Several differences between W$^\pm$ and Z bosons and QED/gluons, most noticeably the fact that W and Z bosons are quite massive

m$_W$ = 80.4 GeV
m$_Z$ = 91.2 GeV

Because they are massive, they can have three spin polarizations (vector boson!), not just two. So we impose Lorentz condition but not the Coulomb gauge

$$\epsilon^\mu p_\mu = 0$$ Lorentz condition

$$\epsilon^0 = 0 \rightarrow \epsilon \cdot \mathbf{p} = 0$$ Coulomb gauge

Weak vector boson propagator:

$$\frac{-i(g_{\mu\nu} - q_\mu q_\nu / M^2)}{q^2 - M^2 + iM\Gamma}$$
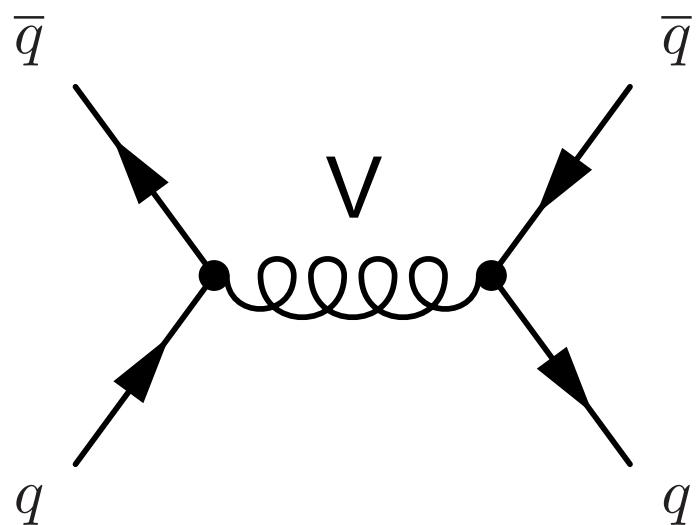
Where last term in denominator is the decay width of the new object For $q^2 << M^2$ and large M we can often reduce this to:

$$\frac{ig_{\mu\nu}}{M^2}$$

$$\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$
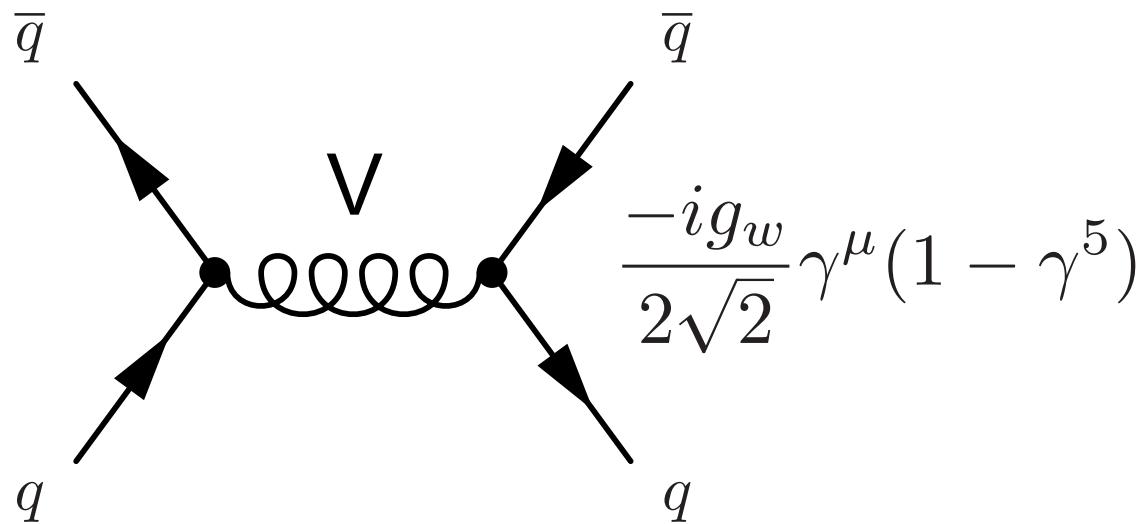
What are the two types of terms that we're adding here? Anyone remember?
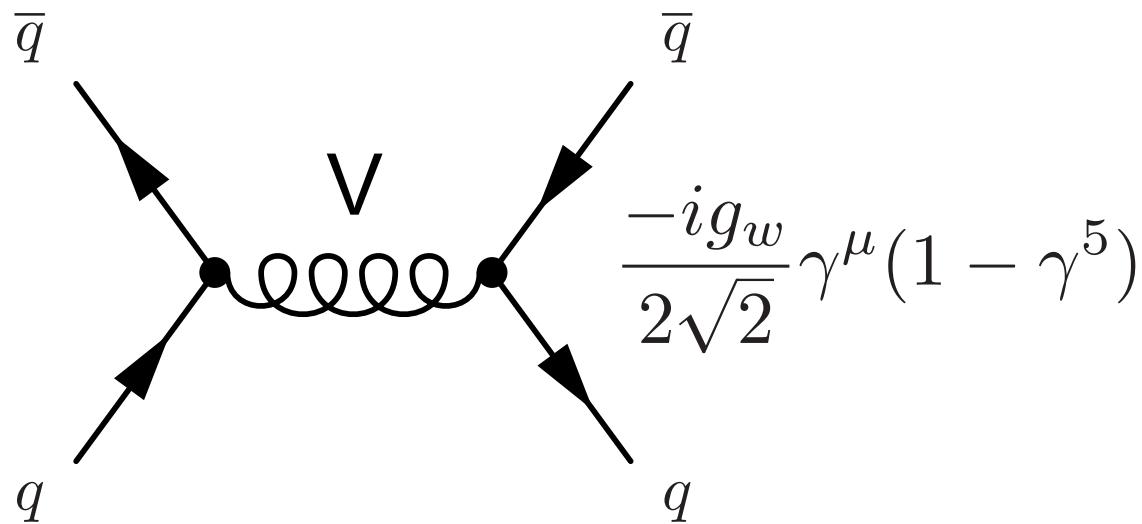
We are adding a vector to an axial vector, which violates parity! It violates it... maximally, as we'll see. (V-A) coupling

