2 + p_y^2 + p_z^2 + m^2$$

$$\frac{dp'_z}{dp_z} = \gamma(1 - \beta \frac{\partial E}{\partial p_z})$$

$$\frac{\partial E}{\partial p_z} = \frac{1}{2} \cdot 2p_z \left(\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}\right)^{-1} = \frac{p_z}{E}$$

$$\frac{dp'_z}{dp_z} = \gamma(1 - \beta \frac{p_z}{E}) = \frac{\gamma}{E}(E - \beta p_z) = \frac{E'}{E}$$

So ...

$$\boxed{d^3\mathbf{p}' = d^3\mathbf{p}\frac{E'}{E} \rightarrow \frac{d^3\mathbf{p}'}{E'} = \frac{d^3\mathbf{p}}{E}}$$

OK, back to the golden rule... for decays

1→2+3+4+....n

$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0)\frac{d^4 p_j}{(2\pi)^4}$$

Start with this delta function. We know that $p_j^2 = (p^0)^2 - \mathbf{p}^2$

$$\delta(p_j^2 - m_j^2) = \delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m^2\right)$$

$$\rightarrow (p^0)_j = \pm\sqrt{\mathbf{p_j}^2 + m^2}$$

Golden rule for decays

$$\delta(p_j^2 - m_j^2) = \delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m^2\right)$$

$$\rightarrow (p^0)_j = \pm\sqrt{\mathbf{p_j}^2 + m^2}$$

$$\frac{d}{dp_j^0}\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = 2p_j^0$$

$$\rightarrow \delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = \frac{1}{2p_j^0}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m_j^2}\right) + \delta\left(p_j^0 + \sqrt{\mathbf{p_j}^2 + m_j^2}\right)\right]$$

Golden rule for decays

$$\delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = \frac{1}{2p_j^0}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m_j^2}\right) + \delta\left(p_j^0 + \sqrt{\mathbf{p_j}^2 + m_j^2}\right)\right]$$

$$\theta(p_j^0)\delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = \frac{\theta(p_j^0)}{2p_j^0}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m_j^2}\right) + \cancel{\delta\left(p_j^0 + \sqrt{\mathbf{p_j}^2 + m_j^2}\right)}\right]$$

Heaviside forces p_j^0 to always be greater than 0

$$\theta(p_j^0)\delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = \frac{1}{2p_j^0}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m_j^2}\right)\right]$$

Golden rule for decays

1→2+3+4+....n

$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n)\times$$

$$\prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0)\frac{d^4p_j}{(2\pi)^4}$$

Delta picks out specific value for p_j^0, no need to integrate

So...

$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n)\times$$

$$\prod_{j=2}^{n} 2\pi\frac{\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m^2}\right)}{2p_j^0}\frac{d^4p_j}{(2\pi)^4}$$

Golden rule for decays

$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi \frac{\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m^2}\right)}{2p_j^0} \frac{d^4 p_j}{(2\pi)^4}$$

Make delta function substitution

$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^4}$$

Golden rule for decays

$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 ... - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^4}$$

And rearrange a bit

$$\boxed{\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 ... - p_n) \times \prod_{j=2}^{n} \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3}}$$

Let's assume only two particles in final state

1→2+3

$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 ... - p_n)\times$$

$$\frac{1}{2\sqrt{\mathbf{p_2}^2 + m_2^2}}\frac{d^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p_3}^2 + m_3^2}}\frac{d^3\mathbf{p}_3}{(2\pi)^3}$$

Rearrange...

$$\Gamma = \frac{1}{32\pi^2 m_1}\int |\mathcal{M}|^2 \delta^4(p_1 - p_2 - p_3 ... - p_n)\times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}}\frac{1}{\sqrt{\mathbf{p_3}^2 + m_3^2}} d^3\mathbf{p}_2 d^3\mathbf{p}_3$$

More delta functions (a common theme)

$$\delta^4(p_1 - p_2 - p_3) = \delta(p_1^0 - p_2^0 - p_3^0)\delta^3(\mathbf{p_1} - \mathbf{p_2} - \mathbf{p_3})$$

Let’s choose reference frame where p$_1$ is at rest, so $\mathbf{p_1=0}$ and p$_1$ = (m$_{1,}$$\mathbf{0}$)

$$\delta^4(p_1 - p_2 - p_3) = \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_3}^2 + m_3^2})\delta^3(\mathbf{p_2} + \mathbf{p_3})$$

$$\Gamma = \frac{1}{32\pi^2 m_1}\int |\mathcal{M}|^2 \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_3}^2 + m_3^2})\delta^3(\mathbf{p_2} + \mathbf{p_3}) \times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}}\frac{1}{\sqrt{\mathbf{p_3}^2 + m_3^2}} d^3\mathbf{p}_2 d^3\mathbf{p}_3$$

Continuing to work this out

$\mathbf{p_3}$=-$\mathbf{p_2}$ (had to be, due to conservation of momentum if $\mathbf{p_1}$=0)

$$\Gamma = \frac{1}{32\pi^2 m_1}\int |\mathcal{M}|^2 \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_3}^2 + m_3^2})\delta^3(\mathbf{p_2} + \mathbf{p_3})\times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}}\frac{1}{\sqrt{\mathbf{p_3}^2 + m_3^2}} d^3\mathbf{p}_2 d^3\mathbf{p}_3$$

$$\Gamma = \frac{1}{32\pi^2 m_1}\int |\mathcal{M}|^2 \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2})\times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}}\frac{1}{\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p}_2$$

Almost there

$$\Gamma = \frac{1}{32\pi^2 m_1}\int |\mathcal{M}|^2 \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2})\times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}}\frac{1}{\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p}_2$$

Let’s rearrange again

$$\Gamma = \frac{1}{32\pi^2 m_1}\int |\mathcal{M}|^2 \frac{\delta(m_1 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2})}{\sqrt{\mathbf{p_2}^2 + m_2^2}\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p}_2$$

Stepping back, this is impressive but not surprising.
Only have integral over momentum of one particle left for phase space (why not surprising?)!

Almost there

$$\Gamma = \frac{1}{32\pi^2 m_1}\int |\mathcal{M}|^2 \frac{\delta(m_1 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2})}{\sqrt{\mathbf{p_2}^2 + m_2^2}\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p}_2$$

Let's go to spherical coordinates, $\mathbf{p_2} = (r,\theta,\phi)$ and $d^3\mathbf{p_2}=r^2\sin\theta\ dr\ d\theta\ d\phi$

Matrix element squared cannot be a function of anything but $|\mathbf{p_2}|$ anymore since object 1 was at rest and $\mathbf{p_3}$ is just $-\mathbf{p_2}$ so angular integrals can be easily done

$$\int \sin\theta d\theta d\phi = 4\pi$$

Almost there

$$\Gamma = \frac{1}{32\pi^2 m_1}\int |\mathcal{M}|^2 \frac{\delta(m_1 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2})}{\sqrt{\mathbf{p_2}^2 + m_2^2}\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p}_2$$

Do substitution and angular integrals

$$\Gamma = \frac{1}{8\pi m_1}\int |\mathcal{M}(r)|^2 \frac{\delta(m_1 - \sqrt{r^2 + m_2^2} - \sqrt{r^2 + m_3^2})}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}} r^2 dr$$

Let’s make another substitution...

$$u = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

Almost there

$$u = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \frac{\delta(m_1 - \sqrt{r^2 + m_2^2} - \sqrt{r^2 + m_3^2})}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}} r^2 dr$$

$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \frac{\delta(m_1 - u)}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}} r^2 dr$$

$$\frac{du}{dr} = \frac{r}{\sqrt{r^2 + m_2^2}} + \frac{r}{\sqrt{r^2 + m_3^2}}$$

$$\frac{du}{dr} = \frac{r\left(\sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}\right)}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}}$$

$$\frac{du}{dr} = \frac{ru}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}}$$

Some more substitutions

$$\Gamma = \frac{1}{8\pi m_1}\int |\mathcal{M}(r)|^2 \frac{\delta(m_1 - u)}{\sqrt{r^2+m_2^2}\sqrt{r^2+m_3^2}} r^2 dr$$

$$\frac{du}{dr} = \frac{ru}{\sqrt{r^2+m_2^2}\sqrt{r^2+m_3^2}}$$

$$dr = du\frac{\sqrt{r^2+m_2^2}\sqrt{r^2+m_3^2}}{ru}$$

$$\Gamma = \frac{1}{8\pi m_1}\int |\mathcal{M}(r)|^2 \delta(m_1 - u)\frac{r}{u} du$$

$u=m_1$

More on that delta function

$$\Gamma = \frac{1}{8\pi m_1}\int |\mathcal{M}(r)|^2 \delta(m_1 - u)\frac{r}{u}du$$

u=m$_1$

What does SR tell us?

$$u = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

$$m_1 = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

$$m_1^2 = r^2 + m_2^2 + r^2 + m_3^2 + 2\sqrt{(r^2 + m_3^2)(r^2 + m_2^2)}$$

$$m_1^2 - 2r^2 - m_2^2 - m_3^2 = 2\sqrt{(r^2 + m_3^2)(r^2 + m_2^2)}$$

$$m_1^4 + 4r^4 + m_2^4 + m_3^4 - 4r^2m_1^2 + 4r^2m_2^2 + 4r^2m_3^2 - 2m_1^2m_2^2 - 2m_1^2m_3^2 + 2m_2^2m_3^2 =$$

$$4r^4 + 4m_2^2m_3^2 + 4r^2m_2^2 + 4r^2m_3^2$$

$$-4r^2m_1^2 + m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2 = 0$$

$$\boxed{r = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2}}$$

Finalizing it

$$r = |\mathbf{p_2}| = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2}$$

$$\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2$$

Note that matrix element factorizes (not always possible, but a pretty nice result!)

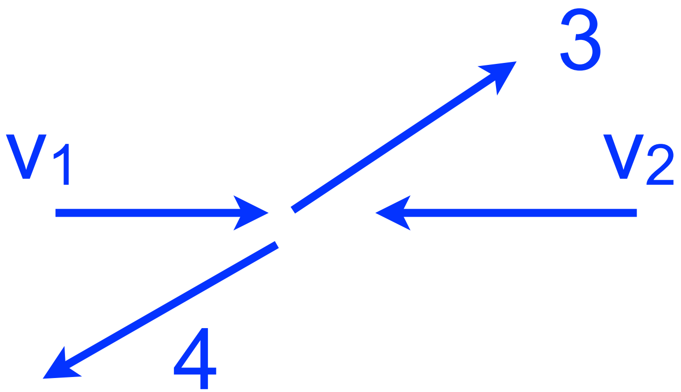
What about scattering?

1+2→3+4

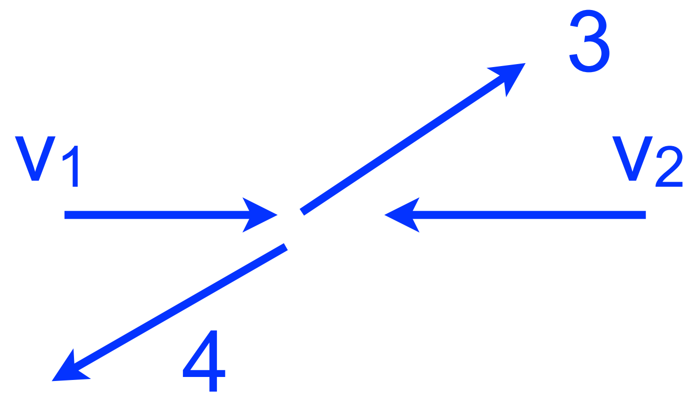
$$\Gamma = \frac{1}{4E_1E_2}\int |\mathcal{M}|^2(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times$$

$$\prod_{j=3}^{4} 2\pi\delta(p_j^2-m_j^2)\theta(p_j^0)\frac{d^4p_j}{(2\pi)^4}$$

This is the rate, though not quite what we're looking for. We are interested in the **cross section (σ)**



Scattering



If we assume one particle per unit volume, then rate = $(v_1+v_2)\sigma$, ie the faster the set of objects 1 and 2 pass through each other, the larger the rate

Be careful (of relative v_1 and v_2 minus signs)

$$\Gamma = (v_1 + v_2)\sigma$$

$$\sigma = \frac{\Gamma}{v_1 + v_2}$$

So the cross section is ...

$$\sigma = \frac{1}{4E_1E_2}\frac{1}{v_1+v_2}\int |\mathcal{M}|^2(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times$$

$$\prod_{j=3}^{4} 2\pi\delta(p_j^2-m_j^2)\theta(p_j^0)\frac{d^4p_j}{(2\pi)^4}$$

Looks a bit odd to have velocity there!
Can that at all be Lorentz invariant?

$$F = 4E_1E_2(v_1+v_2) = 4E_1E_2\left(\frac{|\mathbf{p_1}|}{E_1}+\frac{|\mathbf{p_2}|}{E_2}\right) = 4\left(E_2|\mathbf{p_1}|+E_1|\mathbf{p_2}|\right)$$

$$F^2 = 16\left(E_2^2|\mathbf{p_1}|^2+E_1^2|\mathbf{p_2}|^2+2E_1E_2|\mathbf{p_1}||\mathbf{p_2}|\right)$$

So the cross section is ...

In case where particles 1 and 2 are collinear

$$(p_1 \cdot p_2) = E_1 E_2 + \mathbf{p_1 p_2}$$

Remember the extra minus sign here

$$(p_1 \cdot p_2)^2 = E_1^2 E_2^2 + \mathbf{p_1}^2 \mathbf{p_2}^2 + 2E_1 E_2 \mathbf{p_1 p_2}$$

$$F^2 = 16\left(E_2^2 |\mathbf{p_1}|^2 + E_1^2 |\mathbf{p_2}|^2 + 2E_1 E_2 |\mathbf{p_1}||\mathbf{p_2}|\right)$$

$$F^2 = 16\left(E_2^2 |\mathbf{p_1}|^2 + E_1^2 |\mathbf{p_2}|^2 + (p_1 \cdot p_2)^2 - E_1^2 E_2^2 - \mathbf{p_1}^2 \mathbf{p_2}^2\right)$$

$$F^2 = 16\left[(p_1 \cdot p_2)^2 - (E_1^2 - \mathbf{p_1}^2)(E_2^2 - \mathbf{p_2}^2)\right]$$

Lorentz invariant!

$$F^2 = 16\left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2\right]$$

$$\boxed{F = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}$$

So the cross section is ...