$$(2\pi)^4 \delta^4(p_2 + q - p_4)$$

$$i$$

$$\frac{1}{(2\pi)^4} d^4 q$$

ν_μ p2 u $\frac{-ig_W}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)$ p4 μ^- $\overline{u}$

W q $\frac{ig_{\mu\nu}}{M^2}$

p1 $\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$ p3 $\overline{u}$

e^- u ν_e
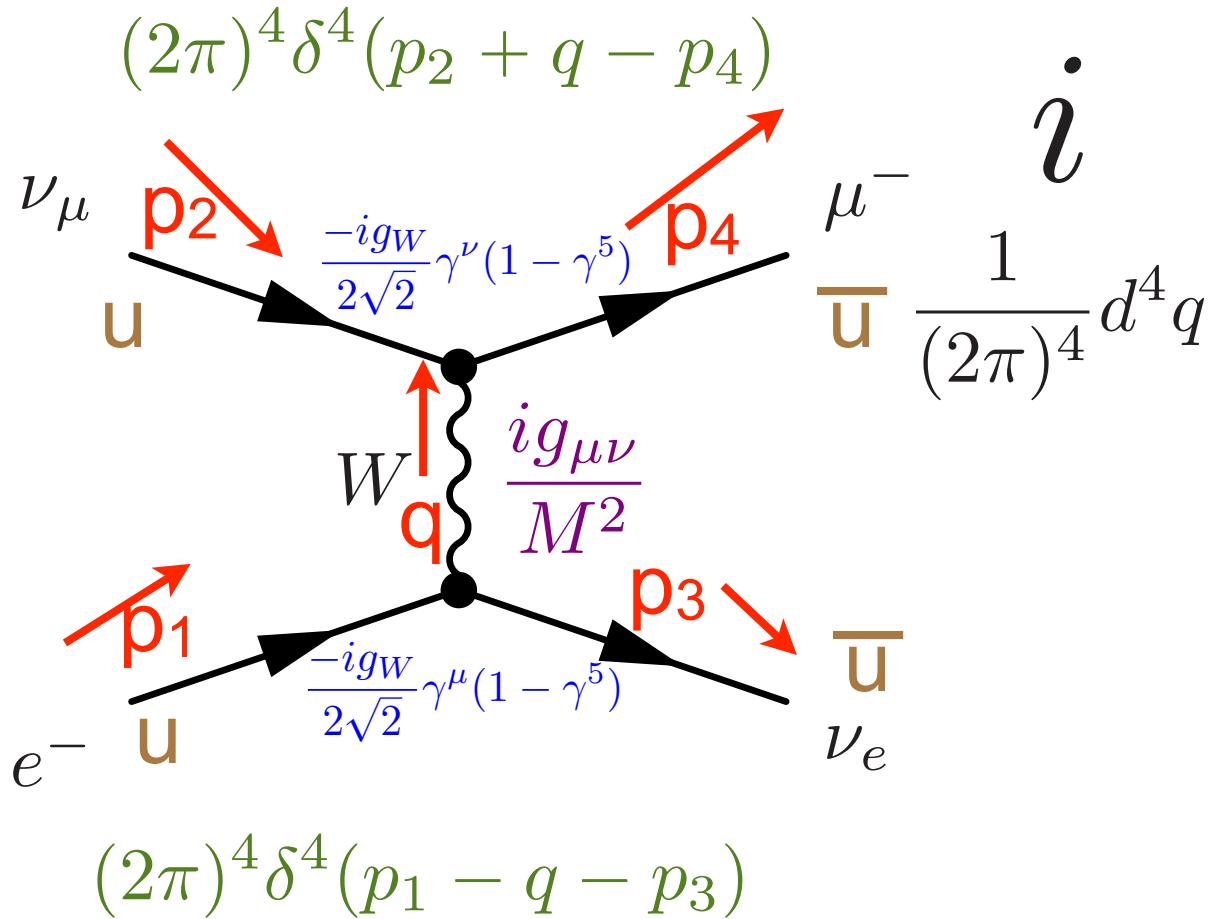
$$(2\pi)^4 \delta^4(p_1 - q - p_3)$$

You might get this from a muon neutrino beam hitting a tank of water

Assume small momentum transfer

$$\mathcal{M} = \int i[\overline{u}(3)][-i\frac{g_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)][u(1)]\frac{ig_{\mu\nu}}{M^2}[\overline{u}(4)][-i\frac{g_W}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)][u(2)]$$

$$(2\pi)^4\delta^4(p_1 - q - p_3)(2\pi)^4\delta^4(p_2 + q - p_4)\frac{d^4q}{(2\pi)^4}$$

$$\mathcal{M} = \int i[\overline{u}(3)][-i\frac{g_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)][u(1)]\frac{ig_{\mu\nu}}{M^2}[\overline{u}(4)][-i\frac{g_W}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)][u(2)]$$

$$(2\pi)^4\delta^4(p_1-q-p_3)(2\pi)^4\delta^4(p_2+q-p_4)\frac{d^4q}{(2\pi)^4}$$

$$\mathcal{M} = [\overline{u}(3)][\frac{g_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)][u(1)]\frac{1}{M_W^2}[\overline{u}(4)][\frac{g_W}{2\sqrt{2}}\gamma_\mu(1-\gamma^5)][u(2)]$$

$$\mathcal{M} = \frac{g_W^2}{8M_W^2}[\overline{u}(3)\gamma^\mu(1-\gamma^5)u(1)][\overline{u}(4)\gamma_\mu(1-\gamma^5)u(2)]$$

Apply Casimir's trick:

$$\sum_{spins}[\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^* = \text{Trace}[\Gamma_1(\not{p}_b+m_b)\overline{\Gamma}_2(\not{p}_a+m_a)]$$

$$\mathcal{M} = \frac{g_W^2}{8M_W^2}[\overline{u}(3)\gamma^\mu(1-\gamma^5)u(1)][\overline{u}(4)\gamma_\mu(1-\gamma^5)u(2)]$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{64M_W^4}[\overline{u}(3)\gamma^\mu(1-\gamma^5)u(1)][\overline{u}(4)\gamma_\mu(1-\gamma^5)u(2)][\overline{u}(3)\gamma^\nu(1-\gamma^5)u(1)]^*[\overline{u}(4)\gamma_\nu(1-\gamma^5)u(2)]^*$$

Apply Casimir's trick:

$$\sum_{spins}[\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^* = \text{Trace}[\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2(\not{p}_a + m_a)]$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{64M_W^4}Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_1)\overline{\gamma^\nu(1-\gamma^5)}(\not{p}_3 + m_3)]Tr[\gamma_\mu(1-\gamma^5)(\not{p}_2 + m_2)\overline{\gamma_\nu(1-\gamma^5)}(\not{p}_4 + m_4)]$$

Neutrino masses ~0 (m$_2$ = m$_3$ = 0)

$$|\mathcal{M}|^2 = \frac{g_W^4}{64M_W^4}Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_e)\overline{\gamma^\nu(1-\gamma^5)}(\not{p}_3)]Tr[\gamma_\mu(1-\gamma^5)(\not{p}_2)\overline{\gamma_\nu(1-\gamma^5)}(\not{p}_4 + m_\mu)]$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{64 M_W^4} Tr[\gamma^\mu (1-\gamma^5)(\not{p}_1 + m_e)\overline{\gamma^\nu (1-\gamma^5)}(\not{p}_3)] Tr[\gamma_\mu (1-\gamma^5)(\not{p}_2)\overline{\gamma_\nu (1-\gamma^5)}(\not{p}_4 + m_\mu)]$$

$$\overline{\gamma}^\mu = \gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$$

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$$\gamma^{5\dagger} = -i\gamma^{3\dagger} \gamma^{2\dagger} \gamma^{1\dagger} \gamma^{0\dagger}$$

$$\gamma^{5\dagger} = -i(-\gamma^3)(-\gamma^2)(-\gamma^1)(\gamma^0)$$

$$\gamma^{5\dagger} = i\gamma^3 \gamma^2 \gamma^1 \gamma^0$$

$$\gamma^{5\dagger} = -i\gamma^2 \gamma^3 \gamma^1 \gamma^0 = i\gamma^2 \gamma^1 \gamma^3 \gamma^0$$

$$\gamma^{5\dagger} = -i\gamma^2 \gamma^1 \gamma^0 \gamma^3 = i\gamma^1 \gamma^2 \gamma^0 \gamma^3$$

$$\gamma^{5\dagger} = -i\gamma^2 \gamma^1 \gamma^0 \gamma^3 = i\gamma^2 \gamma^0 \gamma^1 \gamma^3$$

$$\gamma^{5\dagger} = -i\gamma^0 \gamma^2 \gamma^1 \gamma^3 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^5$$

$$\overline{\gamma_\mu \gamma^5} = \gamma^0 (\gamma_\mu \gamma^5)^\dagger \gamma^0$$

$$\overline{\gamma_\mu \gamma^5} = \gamma^0 \gamma^{5\dagger} \gamma_\mu^\dagger \gamma^0 = \gamma^0 \gamma^5 \gamma_\mu^\dagger \gamma^0$$

$$\overline{\gamma_\mu \gamma^5} = -\gamma^5 \gamma^0 \gamma_\mu^\dagger \gamma^0 = -\gamma^5 \gamma_\mu = \gamma_\mu \gamma^5$$