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4}$$

As before, 1d delta function is easy (and rearranges some 2π's). Heaviside enforces only one solution

$$p_j^0 = E_j = \sqrt{\mathbf{p}_j^2 + m_j^2}$$

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^3}$$

Moving along with cross sections

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

$$p_j^0 = E_j = \sqrt{\mathbf{p}_j^2 + m_j^2}$$

Let’s put back the earlier form for F

$$\sigma = \frac{1}{4E_1 E_2} \frac{1}{v_1 + v_2} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

Moving along with cross sections

$$F = \frac{1}{4E_1E_2}\frac{1}{v_1 + v_2} = \frac{1}{4E_1E_2}\frac{1}{p_1/E_1 + p_2/E_2}$$

$$F = \frac{1}{4}\frac{1}{E_2p_1 + E_1p_2}$$

In Center of Mass frame, $|\mathbf{p}_2| = |\mathbf{p}_1|$

$$F = \frac{1}{4|p_1|}\frac{1}{E_1 + E_2}$$

$$\sigma = \frac{1}{4|p_1|(E_1 + E_2)}\int |\mathcal{M}|^2(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)\times$$

$$\prod_{j=3}^{4}\frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}}\frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

Let’s combine some factors

$$\sigma = \frac{1}{64\pi^2 |p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{\sqrt{\mathbf{p}_j^2 + m_j^2}} d^3\mathbf{p}_j$$

$$p_j^0 = E_j = \sqrt{\mathbf{p}_j^2 + m_j^2}$$

Let’s split up the 4d delta function

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$

Delta functions in action

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$

$$p_j^0 = E_j = \sqrt{\mathbf{p}_j^2 + m_j^2}$$

In CoM frame, we know that **p**$_1$+**p**$_2$ = 0

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_3 + \mathbf{p}_4)$$

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_3 + \mathbf{p}_4) \times \frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2}} \frac{1}{\sqrt{\mathbf{p}_4^2 + m_4^2}} d^3\mathbf{p}_3 d^3\mathbf{p}_4$$

Delta functions in action

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2\delta(E_1+E_2-E_3-E_4)\delta^3(\mathbf{p}_3+\mathbf{p}_4)\times \frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}}\frac{1}{\sqrt{\mathbf{p}_4^2+m_4^2}}d^3\mathbf{p}_3 d^3\mathbf{p}_4$$

The 3D delta function enforces $\mathbf{p}_3 = -\mathbf{p}_4$

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2\delta(E_1+E_2-E_3-E_4)\times \frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}}\frac{1}{\sqrt{\mathbf{p}_3^2+m_4^2}}d^3\mathbf{p}_3$$

Delta functions in action

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)} \int |\mathcal{M}|^2 \delta(E_1+E_2-E_3-E_4) \times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}} \frac{1}{\sqrt{\mathbf{p}_3^2+m_4^2}} d^3\mathbf{p}_3$$

Recall that we had delta functions on E_3 and E_4 (the mass relations)

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)} \int |\mathcal{M}|^2 \delta(E_1+E_2-\sqrt{\mathbf{p}_3^2+m_3^2}-\sqrt{\mathbf{p}_4^2+m_4^2}) \times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}} \frac{1}{\sqrt{\mathbf{p}_3^2+m_4^2}} d^3\mathbf{p}_3$$

Delta functions in action

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2\delta(E_1+E_2-\sqrt{\mathbf{p}_3^2+m_3^2}-\sqrt{\mathbf{p}_4^2+m_4^2})\times \frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}}\frac{1}{\sqrt{\mathbf{p}_3^2+m_4^2}}d^3\mathbf{p}_3$$

The 3D delta function enforced $\mathbf{p}_3 = -\mathbf{p}_4$

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2\delta(E_1+E_2-\sqrt{\mathbf{p}_3^2+m_3^2}-\sqrt{\mathbf{p}_3^2+m_4^2})\times \frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}}\frac{1}{\sqrt{\mathbf{p}_3^2+m_4^2}}d^3\mathbf{p}_3$$

As before, we change coordinate systems

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2\delta(E_1+E_2-\sqrt{\mathbf{p}_3^2+m_3^2}-\sqrt{\mathbf{p}_3^2+m_4^2})\times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}}\frac{1}{\sqrt{\mathbf{p}_3^2+m_4^2}}d^3\mathbf{p}_3$$

$$d^3\mathbf{p}_3 = r^2 dr d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2\delta(E_1+E_2-\sqrt{r^2+m_3^2}-\sqrt{r^2+m_4^2})\times$$

$$\frac{1}{\sqrt{r^2+m_3^2}}\frac{1}{\sqrt{r^2+m_4^2}}r^2 dr$$

Evaluating the integral

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2\delta(E_1+E_2-\sqrt{r^2+m_3^2}-\sqrt{r^2+m_4^2})\times \frac{1}{\sqrt{r^2+m_3^2}}\frac{1}{\sqrt{r^2+m_4^2}}r^2dr$$

Same exact form as last ugly integral, so nothing new here

Differential cross section

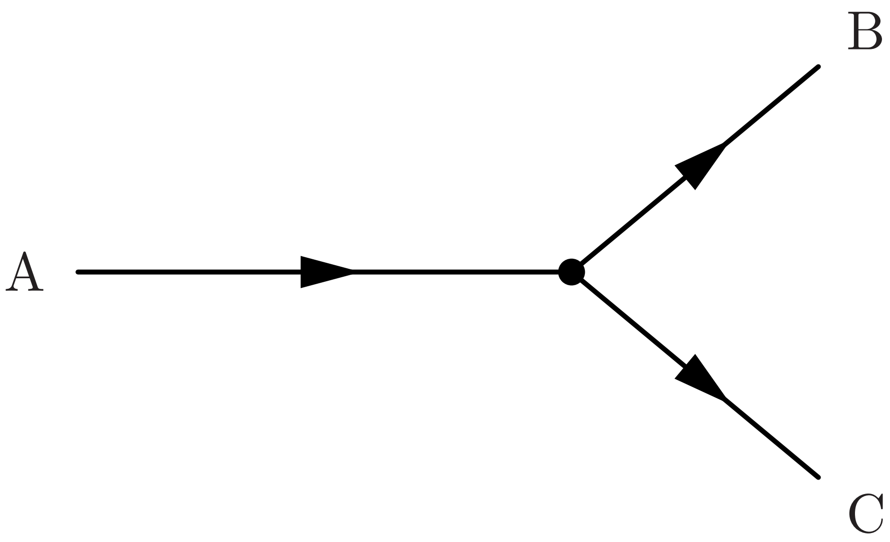
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{|M|^2}{(E_1+E_2)^2}\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}}$$

On to (toy) theories of particle physics!

I think this is where Griffiths does a really nice job. We won’t dive into QED, but will instead start with a simpler theory.

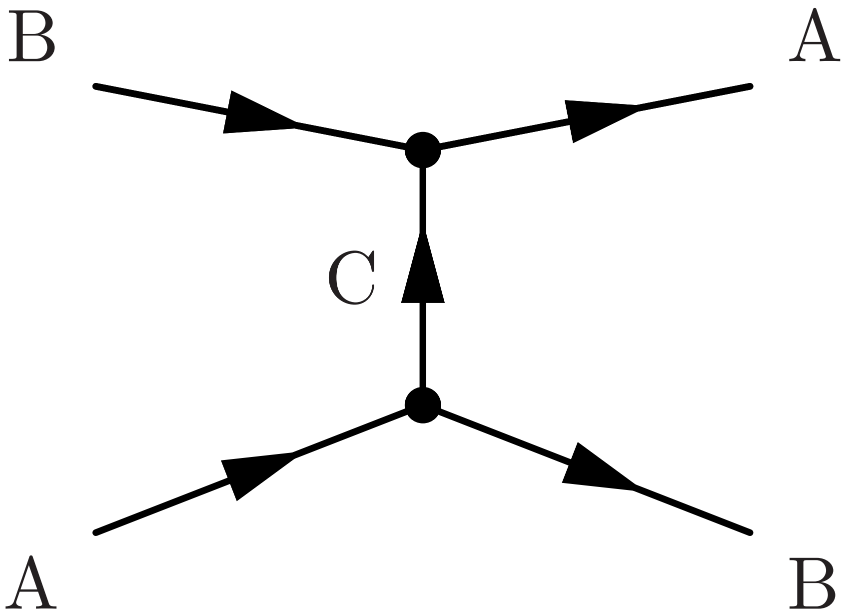
Feynman’s calculus/his rules tell us how to calculate the matrix elements (why? we won’t be diving into QFT, so for now please just accept them, as unappealing as that might be)

Our toy theory



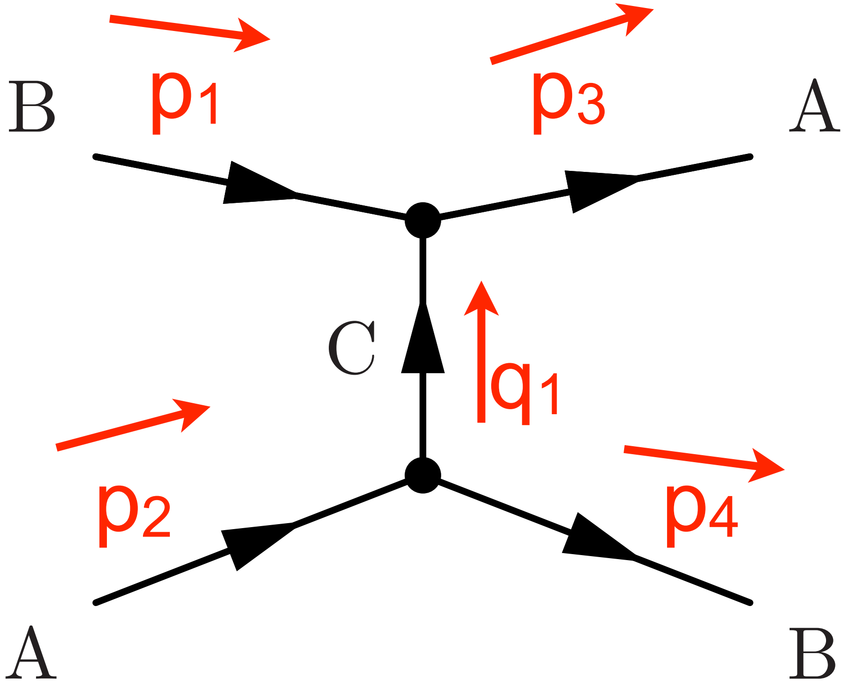
Our toy theory has 3 types of spin-0 particles, A, B and C. Let’s assume that $m_A > m_B + m_C$. Here, A is incoming, and B and C are outgoing. This is a decay vertex

Our toy theory



Here we have scattering A+B→A+B (one example diagram)

Feynman rules (1)



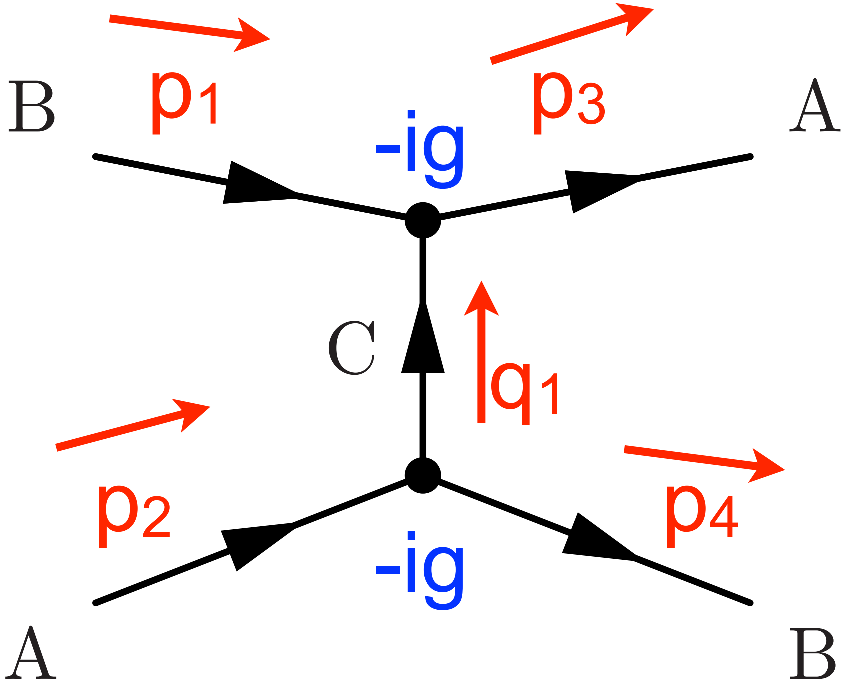
p vs q is pure convention!

Label all incoming and outgoing lines with p_1, p_2, ... p_n

Internal lines can go either way

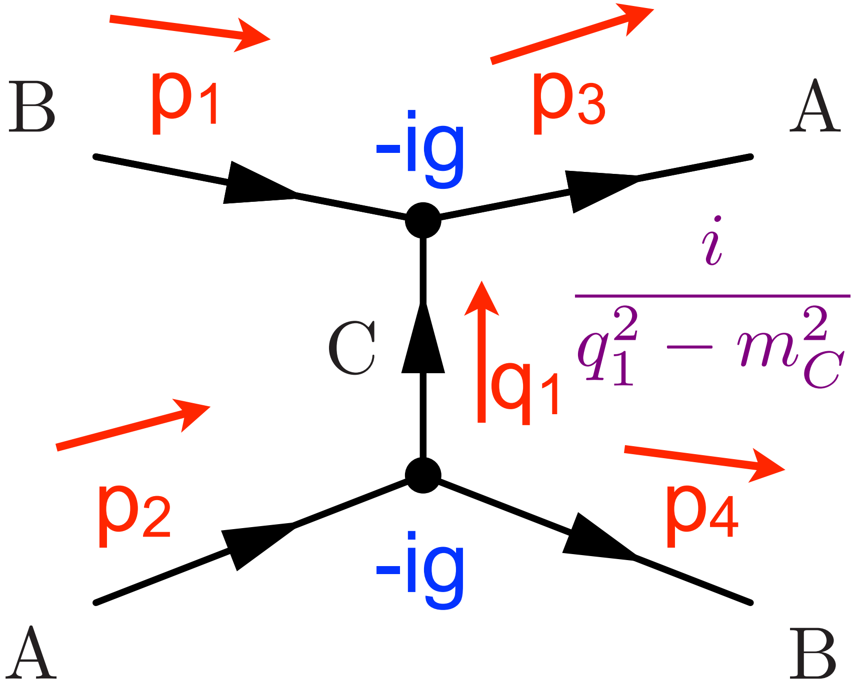
Use arrows to keep track of what is going in and out (here this looks trivial, but can be more tricky with anti-particles)

Feynman rules (2)



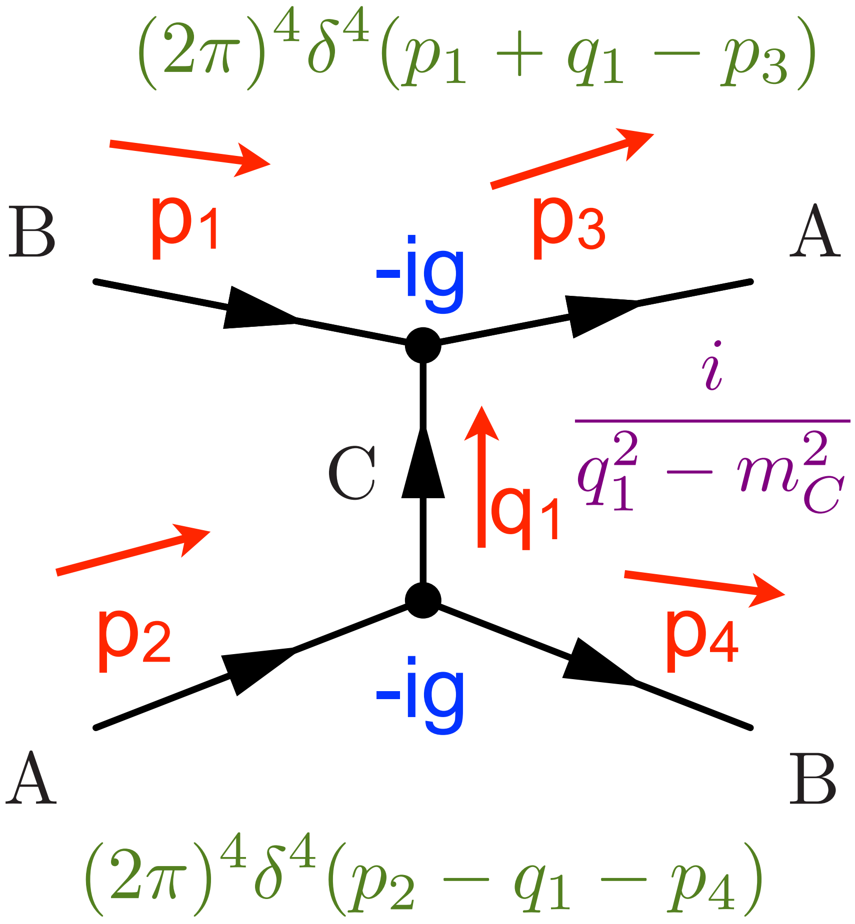
Add factors of -ig for each vertex, specifying the coupling constants

Feynman rules (3)



For each internal line add a factor for the propagator (note that we don't have to be on-shell here!)

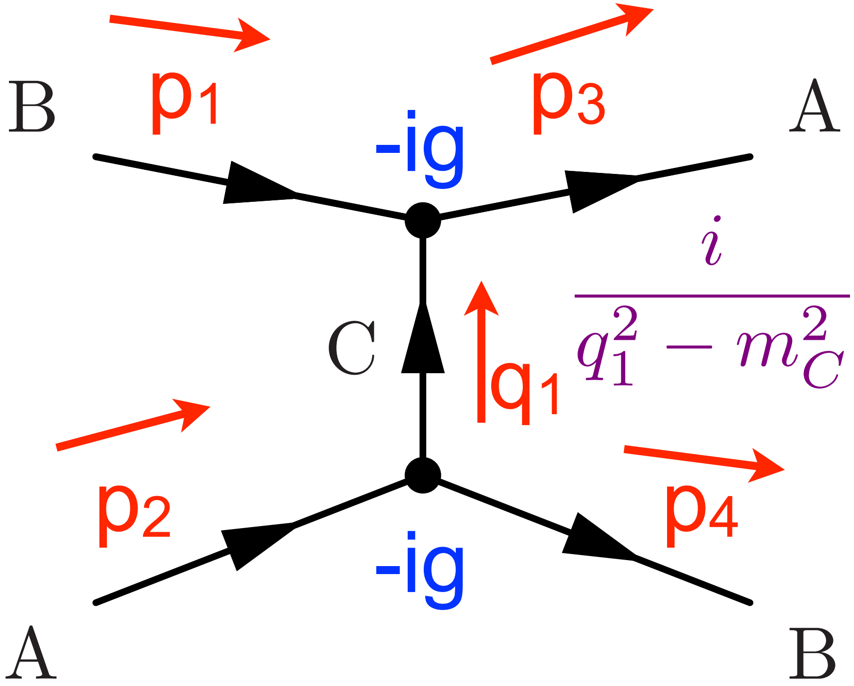
Feynman rules (4)



Impose conservation of energy and momentum at each vertex with 4d Dirac Delta function (with appropriate 2pi normalization)

Feynman rules (5)

$$(2\pi)^4\delta^4(p_1+q_1-p_3)$$



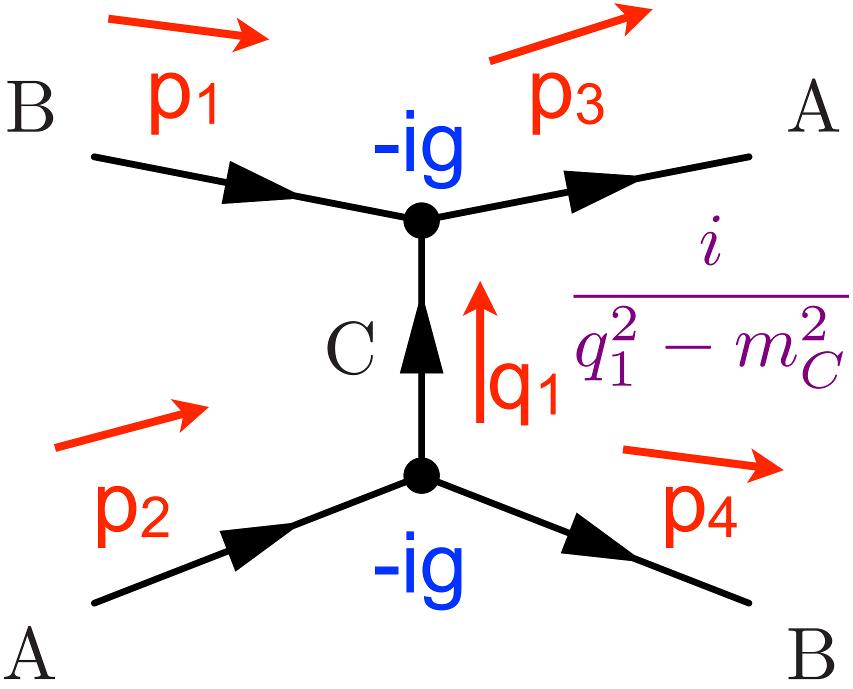
$$\frac{i}{q_1^2-m_C^2}\,\frac{1}{(2\pi)^4}d^4q_1$$

$$(2\pi)^4\delta^4(p_2-q_1-p_4)$$

Integrate over 4-momentum of internal lines with appropriate 2pi normalization factor

Feynman rules (6)

$$(2\pi)^4\delta^4(p_1+q_1-p_3)$$



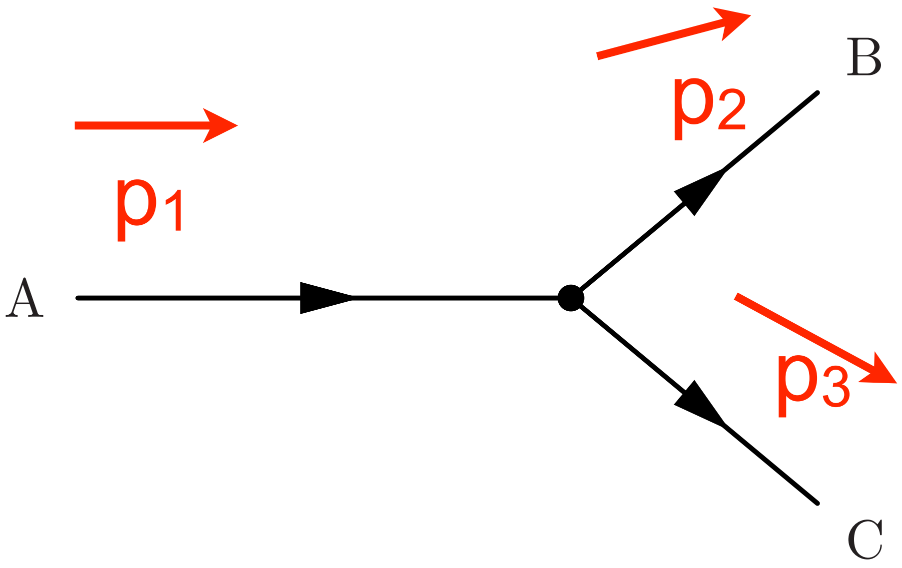
$$i$$

$$\frac{i}{q_1^2-m_C^2}\,\frac{1}{(2\pi)^4}d^4q_1$$

$$(2\pi)^4\delta^4(p_2-q_1-p_4)$$

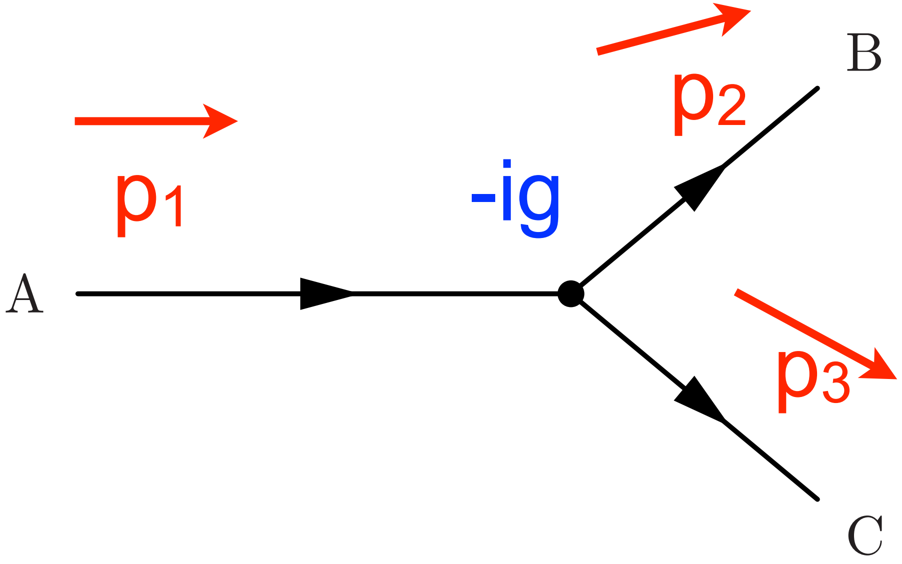
Cancel remaining delta function and add a factor of i, and you have the matrix element

Feynman rules for decay of object A (1)



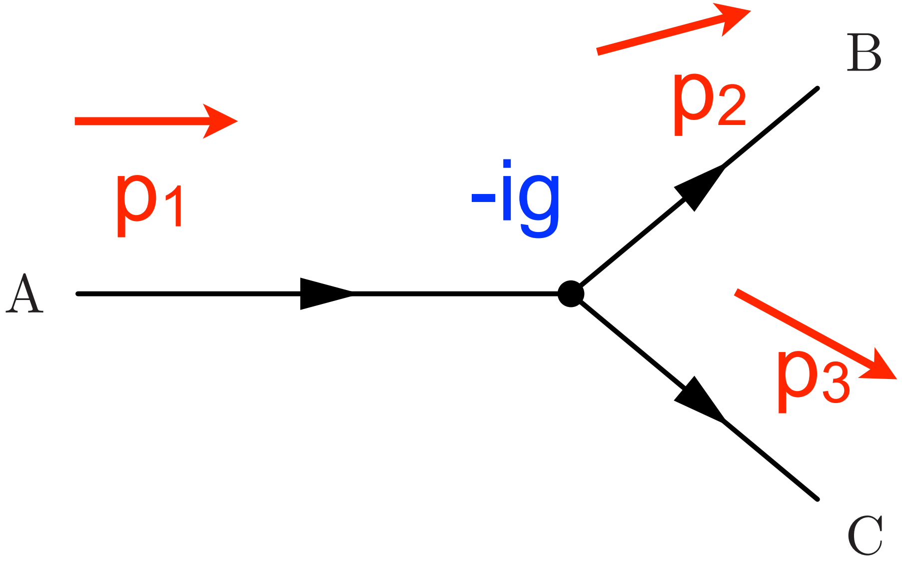
Label external (p) and internal lines (q) and draw arrows. Here we have no internal lines

Feynman rules for decay of object A (2)



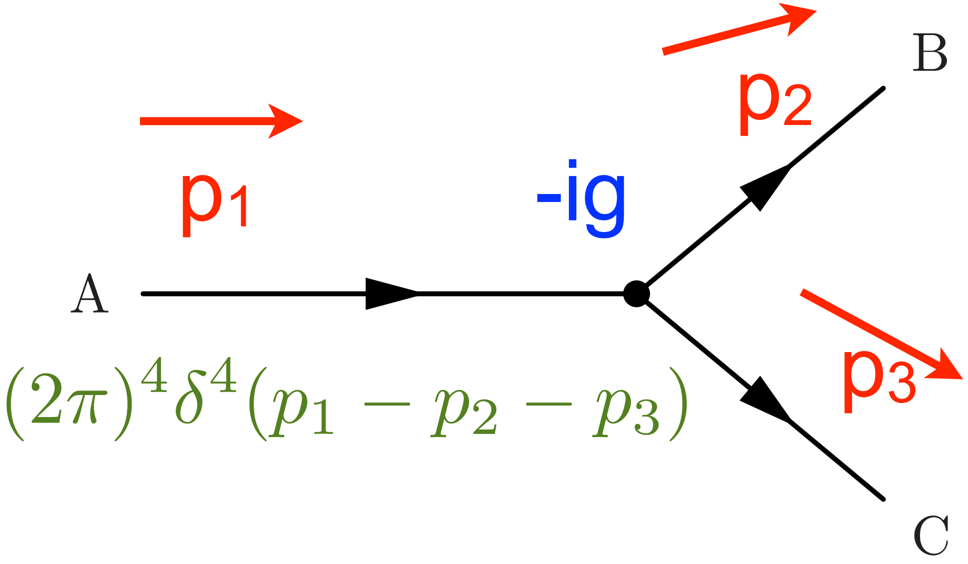
Single factor of -ig for our one vertex

Feynman rules for decay of object A (3)



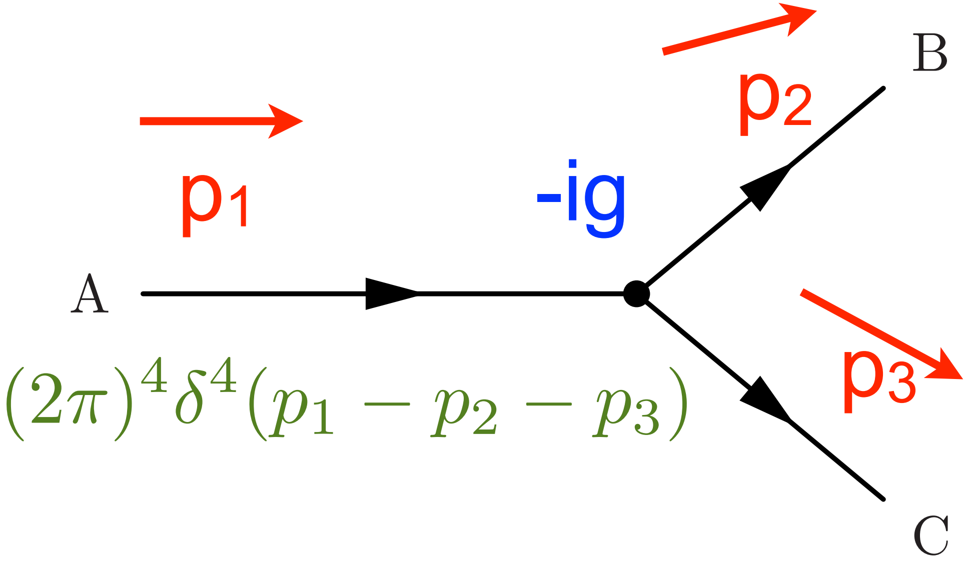
For each internal line add a factor for the propagator, but we don’t have one here! (It’s nice when things are simple)