Inverse muon decay

$$|\mathcal{M}|^2 = \frac{g_W^4}{64 M_W^4} Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_e)\overline{\gamma^\nu(1-\gamma^5)}(\not{p}_3)]Tr[\gamma_\mu(1-\gamma^5)(\not{p}_2)\overline{\gamma_\nu(1-\gamma^5)}(\not{p}_4 + m_\mu)]$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{64 M_W^4} Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_e)(\gamma^\nu - \gamma^\nu\gamma^5)(\not{p}_3)]Tr[\gamma_\mu(1-\gamma^5)(\not{p}_2)(\gamma_\nu - \gamma_\nu\gamma^5)(\not{p}_4 + m_\mu)]$$

Focus on first trace

$$Tr[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3 - \gamma^\mu\not{p}_1\gamma^\nu\gamma^5\not{p}_3 + \gamma^\mu m_e\gamma^\nu\not{p}_3 - \gamma^\mu m_e\gamma^\nu\gamma^5\not{p}_3 - \gamma^\mu\gamma^5\not{p}_1\gamma^\nu\not{p}_3 +$$
$$\gamma^\mu\gamma^5\not{p}_1\gamma^\nu\gamma^5\not{p}_3 - \gamma^\mu\gamma^5 m_e\gamma^\nu\not{p}_3 + \gamma^\mu\gamma^5 m_e\gamma^\nu\gamma^5\not{p}_3]$$

Reminders:

$$Tr(\gamma^\mu\gamma^\nu\gamma^\sigma) = 0$$
$$Tr(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda\gamma^\beta) = 0$$
$$Tr(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda\gamma^\beta\gamma^\alpha\gamma^\delta) = 0$$
$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$
$$Tr(\gamma^5) = Tr(\gamma^5\gamma^\mu\gamma^\nu) = 0$$

Any odd number

Inverse muon decay

$$Tr[\gamma^{\mu}\not{p}_1\gamma^{\nu}\not{p}_3 - \gamma^{\mu}\not{p}_1\gamma^{\nu}\gamma^5\not{p}_3 - \gamma^{\mu}\gamma^5\not{p}_1\gamma^{\nu}\not{p}_3 + \gamma^{\mu}\gamma^5\not{p}_1\gamma^{\nu}\gamma^5\not{p}_3]$$

Use: $$\gamma^{\mu}\gamma^5 = -\gamma^5\gamma^{\mu} \qquad \gamma^5\gamma^5 = 1$$

$$Tr[\gamma^{\mu}\not{p}_1\gamma^{\nu}\not{p}_3 - \gamma^{\mu}\not{p}_1\gamma^{\nu}\gamma^5\not{p}_3 - \gamma^{\mu}\gamma^5\not{p}_1\gamma^{\nu}\not{p}_3 + \gamma^{\mu}\not{p}_1\gamma^{\nu}\not{p}_3]$$

$$2Tr[\gamma^{\mu}\not{p}_1\gamma^{\nu}\not{p}_3 + \gamma^5\gamma^{\mu}\not{p}_1\gamma^{\nu}\not{p}_3]$$

-1 for even permutation of 0123
+1 for odd permutation of 0123
0 for repeated indices

Use: $$Tr(\gamma^5\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\sigma}) = 4i\epsilon^{\mu\nu\lambda\sigma}$$

$$2Tr[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 + \gamma^5 \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3]$$

Commute
for nicer
ordering

$$2Tr[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 - \gamma^5 \gamma^\mu \gamma^\nu \not{p}_1 \not{p}_3]$$

Then we get:

$$8(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} p_1 \cdot p_3 - i\epsilon^{\mu\nu\lambda\sigma} p_{1\lambda} p_{3\sigma})$$

And
second trace must be:

$$8(p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} - g_{\mu\nu} p_2 \cdot p_4 - i\epsilon_{\mu\nu\alpha\beta} p_2^\alpha p_4^\beta)$$

So total matrix element (almost there)