Feynman rules for decay of object A (4)



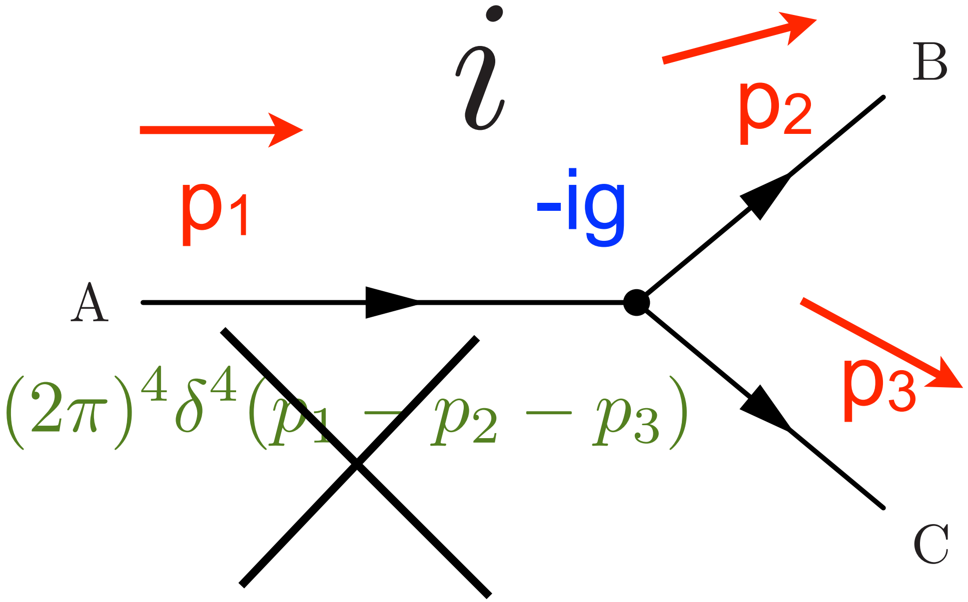
Impose conservation of energy and momentum at each vertex with 4d Dirac Delta function (with appropriate 2pi normalization)

Feynman rules for decay of object A (5)



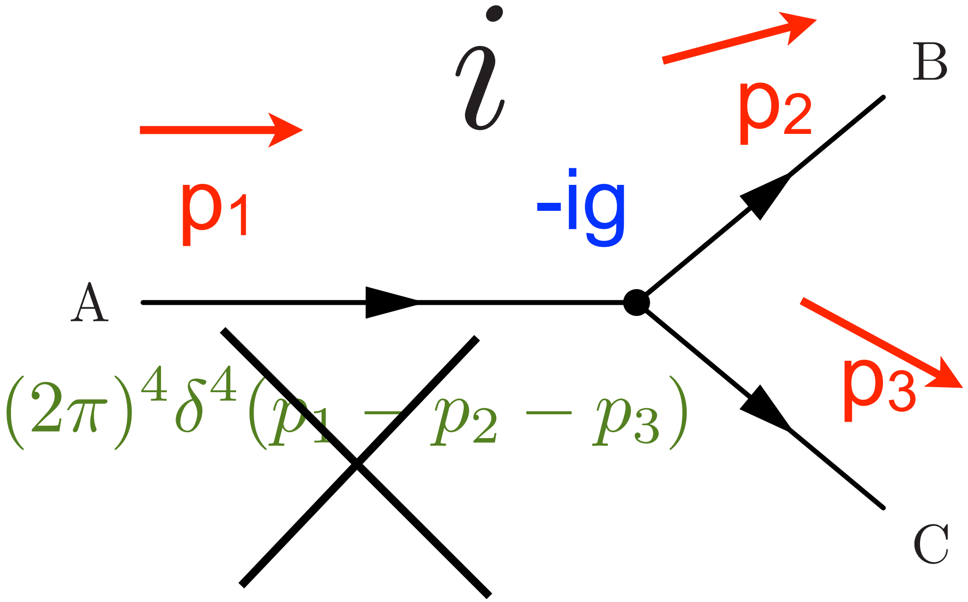
Integrate over 4-momentum of internal lines with appropriate 2pi normalization factor (here, none)

Feynman rules for decay of object A (6)



Add factor of i and cancel remaining delta function

Feynman rules for decay of object A



$$\xcancel{(2\pi)^4\delta^4(p_1 - p_2 - p_3)}$$

We're left only with M = i(-ig) = g

$$\boxed{\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2 = \frac{g^2|\mathbf{p}|}{8\pi m_A^2}}$$

$$|\mathbf{p}| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$$

Lifetime of object A

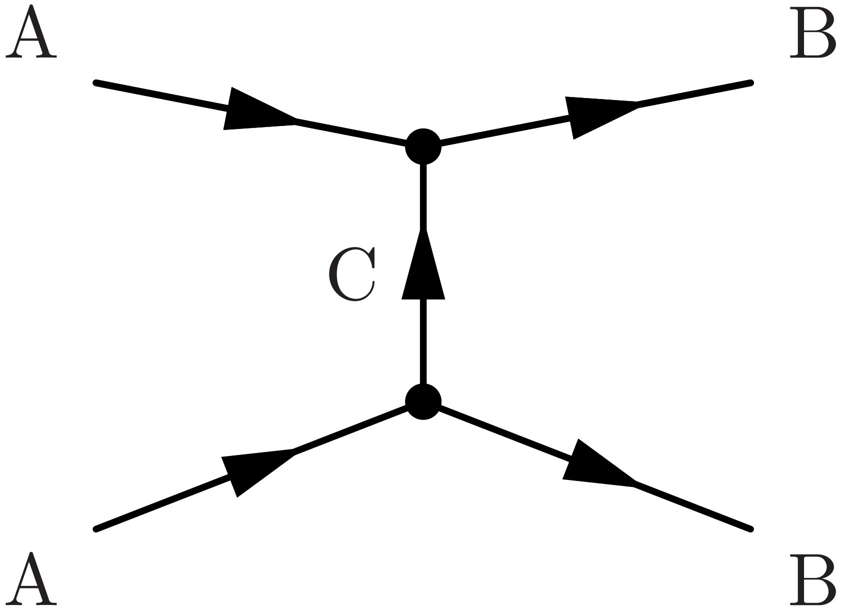
$$\boxed{\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2 = \frac{g^2|\mathbf{p}|}{8\pi m_A^2}}$$

$$|\mathbf{p}| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$$

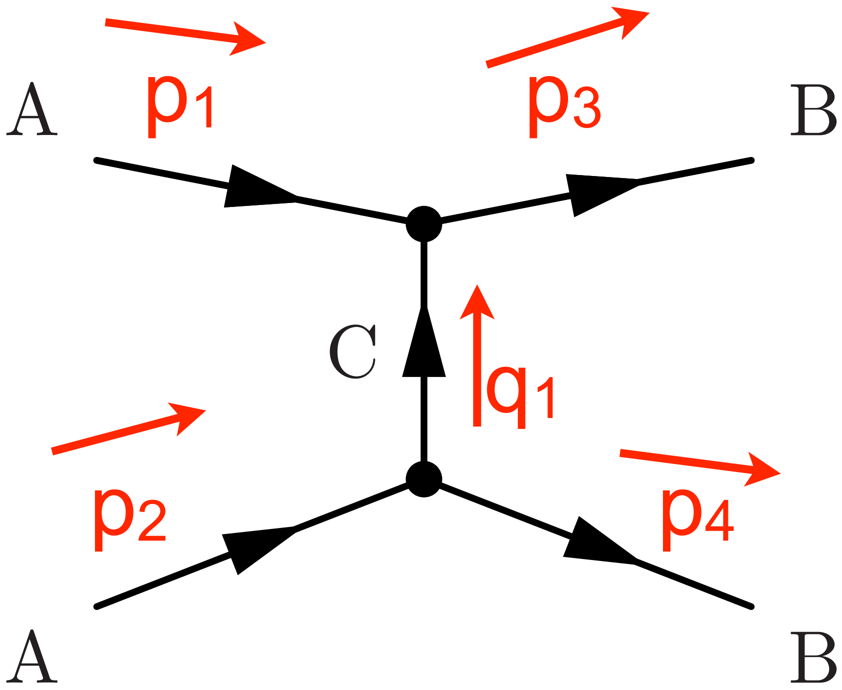
$$\tau = \frac{1}{\Gamma} = \frac{8\pi m_A^2}{g^2|\mathbf{p}|}$$

Let’s calculate matrix element for AA→BB

Leading order diagram (there are others at higher order, as we’ll see)



AA→BB Rule 1

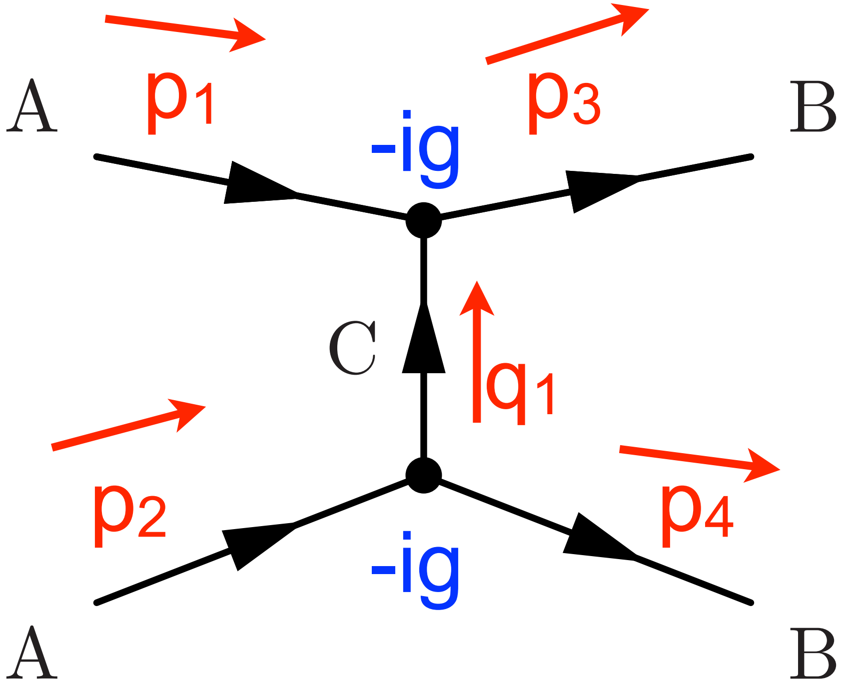


Label all incoming and outgoing lines with p_1, p_2, ... p_n

Internal lines can go either way

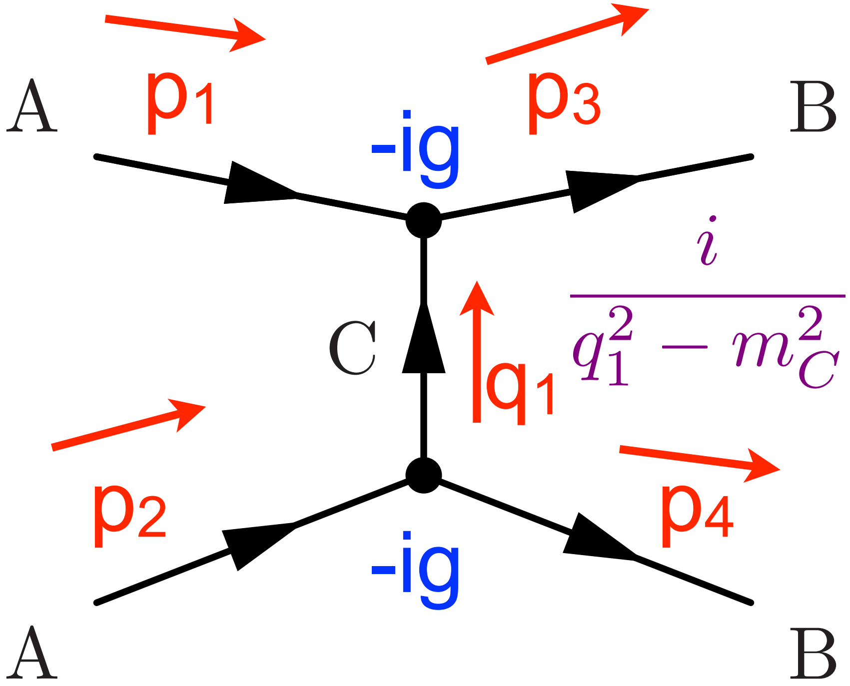
Use arrows to keep track of what is going in and out (here this looks trivial, but can be more tricky with anti-particles)

AA→BB Rule 2



Add factors of -ig for each vertex, specifying the coupling constants

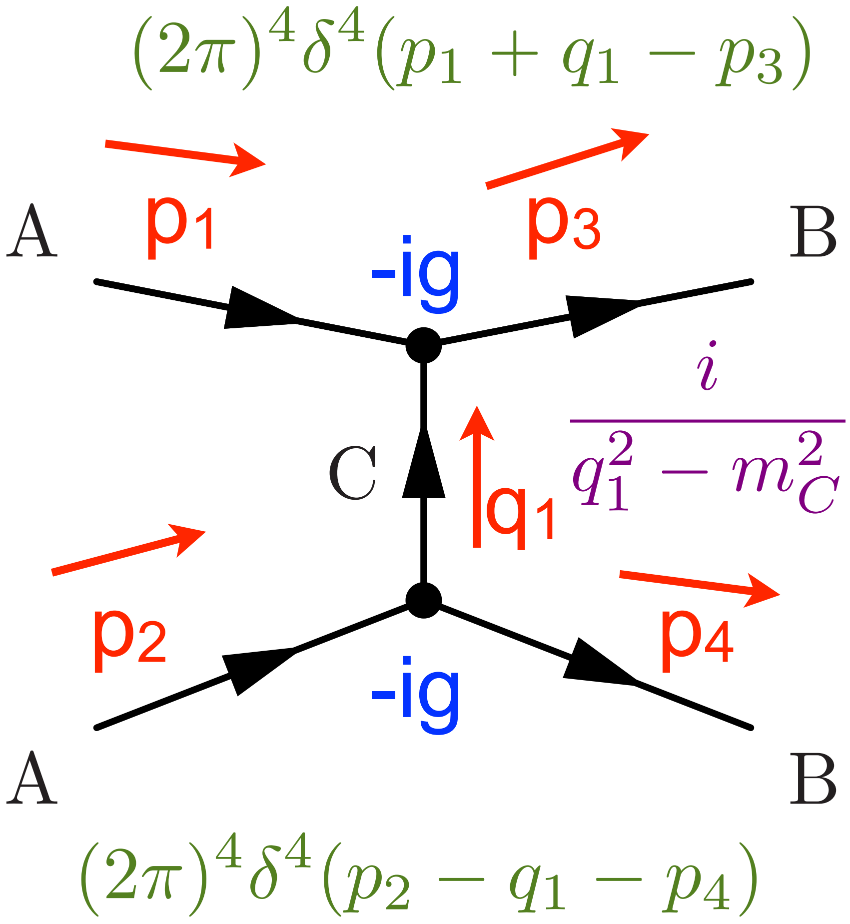
AA→BB Rule 3



$$\frac{i}{q_1^2 - m_C^2}$$

For each internal line add a factor for the propagator (note that we don't have to be on-shell here!)