Finally:

$$|\mathcal{M}|^2 = \frac{g_W^4}{64 M_W^4} 8(p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} - g_{\mu\nu} p_2 \cdot p_4 - i\epsilon_{\mu\nu\alpha\beta} p_2^\alpha p_4^\beta) \times$$

$$8(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} p_1 \cdot p_3 - i\epsilon^{\mu\nu\lambda\sigma} p_{1\lambda} p_{3\sigma})$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4} (p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} - g_{\mu\nu} p_2 \cdot p_4 - i\epsilon_{\mu\nu\alpha\beta} p_2^\alpha p_4^\beta) \times$$

$$(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} p_1 \cdot p_3 - i\epsilon^{\mu\nu\lambda\sigma} p_{1\lambda} p_{3\sigma})$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}\left(p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu}p_2 \cdot p_4 - i\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta\right)\times$$

$$\left(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}p_1 \cdot p_3 - i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\right)$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}\left((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_2 \cdot p_4)(p_1 \cdot p_3) + (p_2 \cdot p_3)(p_1 \cdot p_4) + (p_1 \cdot p_2)(p_3 \cdot p_4) -\right.$$

$$(p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4) + 4(p_1 \cdot p_3)(p_2 \cdot p_4)$$

$$-i\epsilon_{\mu\nu\lambda\sigma}p_2^\lambda p_4^\sigma \left(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}p_1 \cdot p_3\right)$$

$$-i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\left(p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu}p_2 \cdot p_4\right)$$

$$\left.-\epsilon_{\mu\nu\lambda\sigma}p_2^\lambda p_4^\sigma \epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\right)$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}\left((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_2 \cdot p_4)(p_1 \cdot p_3) + (p_2 \cdot p_3)(p_1 \cdot p_4) + (p_1 \cdot p_2)(p_3 \cdot p_4) -\right.$$

$$(p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4) + 4(p_1 \cdot p_3)(p_2 \cdot p_4)$$

$$-i\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta \left(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}p_1 \cdot p_3\right)$$

$$-i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\left(p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu}p_2 \cdot p_4\right)$$

$$\left.-\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta \epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\right)$$

Simplifying the mess

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}\big(2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3)$$

$$-i\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta \left(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}p_1 \cdot p_3\right)$$

$$-i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\left(p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu}p_2 \cdot p_4\right)$$

$$-\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_4^\beta \epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\big)$$

First terms in () the same, but epsilons have opposite sign! These all cancel

Look at terms like:

$$i\epsilon_{01\alpha\beta}p_2^\alpha p_4^\beta \left(p_1^0 p_3^1 + p_1^1 p_3^0\right) +$$

$$i\epsilon_{10\alpha\beta}p_2^\alpha p_4^\beta \left(p_1^1 p_3^0 + p_1^0 p_3^1\right) + ...$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4} \left(2(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) \right.$$

$$+ i\epsilon_{\mu\nu\alpha\beta} p_2^\alpha p_4^\beta \left(g^{\mu\nu} p_1 \cdot p_3 \right)$$

$$+ i\epsilon^{\mu\nu\lambda\sigma} p_{1\lambda} p_{3\sigma} \left(g_{\mu\nu} p_2 \cdot p_4 \right)$$

$$\left. - \epsilon_{\mu\nu\alpha\beta} p_2^\alpha p_4^\beta \epsilon^{\mu\nu\lambda\sigma} p_{1\lambda} p_{3\sigma} \right)$$

Terms like

$$+ i\epsilon_{\mu\nu\alpha\beta} p_2^\alpha p_4^\beta \left(g^{\mu\nu} p_1 \cdot p_3 \right)$$

Epsilon only non-zero when µ!=ν, but term in () only non-zero when µ==ν

Simplifying the mess

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}(2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3)$$

$$-\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\lambda\sigma}p_2^\alpha p_4^\beta p_{1\lambda}p_{3\sigma})$$

What is? $\quad \epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\lambda\sigma}$

α must be λ or σ for terms to be non-zero. And it is anti-symmetric in λ and σ

$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\lambda\sigma} = A[\delta_\alpha^\lambda\delta_\beta^\sigma - \delta_\beta^\lambda\delta_\alpha^\sigma]$$

To find A, let's try λ=2, σ=3

$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu23} = A[\delta_\alpha^2\delta_\beta^3 - \delta_\beta^2\delta_\alpha^3]$$

Clear?

$$\epsilon_{\mu\nu23}\epsilon^{\mu\nu23} = A[\delta_2^2\delta_3^3 - \delta_3^2\delta_2^3] = A$$

$$\epsilon_{\mu\nu23}\epsilon^{\mu\nu23} = \epsilon_{0123}\epsilon^{0123} + \epsilon_{1023}\epsilon^{1023} = +1(-1) + -1(+1) = -2 = A$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}(2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3)$$

$$-\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\lambda\sigma}p_2^\alpha p_4^\beta p_{1\lambda}p_{3\sigma})$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}(2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3)$$

$$+2p_2^\alpha p_4^\beta p_{1\lambda}p_{3\sigma}(\delta_\alpha^\lambda\delta_\beta^\sigma - \delta_\beta^\lambda\delta_\alpha^\sigma))$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}(2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) + 2(p_1 \cdot p_2)(p_3 \cdot p_4) - 2(p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$|\mathcal{M}|^2 = \frac{4g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4))$$

$$|\mathcal{M}|^2 = \frac{4g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4))$$