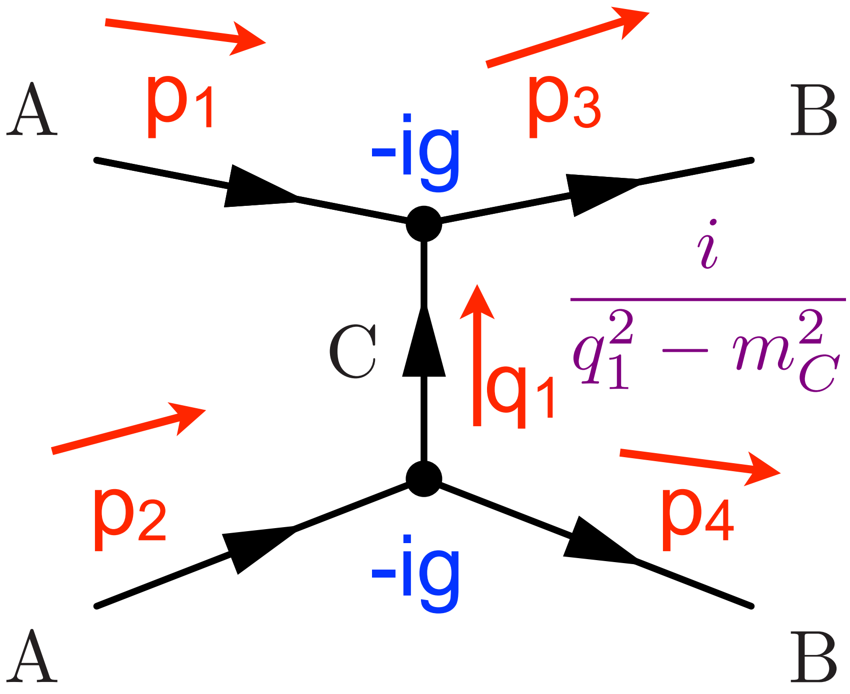
AA→BB Rule 4



Impose conservation of energy and momentum at each vertex with 4d Dirac Delta function (with appropriate 2pi normalization)

AA→BB Rule 5

$$(2\pi)^4 \delta^4(p_1 + q_1 - p_3)$$

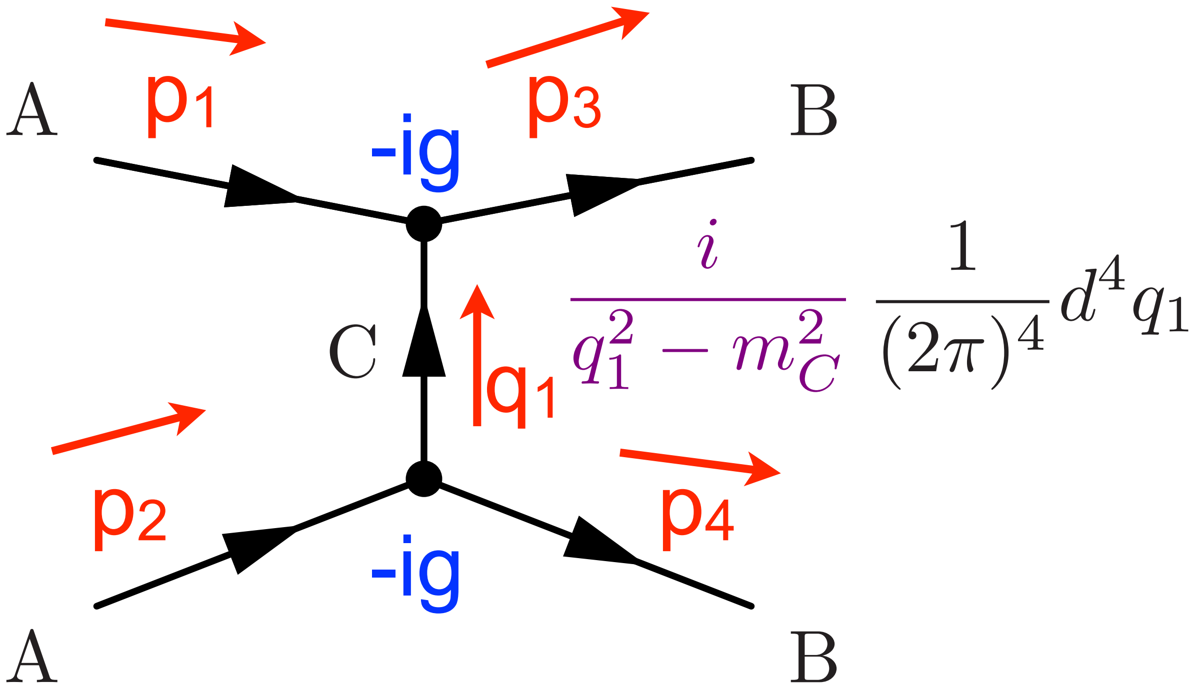


$$\frac{i}{q_1^2 - m_C^2} \frac{1}{(2\pi)^4} d^4 q_1$$

$$(2\pi)^4 \delta^4(p_2 - q_1 - p_4)$$

Integrate over 4-momentum of internal lines with appropriate 2pi normalization factor

AA→BB Rule 5, putting it together



$$(2\pi)^4\delta^4(p_1 + q_1 - p_3)$$

$$\frac{i}{q_1^2 - m_C^2} \quad \frac{1}{(2\pi)^4} d^4q_1$$

$$(2\pi)^4\delta^4(p_2 - q_1 - p_4)$$

$$\boxed{\int (-ig)(-ig)\frac{i}{q_1^2 - m_C^2}(2\pi)^4\delta^4(p_1 + q_1 - p_3)(2\pi)^4\delta^4(p_2 - q_1 - p_4)\frac{d^4q_1}{(2\pi)^4}}$$

AA→BB Rule 5, simplification

$$\boxed{\int(-ig)(-ig)\frac{i}{q_1^2-m_C^2}(2\pi)^4\delta^4(p_1+q_1-p_3)(2\pi)^4\delta^4(p_2-q_1-p_4)\frac{d^4q_1}{(2\pi)^4}}$$

$$(2\pi)^4(-g^2)\int\frac{i}{q_1^2-m_C^2}\delta^4(p_1+q_1-p_3)\delta^4(p_2-q_1-p_4)d^4q_1$$

Integral is over 4-momentum of q_1 but this gets picked up by the Delta function. Let's use the first one, so $q_1 = p_3-p_1$

AA→BB Rule 5, simplification

$$(2\pi)^4(-g^2)\int \frac{i}{q_1^2 - m_C^2}\delta^4(p_1 + q_1 - p_3)\delta^4(p_2 - q_1 - p_4)d^4q_1$$

Integral is over 4-momentum of q_1 but this gets picked up by the Delta function. Let's use the first one, so $q_1 = p_3 - p_1$ (note that this is true for each component of q_1, as we're using some notation shorthand here)

$$(2\pi)^4(-g^2)\frac{i}{(p_3 - p_1)^2 - m_C^2}\delta^4(p_2 + p_1 - p_3 - p_4)$$

Conservation of **p** and E for total system

AA→BB Rule 6

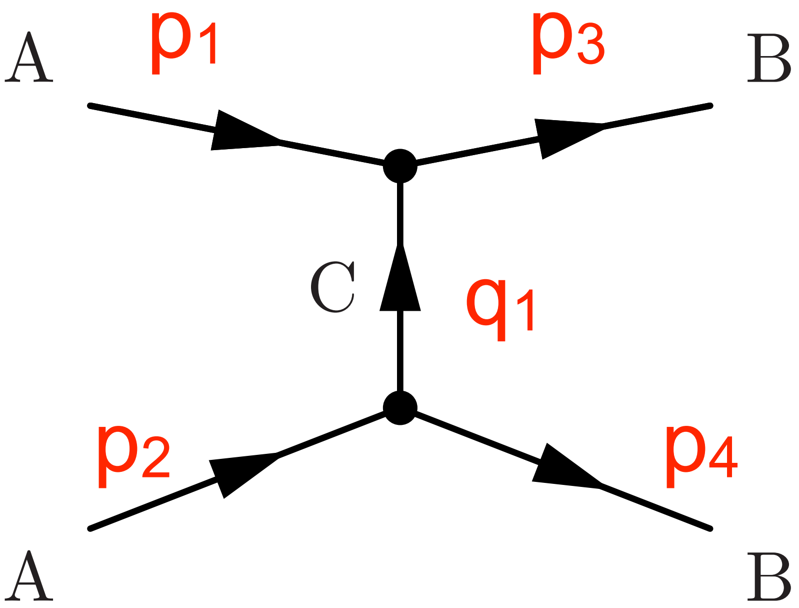
$$(2\pi)^4(-g^2)\frac{i}{(p_3-p_1)^2-m_C^2}\delta^4(p_2+\cancel{p_1}-p_3-p_4)$$

Rule 6: cancel delta function (and 2pi)^4 and multiply by i to get Matrix Element

$$\mathcal{M}=i\cancel{(2\pi)}^4(-g^2)\frac{i}{(p_3-p_1)^2-m_C^2}$$

$$\boxed{\mathcal{M}=\frac{g^2}{(p_3-p_1)^2-m_C^2}}$$

First complication for AA→BB scattering

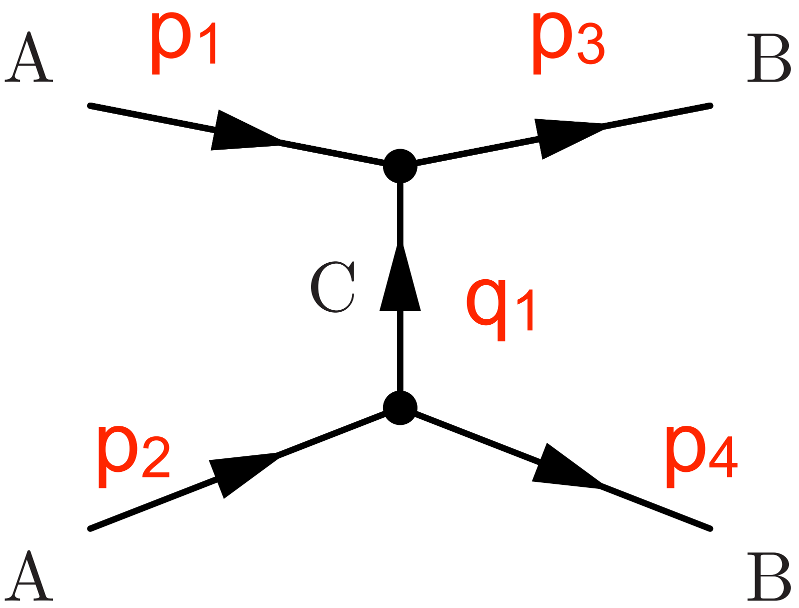


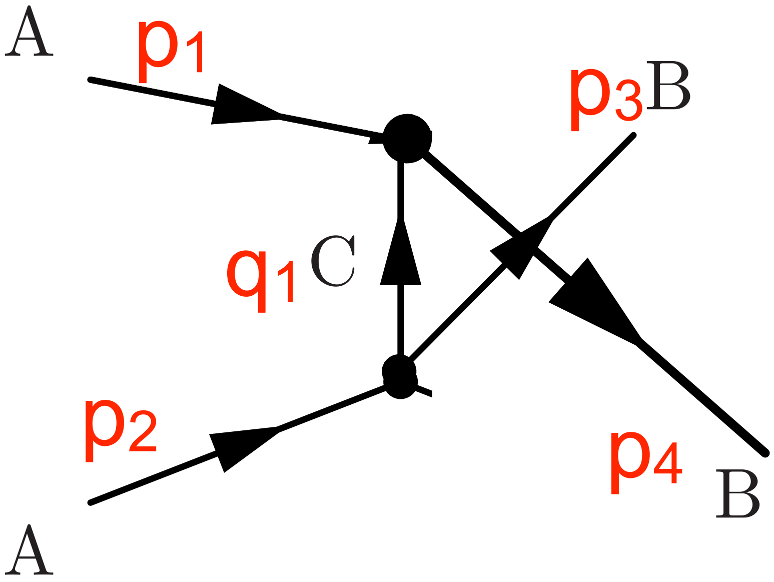
$$\mathcal{M} = \frac{g^2}{(p_3 - p_1)^2 - m_C^2}$$

This is not the only diagram. There is another similar one with the same initial state and the same final state. So they must be added together, as they interfere!