Finally, we want to average over initial spins, but massless neutrino has only one spin configuration, so divide by 2:

$$|\mathcal{M}|^2 = \frac{2g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4))$$

$$|\mathcal{M}|^2 = \frac{2g_W^4}{M_W^4}\left((p_1 \cdot p_2)(p_3 \cdot p_4)\right)$$

$|\mathbf{p}_1|=|\mathbf{p}_2|=p_i$

$|\mathbf{p}_3|=|\mathbf{p}_4|=p_f$

$E_1+E_2 = 2E$

$2E = E_3 + E_4$

$|\mathbf{p}_i| = E$

$|\mathbf{p}_f| = \sqrt{E_4^2 - m_\mu^2} = E_3$

$2E = \sqrt{E_4^2 - m_\mu^2} + E_4$

$2E - E_4 = \sqrt{E_4^2 - m_\mu^2}$

$4E^2 + E_4^2 - 4EE_4 = E_4^2 - m_\mu^2$

$4E^2 - 4EE_4 = -m_\mu^2$

$4EE_4 = 4E^2 + m_\mu^2$

$E_4 = E + m_\mu^2/4E$

$E_3 = E - m_\mu^2/4E$

Assume massless electron. Pick center of mass frame where $E_1 = E_2 = E$
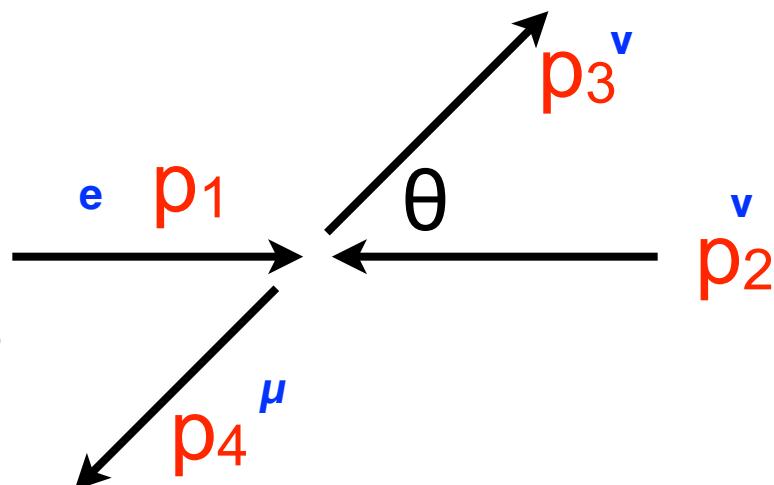
e p_1 θ p_3 **v**

v p_2

p_4 **μ**

$p_1 \cdot p_2 = E^2 + \mathbf{p}_i^2 = 2E^2$

$p_3 \cdot p_4 = E_3E_4 + \mathbf{p}_f^2 = E_3E_4 + E_3^2$

$p_3 \cdot p_4 = (E + m_\mu^2/4E)(E - m_\mu^2/4E) + (E - m_\mu^2/4E)^2$

$$|\mathcal{M}|^2 = \frac{2g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4))$$

$$p_1 \cdot p_2 = E^2 + \mathbf{p}_i^2 = 2E^2$$

$$p_3 \cdot p_4 = E_3 E_4 + \mathbf{p}_f^2 = E_3 E_4 + E_3^2$$

$$p_3 \cdot p_4 = (E + m_\mu^2/4E)(E - m_\mu^2/4E) + (E - m_\mu^2/4E)^2$$

$$p_3 \cdot p_4 = E^2 - m_\mu^4/(16E^2) + E^2 + m_\mu^4/(16E^2) - m_\mu^2/2$$

$$p_3 \cdot p_4 = 2E^2 - m_\mu^2/2$$

$$|\mathcal{M}|^2 = \frac{2g_W^4}{M_W^4}(2E^2)(2E^2 - m_\mu^2/2)$$

$$|\mathcal{M}|^2 = \frac{8E^4 g_W^4}{M_W^4}(1 - \frac{m_\mu^2}{4E^2})$$

$$|\mathcal{M}|^2 = \frac{8E^4 g_W^4}{M_W^4}\left(1 - \frac{m_\mu^2}{4E^2}\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{|M|^2}{(E_1 + E_2)^2}\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{|M|^2}{(2E)^2}\frac{E - m_{\mu^2}/4E}{E}$$

$$\frac{d\sigma}{d\Omega} = \frac{8E^4 g_W^4}{M_W^4}\left(1 - \frac{m_\mu^2}{4E^2}\right)\frac{1}{64\pi^2}\frac{1}{(2E)^2}\frac{E - m_{\mu^2}/4E}{E}$$