Second AA→BB diagram

Let's work this out together on the board

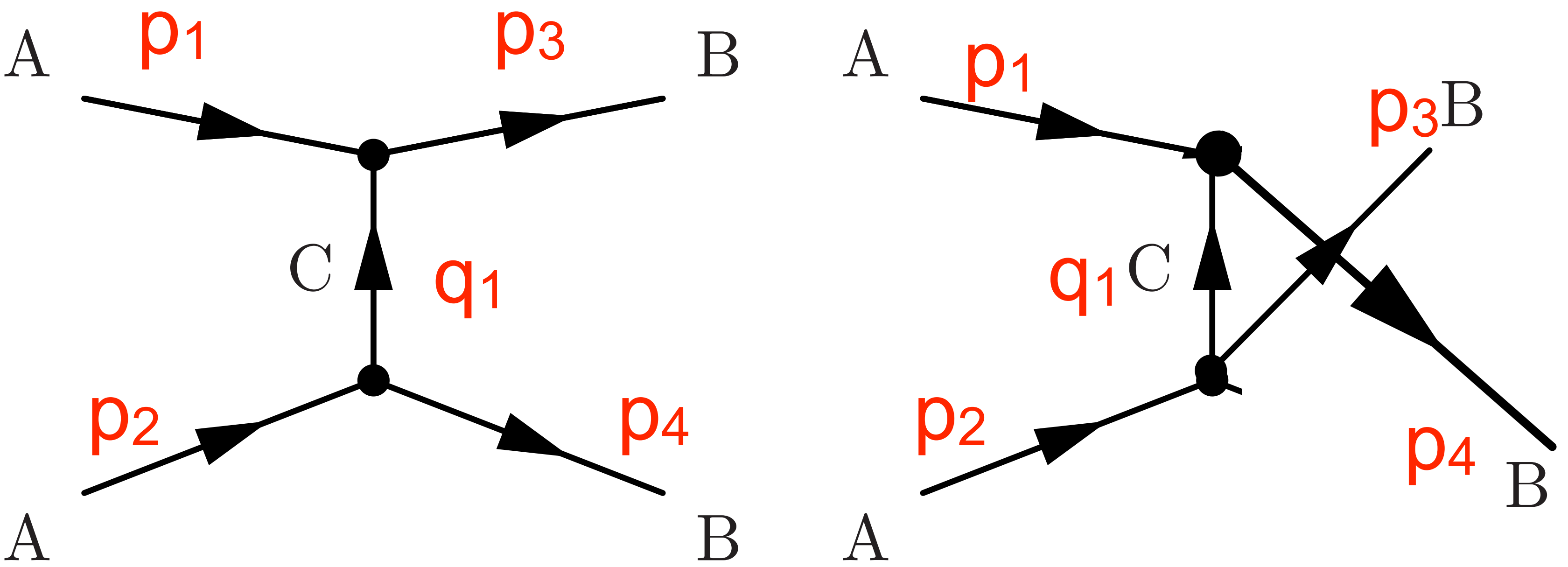




$$\mathcal{M} = \frac{g^2}{(p_3 - p_1)^2 - m_C^2}$$

Same diagram except that p_1 connects to p_4, not to p_3

Second AA→BB diagram



$$\boxed{\mathcal{M} = \frac{g^2}{(p_3 - p_1)^2 - m_C^2}}$$

$$\boxed{\mathcal{M} = \frac{g^2}{(p_4 - p_1)^2 - m_C^2}}$$

Does this look at all similar (notation-wise to your previous homework)?

So total AA→BB ME is

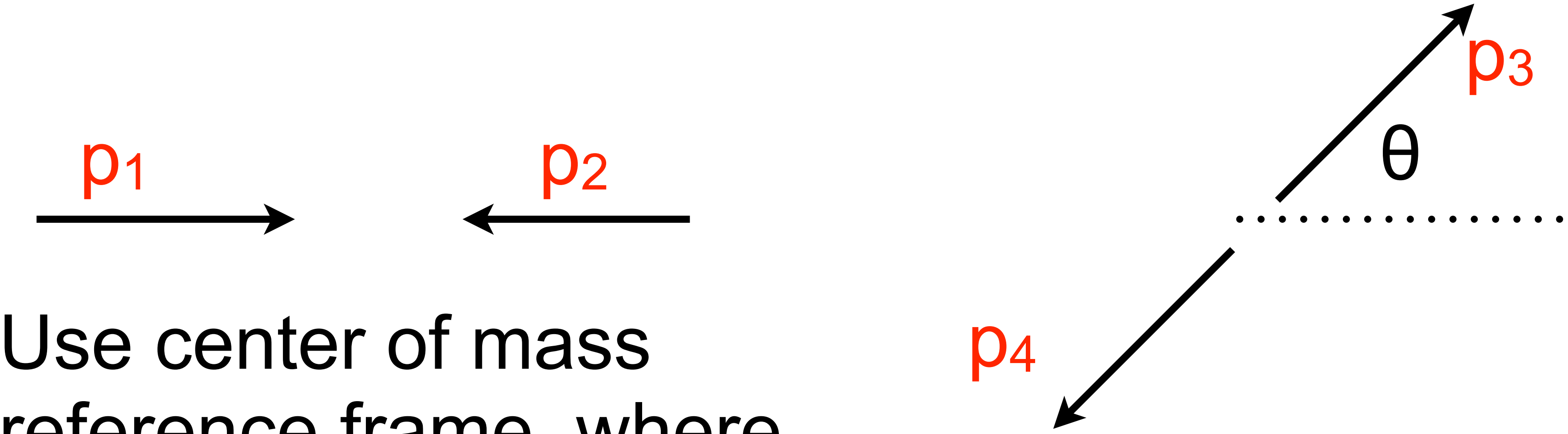
$$\boxed{\mathcal{M} = g^2 \left[\frac{1}{(p_3 - p_1)^2 - m_C^2} + \frac{1}{(p_4 - p_1)^2 - m_C^2} \right]}$$

As in Griffiths, let’s assume m_C is zero to simplify things, $m_A=m_B=m$

$$\mathcal{M} = g^2 \left[\frac{1}{(p_3 - p_1)^2} + \frac{1}{(p_4 - p_1)^2} \right]$$

AA→BB ME for toy with massless m_C

$$\mathcal{M} = g^2 \left[\frac{1}{(p_3 - p_1)^2} + \frac{1}{(p_4 - p_1)^2} \right]$$



Use center of mass reference frame, where

$|\mathbf{p}_1|=|\mathbf{p}_2|$, $|\mathbf{p}_3|=|\mathbf{p}_4|$

Since $m_A=m_B$ this means

$|\mathbf{p}_1|=|\mathbf{p}_2|=|\mathbf{p}_3|=|\mathbf{p}_4|=p$ $E_1=E_2=E_3=E_4=E$

That clear?

AA→BB ME for toy with massless mC

$$(p_4 - p_1)^2 = p_1^2 + p_4^2 - 2p_4 \cdot p_1 = 2m^2 - 2E_4E_1 + 2\mathbf{p}_4 \cdot \mathbf{p}_1$$

$$(p_4 - p_1)^2 = 2m^2 - 2E^2 + 2\mathbf{p}_4 \cdot \mathbf{p}_1 = 2(E^2 - \mathbf{p}^2) - 2E^2 + 2\mathbf{p}_4 \cdot \mathbf{p}_1$$

$$(p_4 - p_1)^2 = 2(E^2 - \mathbf{p}^2) - 2E^2 + 2\mathbf{p}_4 \cdot \mathbf{p}_1 = -2\mathbf{p}^2 + 2\mathbf{p}_4 \cdot \mathbf{p}_1$$

$$(p_4 - p_1)^2 = -2\mathbf{p}^2 - 2\mathbf{p}^2 \cos\theta = -2\mathbf{p}^2(1 + \cos\theta)$$

AA→BB ME for toy with massless mC

$$\mathcal{M} = g^2 \left[\frac{1}{(p_3 - p_1)^2} + \frac{1}{(p_4 - p_1)^2} \right]$$

$$(p_4 - p_1)^2 = -2\mathbf{p}^2(1 + \cos\theta)$$

$$(p_3 - p_1)^2 = -2\mathbf{p}^2(1 - \cos\theta)$$

$$\mathcal{M} = g^2 \left[\frac{1}{-2\mathbf{p}^2(1 - \cos\theta)} + \frac{1}{-2\mathbf{p}^2(1 + \cos\theta)} \right]$$

$$\mathcal{M} = \frac{g^2}{-2\mathbf{p}^2} \left[\frac{1}{(1 - \cos\theta)} + \frac{1}{(1 + \cos\theta)} \right]$$

$$\mathcal{M} = \frac{g^2}{-2\mathbf{p}^2} \left[\frac{(1 + \cos\theta) + (1 - \cos\theta)}{(1 - \cos\theta)(1 + \cos\theta)} \right]$$

$$\mathcal{M} = \frac{g^2}{-2\mathbf{p}^2} \left[\frac{2}{1 - \cos^2\theta} \right]$$

$$\boxed{\mathcal{M} = \frac{-g^2}{\mathbf{p}^2 \sin^2\theta}}$$

Differential cross section here

$$\mathcal{M} = \frac{-g^2}{\mathbf{p}^2 \sin^2 \theta}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|M|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

$$|\mathbf{p}_f| = |\mathbf{p}_i| = |\mathbf{p}|$$

$$E_1 = E_2 = E$$

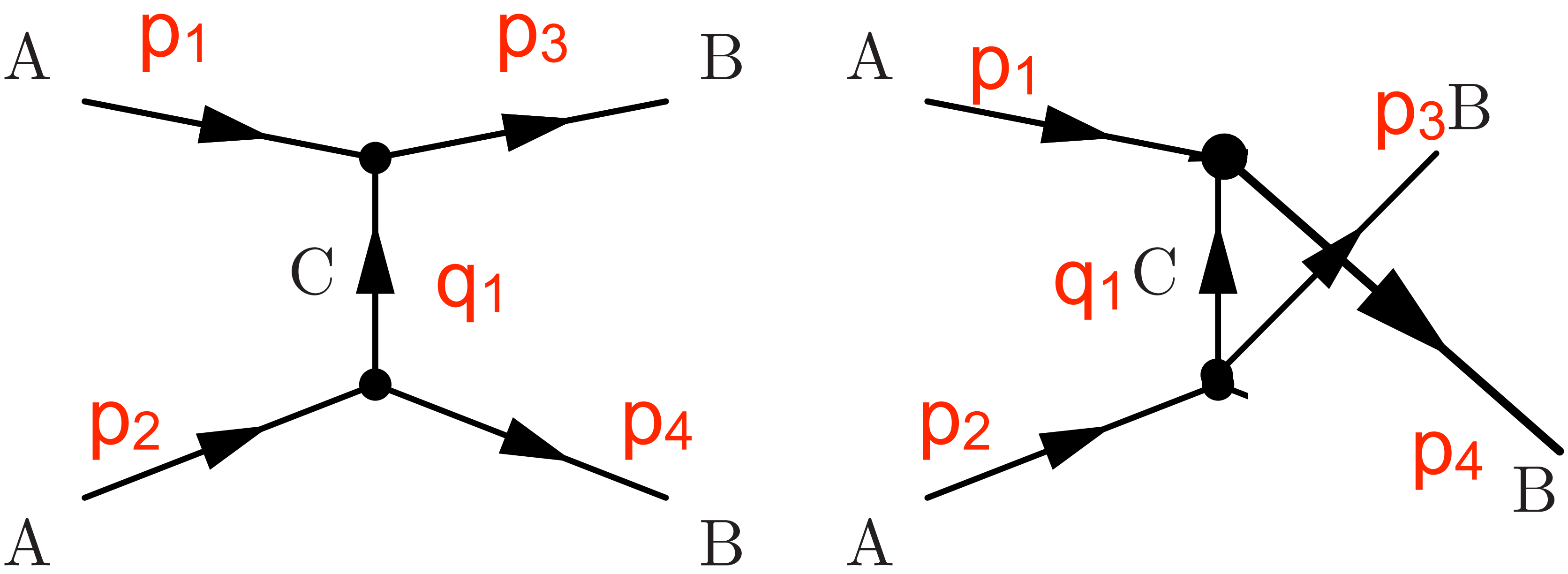
$$|\mathcal{M}|^2 = \frac{g^4}{\mathbf{p}^4 \sin^4 \theta}$$

Infinite cross section as θ→0

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{g^4}{(2E)^2 \mathbf{p}^4 \sin^4 \theta}}$$

We are very close except for one final thing

Here we have two instances of particle B in the final state. For every s identical particles, we add factor of 1/(s!) to account for this, or else we have over-counted the phase space



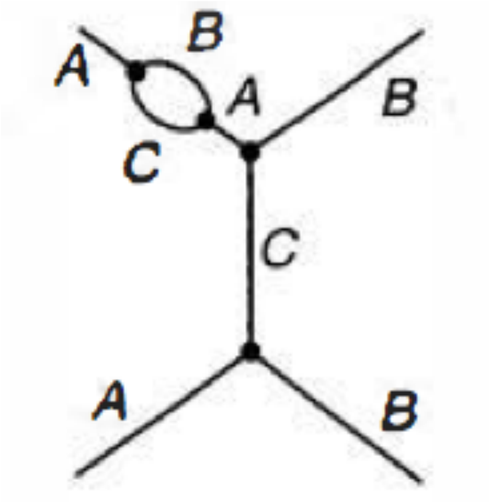
The final differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{2!}\frac{1}{64\pi^2}\frac{g^4}{(2E)^2 2 * \mathbf{p}^4 \sin^4\theta}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{1024\pi^2}\frac{g^4}{E^2 \mathbf{p}^4 \sin^4\theta}}$$

What about beyond leading order

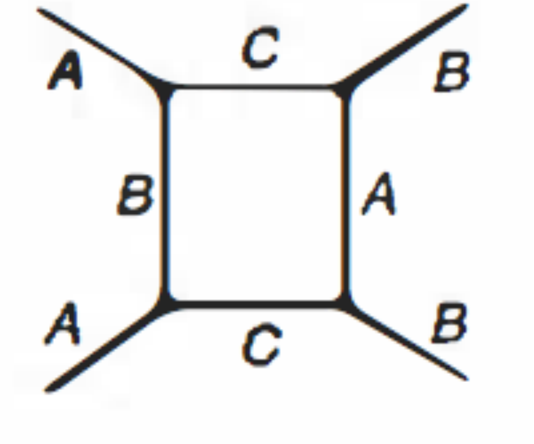
(From Griffiths)



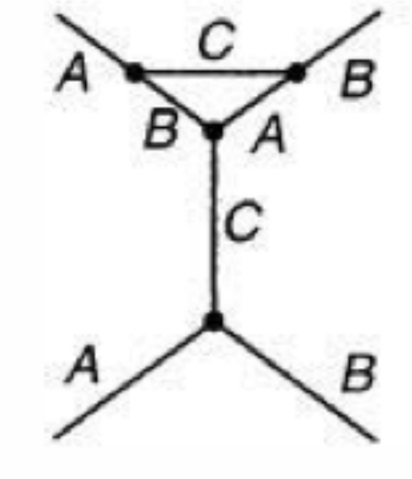
“Self-energy” diagrams

There are many of these. And even more at higher order

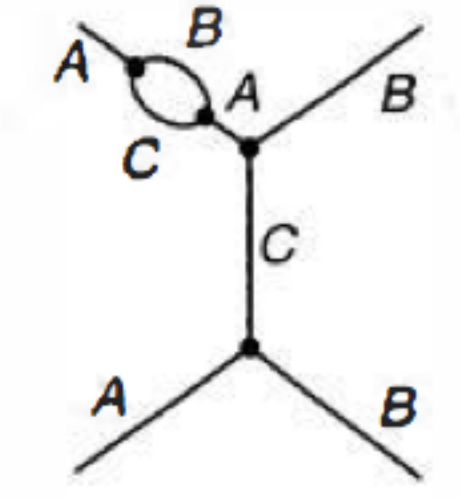
Box diagram



Vertex correction diagrams

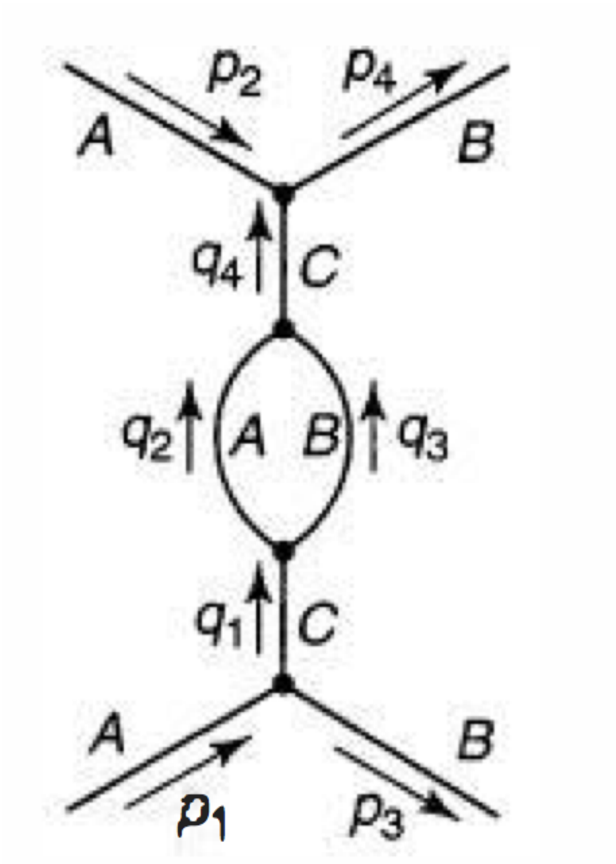


The good thing...