Total cross section

$$\frac{d\sigma}{d\Omega} = \frac{8E^4 g_W^4}{M_W^4}(1 - \frac{m_\mu^2}{4E^2})\frac{1}{64\pi^2}\frac{1}{(2E)^2}\frac{E - m_{\mu^2}/4E}{E}$$

$$\frac{d\sigma}{d\Omega} = \frac{E^2 g_W^4}{32\pi^2 M_W^4}(1 - \frac{m_\mu^2}{4E^2})\frac{E - m_{\mu^2}/4E}{E}$$

$$\frac{d\sigma}{d\Omega} = \frac{E^2 g_W^4}{32\pi^2 M_W^4}(1 - \frac{m_\mu^2}{4E^2})(1 - m_{\mu^2}/4E^2)$$

$$\frac{d\sigma}{d\Omega} = \frac{E^2 g_W^4}{32\pi^2 M_W^4}(1 - \frac{m_\mu^2}{4E^2})^2$$

$$\boxed{\sigma = \frac{E^2 g_W^4}{8\pi m_W^4}\left(1 - \frac{m_\mu^2}{4E^2}\right)^2}$$

Any suggestions, other than… PHEW?

$$(2\pi)^4\delta^4(q - p_2 - p_4)$$

$$i$$

$$\frac{1}{(2\pi)^4}d^4q$$

e

p_4

$\overline{u}$

$\frac{ig_{\mu\nu}}{M^2}$

$\frac{-ig_W}{2\sqrt{2}}\gamma^\nu(1 - \gamma^5)$

u

q

μ

ν_e v

p_1

$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1 - \gamma^5)$

W

p_2

Assume small momentum transfer

$\overline{u}$

p_3

ν_μ
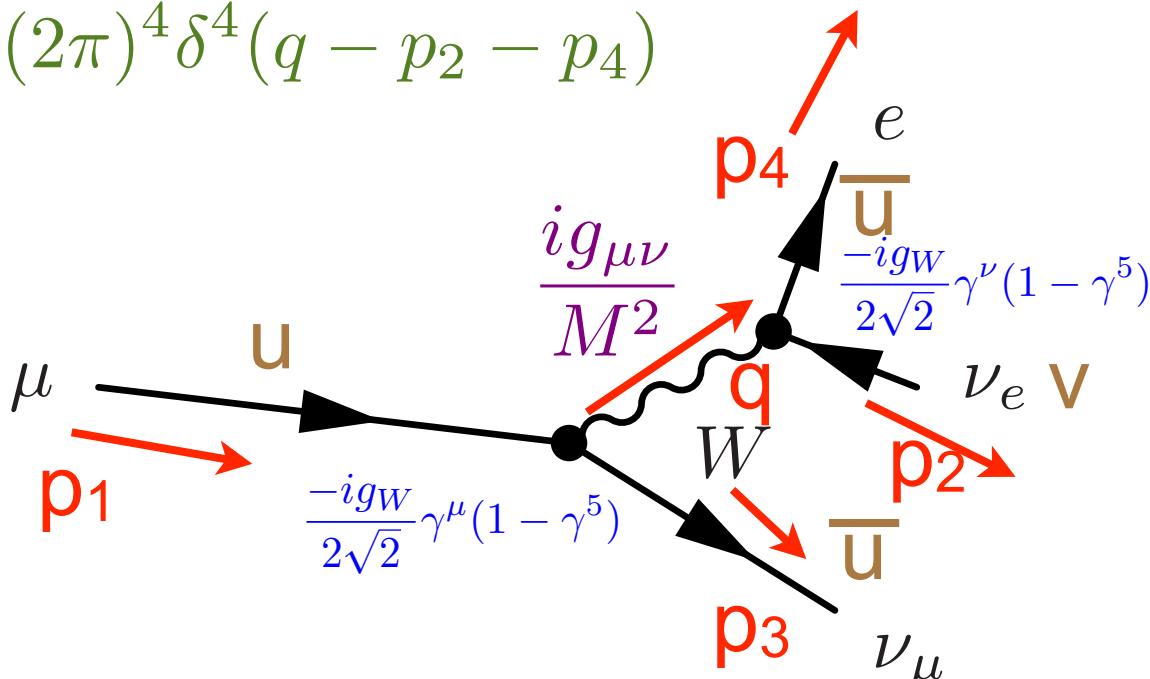
$$(2\pi)^4\delta^4(p_1 - q - p_3)$$

$$\mathcal{M} = \int i[\overline{u}(3)][-i\frac{g_W}{2\sqrt{2}}\gamma^\mu(1 - \gamma^5)][u(1)]\frac{ig_{\mu\nu}}{M^2}[\overline{u}(4)][-i\frac{g_W}{2\sqrt{2}}\gamma^\nu(1 - \gamma^5)][v(2)]$$

$$(2\pi)^4\delta^4(p_1 - q - p_3)(2\pi)^4\delta^4(q - p_2 - p_4)\frac{d^4q}{(2\pi)^4}$$

u(2) became v(2), which changes the sign on the mass term of a neutrino, which is zero anyway