Each vertex carries a factor of “g” in the matrix element, so g^2 for physical quantities. So diagrams with extra vertices should be sub-dominant corrections, or so we hope

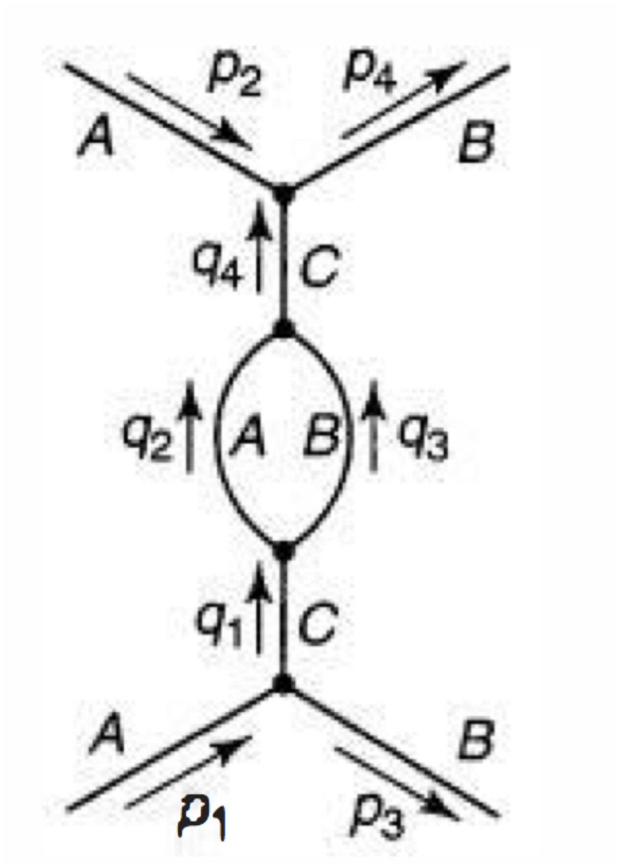
Griffiths' suggested diagram to calculate



1. Label your p's and q's
2. Vertex factors
3. Propagators
4. Momentum and Energy conservation
5. Internal momentum integration
6. Cancel delta function and add extra 'i'

Let’s try this one

$$(-ig)^4$$



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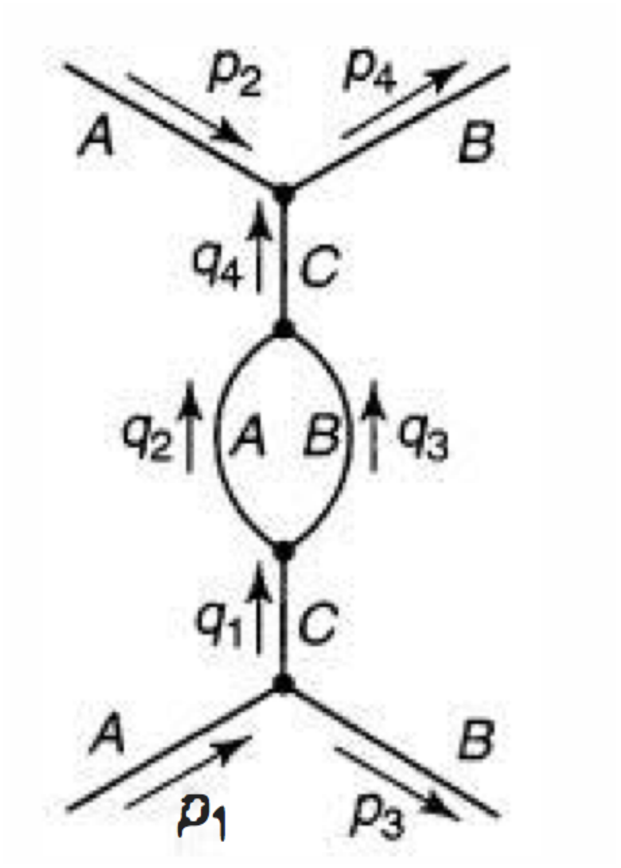
Let’s try this one

$$(-ig)^4 \quad \frac{i}{q_4^2 - m_C^2} \frac{i}{q_2^2 - m_A^2} \frac{i}{q_3^2 - m_B^2} \frac{i}{q_1^2 - m_C^2}$$



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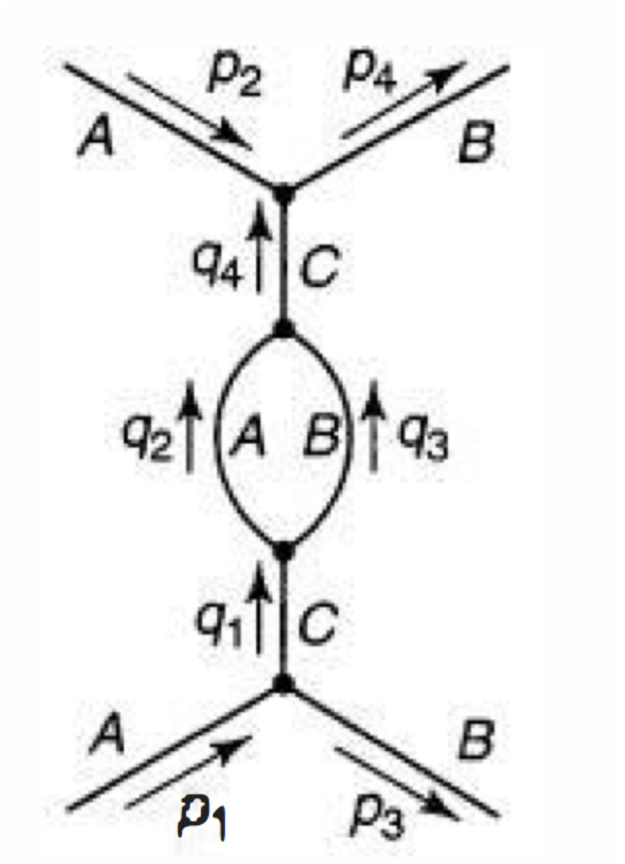
$$(2\pi)^4\delta^4(p_2 + q_4 - p_4)(2\pi)^4\delta^4(q_2 + q_3 - q_4)(2\pi)^4\delta^4(q_1 - q_2 - q_3)(2\pi)^4\delta^4(p_1 - q_1 - p_3)$$



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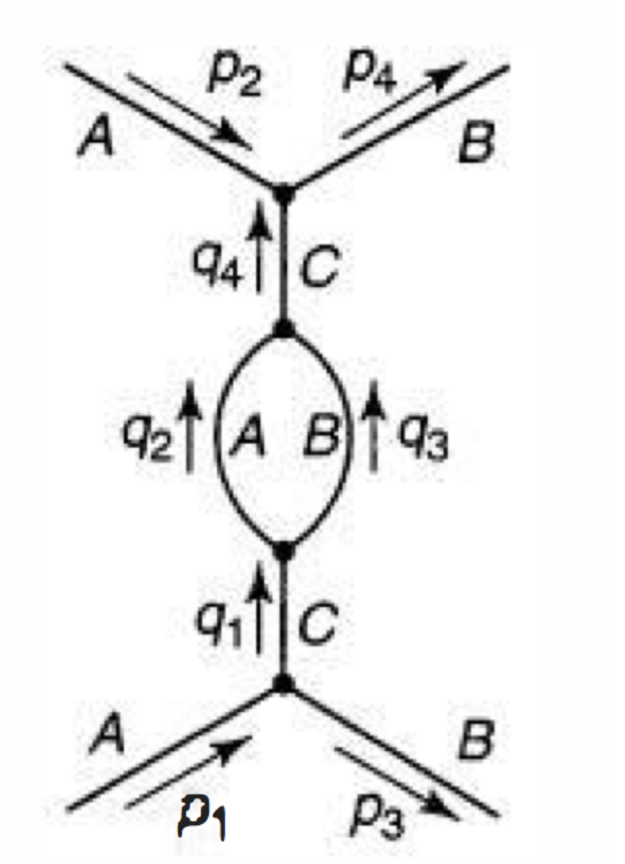
Let's try this one

$$\int (-ig)^4 \quad \frac{i}{q_4^2 - m_C^2} \frac{i}{q_2^2 - m_A^2} \frac{i}{q_3^2 - m_B^2} \frac{i}{q_1^2 - m_C^2}$$

$$(2\pi)^4\delta^4(p_2 + q_4 - p_4)(2\pi)^4\delta^4(q_2 + q_3 - q_4)(2\pi)^4\delta^4(q_1 - q_2 - q_3)(2\pi)^4\delta^4(p_1 - q_1 - p_3)$$

$$\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\frac{d^4q_4}{(2\pi)^4}$$



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Let’s combine / cancel terms

$$\int (-ig)^4 \quad \frac{i}{q_4^2 - m_C^2}\frac{i}{q_2^2 - m_A^2}\frac{i}{q_3^2 - m_B^2}\frac{i}{q_1^2 - m_C^2}$$

$$(2\pi)^4\delta^4(p_2 + q_4 - p_4)(2\pi)^4\delta^4(q_2 + q_3 - q_4)(2\pi)^4\delta^4(q_1 - q_2 - q_3)(2\pi)^4\delta^4(p_1 - q_1 - p_3)$$

$$\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\frac{d^4q_4}{(2\pi)^4}$$

$$\boxed{\begin{gathered} g^4\int \frac{1}{q_4^2 - m_C^2}\frac{1}{q_2^2 - m_A^2}\frac{1}{q_3^2 - m_B^2}\frac{1}{q_1^2 - m_C^2} \\ \delta^4(p_2 + q_4 - p_4)\delta^4(q_2 + q_3 - q_4)\delta^4(q_1 - q_2 - q_3)\delta^4(p_1 - q_1 - p_3)d^4q_1d^4q_2d^4q_3d^4q_4 \end{gathered}}$$

Now let’s use those delta functions

Delta functions to the rescue

$$g^4 \int \frac{1}{q_4^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{q_1^2 - m_C^2}$$

$$\delta^4(p_2 + q_4 - p_4)\delta^4(q_2 + q_3 - q_4)\delta^4(q_1 - q_2 - q_3)\delta^4(p_1 - q_1 - p_3)d^4q_1 d^4q_2 d^4q_3 d^4q_4$$

First eliminate q_1

$$g^4 \int \frac{1}{q_4^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2}$$

$$\delta^4(p_2 + q_4 - p_4)\delta^4(q_2 + q_3 - q_4)\delta^4(p_1 - p_3 - q_2 - q_3)d^4q_2 d^4q_3 d^4q_4$$

Now q_4

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2}$$

$$\delta^4(q_2 + q_3 + p_2 - p_4)\delta^4(p_1 - p_3 - q_2 - q_3)d^4q_2 d^4q_3$$

Delta functions to the rescue

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2}$$

$$\delta^4(q_2 + q_3 + p_2 - p_4)\delta^4(p_1 - p_3 - q_2 - q_3)d^4q_2 d^4q_3$$

Now eliminate q_2

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2}$$

$$\delta^4(p_1 - p_3 - q_3 + q_3 + p_2 - p_4)d^4q_3$$

Rearrange

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2} \delta^4(p_1 + p_2 - p_3 - p_4)d^4q_3$$

A bit more rearranging

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2} \delta^4(p_1 + p_2 - p_3 - p_4) d^4 q_3$$

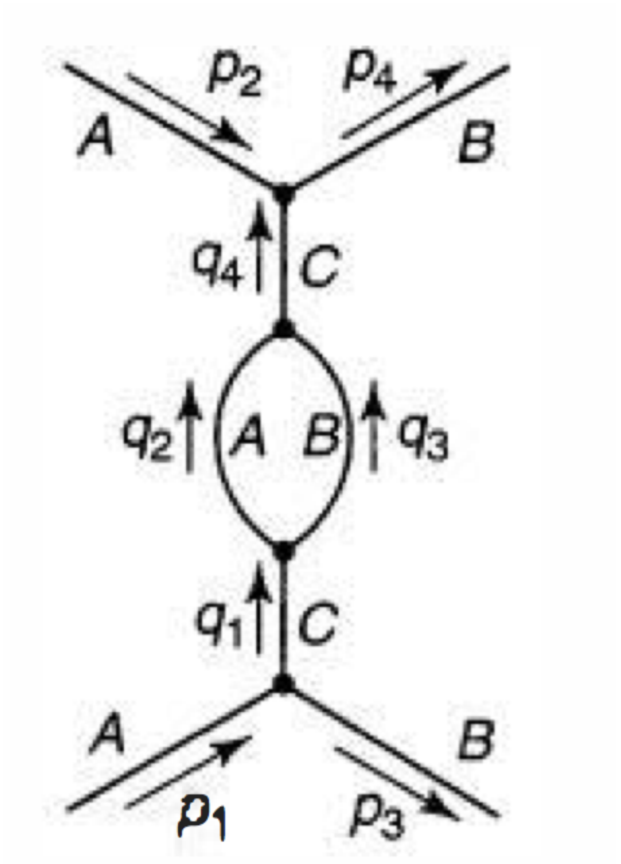
Delta function implies (p_4-p_2) = (p_1-p_3)

$$g^4 \int \frac{1}{(p_1 - p_3)^2 - m_C^2} \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2} \delta^4(p_1 + p_2 - p_3 - p_4) d^4 q_3$$

$$g^4 \int \left(\frac{1}{(p_1 - p_3)^2 - m_C^2} \right)^2 \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \delta^4(p_1 + p_2 - p_3 - p_4) d^4 q_3$$

We just finished step 5

$$g^4 \int \left(\frac{1}{(p_1 - p_3)^2 - m_C^2} \right)^2 \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \delta^4(p_1 + p_2 - p_3 - p_4) d^4 q_3$$



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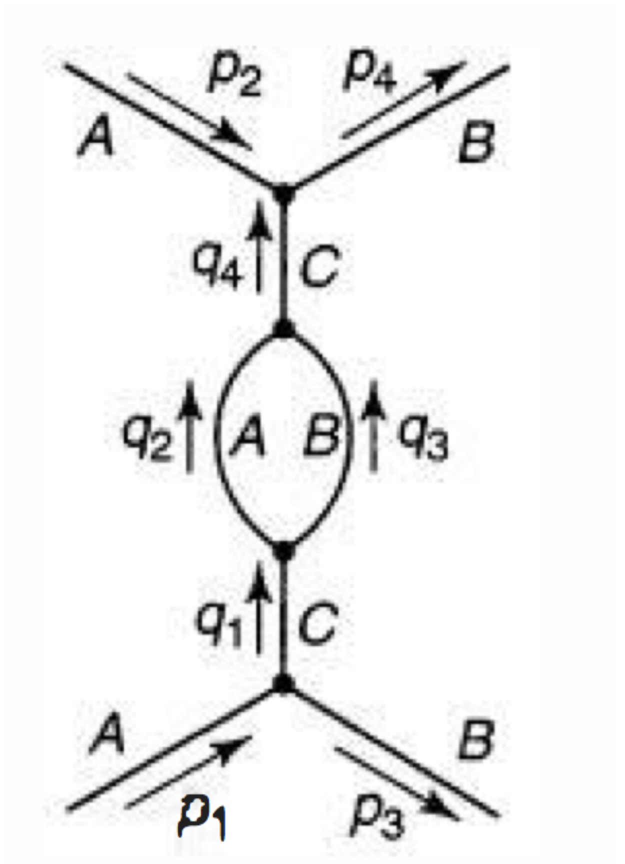
Last step

$$i\left(\frac{g}{2\pi}\right)^4\left(\frac{1}{(p_1-p_3)^2-m_C^2}\right)^2\int\frac{1}{((p_1-p_3-q)^2-m_A^2)\left(q^2-m_B^2\right)}d^4q$$

(2pi)$^{-4}$ from canceling delta



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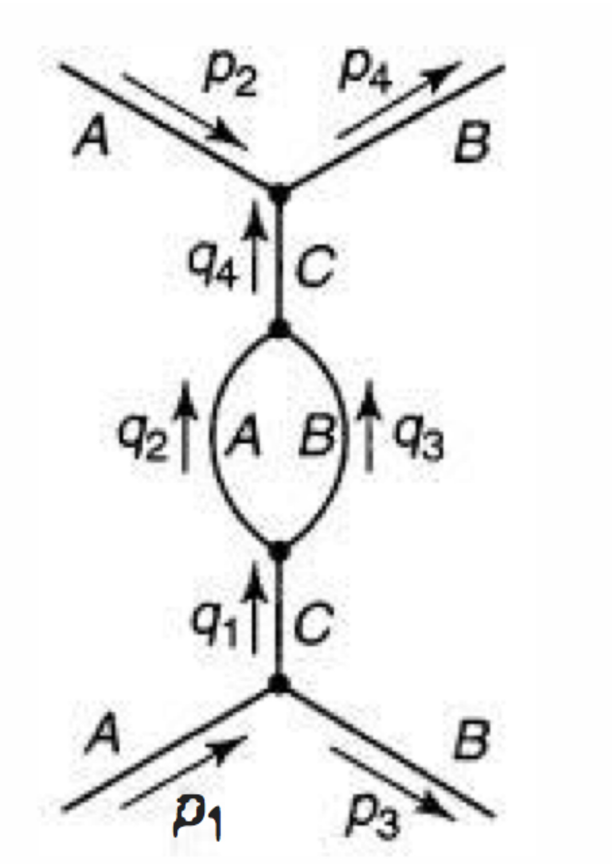
Phew

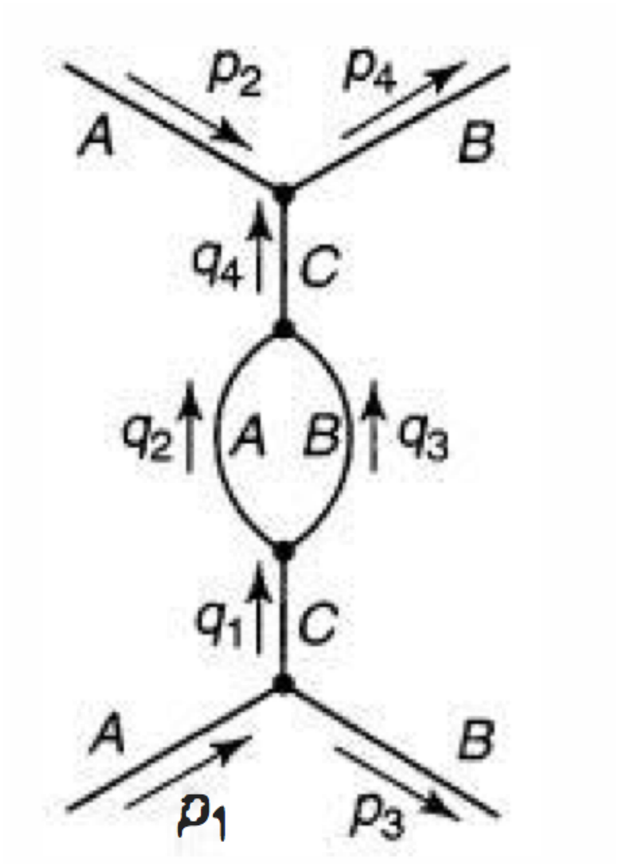
$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \left(\frac{1}{(p_1 - p_3)^2 - m_C^2}\right)^2 \int \frac{1}{((p_1 - p_3 - q)^2 - m_A^2)\,(q^2 - m_B^2)} d^4q$$



Can try and evaluate that ugly integral, but we find that it diverges at large momentum for internal q!

How to deal with infinities?!?!

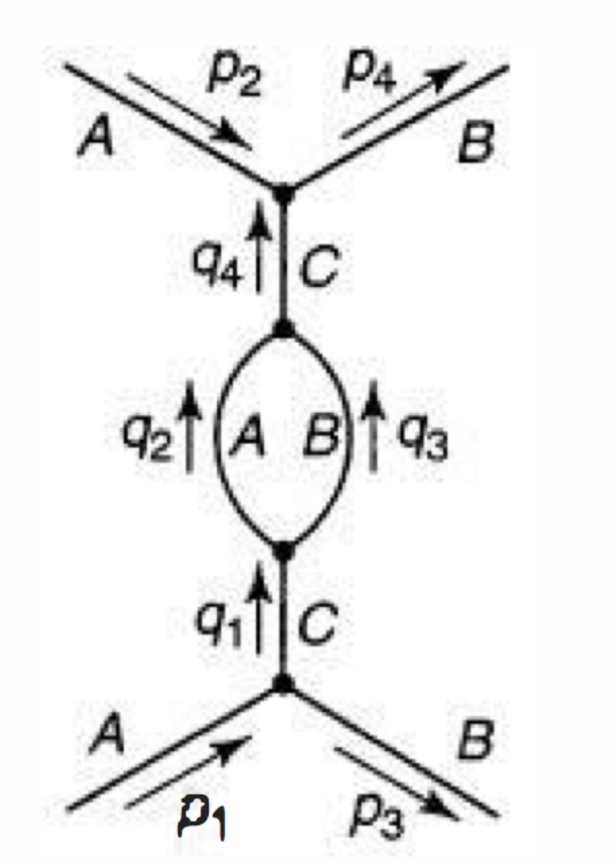
$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \left(\frac{1}{(p_1 - p_3)^2 - m_C^2}\right)^2 \int \frac{1}{((p_1 - p_3 - q)^2 - m_A^2)\,(q^2 - m_B^2)} d^4q$$



More thorough investigation finds that the infinities are really just affecting the masses of objects and the coupling constants - but these are measured quantities anyway, so we can “use” the measured values. Such theories are **renormalizable**

How to deal with infinities?!?!

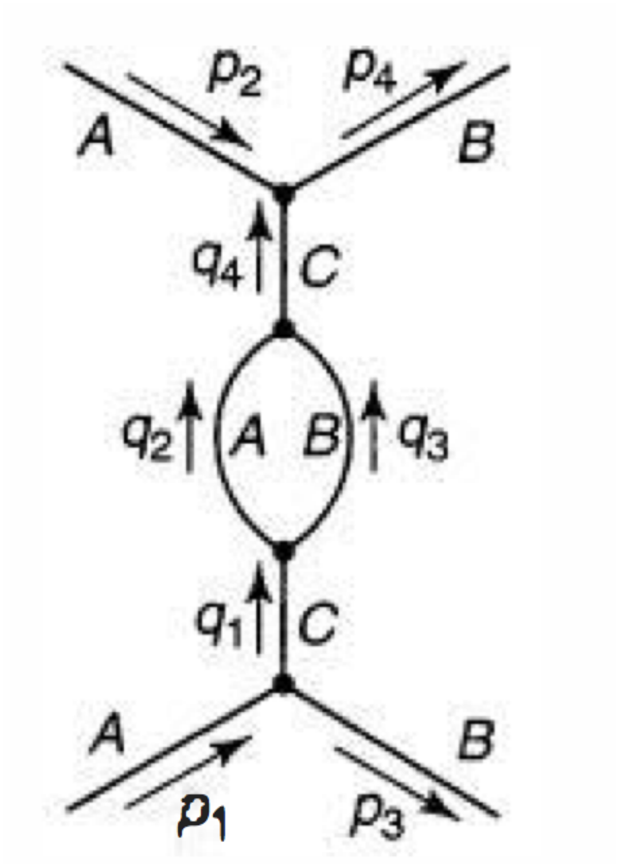
$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \left(\frac{1}{(p_1 - p_3)^2 - m_C^2}\right)^2 \int \frac{1}{((p_1 - p_3 - q)^2 - m_A^2)\,(q^2 - m_B^2)} d^4q$$



We can’t assume that physics inside those loops can have infinite momentum - there must be a “cutoff” at which new physics appears (of course, this is just a toy theory, but the same thing happens in real theories!)

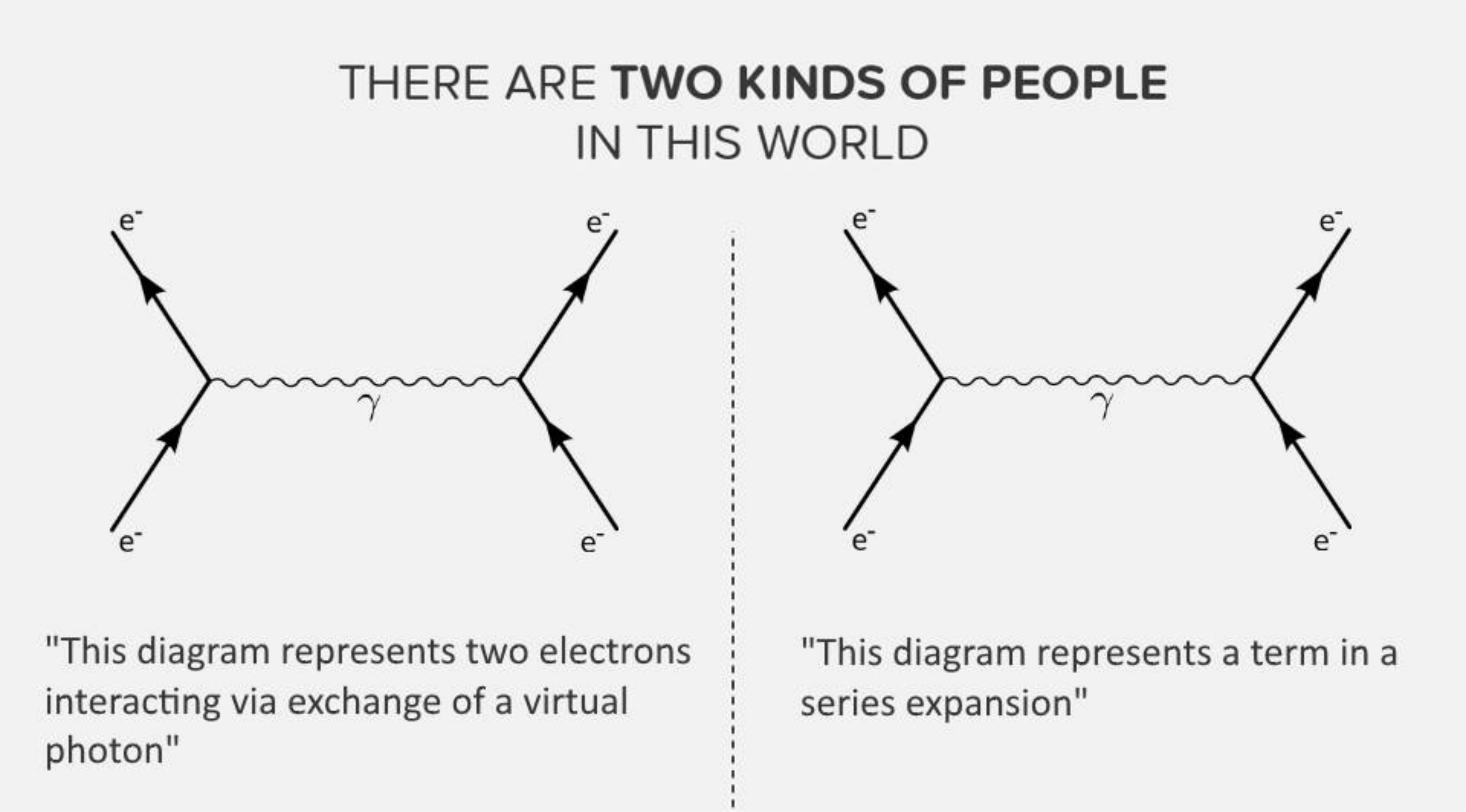
Beyond the infinities

$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \left(\frac{1}{(p_1 - p_3)^2 - m_C^2}\right)^2 \int \frac{1}{\left((p_1 - p_3 - q)^2 - m_A^2\right)\left(q^2 - m_B^2\right)} d^4q$$



There **are** modifications to the matrix element that are not infinite that do provide corrections to m and g. Implies that there is energy dependence to masses and couplings! (See running of the couplings from earlier in the semester)

Well said



THERE ARE **TWO KINDS OF PEOPLE** IN THIS WORLD

"This diagram represents two electrons interacting via exchange of a virtual photon"

"This diagram represents a term in a series expansion"

Some homework for you

6.8,6.12,6.14 (focus on non-relativistic limit only, and use results of problem 6.9 to help) + 6.15

Also: if a 1 GeV muon neutrino is fired at a 1 m-thick block of iron-56 with density 7.87x10^3 kg/m^3 and the average neutrino-nucleon σ=8x10^{-39} cm^2, what is the probability that the neutrino interacts in the block?

Nothing due related to your final assignment, but you should be starting to think about your talk (and you should be asking me for help if we need to understand something in the analysis/paper/result)