One of the delta functions is also different, but it gets canceled. So at least this part is simple :)

$$|\mathcal{M}|^2 = \frac{2g_W^4}{M_W^4}((p_1 \cdot p_2)(p_3 \cdot p_4))$$

Let's evaluate in muon rest frame, and assume zero electron mass (valid vs muon mass) so:

$$p_1 = (m_\mu, \mathbf{0}) \rightarrow (p_1 \cdot p_2) = m_\mu E_2$$

$$p_1 = p_2 + p_3 + p_4 \rightarrow (p_3 + p_4)^2 = (p_1 - p_2)^2$$

$$(p_1 - p_2)^2 = p_1^2 + p_2^2 - 2(p_1 \cdot p_2) = m_\mu^2 - 2m_\mu E_2$$

$$(p_3 + p_4)^2 = p_3^2 + p_4^2 + 2(p_3 \cdot p_4) = 2(p_3 \cdot p_4) = m_\mu^2 - 2m_\mu E_2$$

$$p_3 \cdot p_4 = m_\mu^2/2 - m_\mu E_2$$

$$|\mathcal{M}|^2 = \frac{2g_W^4}{M_W^4}(m_\mu E_2)(m_\mu^2/2 - m_\mu E_2))$$

$$|\mathcal{M}|^2 = \frac{g_W^4}{M_W^4}(m_\mu E_2)(m_\mu^2 - 2m_\mu E_2))$$

$$|\mathcal{M}|^2 = \frac{g_W^4 m_\mu^2}{M_W^4}|\mathbf{p}_2|(m_\mu - 2|\mathbf{p}_2|))$$

Way way back we found the decay rate for any particle from Fermi's Golden Rule:

$$d\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3... - p_n) \times$$

$$\prod_{j=2}^{n} \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

If decay products are massless, that simplifies things a bit. Here n=4 in total, so...

$$d\,\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3... - p_n) \times$$

$$\prod_{j=2}^{n} \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

$$d\,\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 - p_4) \times$$

$$\frac{d^3\mathbf{p}_2}{(2\pi)^3 2|\mathbf{p_2}|} \frac{d^3\mathbf{p}_3}{(2\pi)^3 2|\mathbf{p_3}|} \frac{d^3\mathbf{p}_4}{(2\pi)^3 2|\mathbf{p_4}|}$$

$$\delta^4(p_1 - p_2 - p_3 - p_4) = \delta(E_1 - E_2 - E_3 - E_4)\delta^3(\mathbf{p_1} - \mathbf{p_2} - \mathbf{p_3} - \mathbf{p_4})$$

$$\mathbf{p_1} = 0$$

$$E_1 = m_\mu$$

$$E_2 = |\mathbf{p_2}|$$

$$E_3 = |\mathbf{p_3}|$$

$$E_4 = |\mathbf{p_4}|$$

$$\delta^4(p_1 - p_2 - p_3 - p_4) = \delta(m_\mu - |\mathbf{p_2}| - |\mathbf{p_3}| - |\mathbf{p_4}|)\delta^3(\mathbf{p_2} + \mathbf{p_3} + \mathbf{p_4})$$

$$\mathsf{d}\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta(m_\mu - |\mathbf{p_2}| - |\mathbf{p_3}| - |\mathbf{p_4}|)\delta^3(\mathbf{p_2} + \mathbf{p_3} + \mathbf{p_4}) \times$$

$$\frac{d^3\mathbf{p_2}}{(2\pi)^3 2|\mathbf{p_2}|} \frac{d^3\mathbf{p_3}}{(2\pi)^3 2|\mathbf{p_3}|} \frac{d^3\mathbf{p_4}}{(2\pi)^3 2|\mathbf{p_4}|}$$

with $\quad |\mathcal{M}|^2 = \dfrac{g_W^4 m_\mu^2}{M_W^4}|\mathbf{p_2}|(m_\mu - 2|\mathbf{p_2}|))$

$$d\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta(m_\mu - |\mathbf{p}_2| - |\mathbf{p}_3| - |\mathbf{p}_4|)\delta^3(\mathbf{p_2} + \mathbf{p_3} + \mathbf{p_4}) \times$$

$$\frac{d^3\mathbf{p}_2}{(2\pi)^3 2|\mathbf{p_2}|} \frac{d^3\mathbf{p}_3}{(2\pi)^3 2|\mathbf{p_3}|} \frac{d^3\mathbf{p}_4}{(2\pi)^3 2|\mathbf{p_4}|}$$

$\mathbf{p_3}$ integral gives

$$d\Gamma = \frac{|\mathcal{M}|^2}{16(2\pi)^5 m_1} \int \delta(m_\mu - |\mathbf{p}_2| - |\mathbf{p}_2 + \mathbf{p}_4| - |\mathbf{p}_4|) \times$$

$$\frac{d^3\mathbf{p}_2 d^3\mathbf{p}_4}{|\mathbf{p_2}||\mathbf{p_4}||\mathbf{p_2} + \mathbf{p_4}|}$$

Two integrals left. Let's point **p4** along z axis so:

$$d^3\mathbf{p}_2 = |\mathbf{p}_2|^2 d|\mathbf{p}_2| \sin\theta d\phi d\theta$$

Let's work it out...

$$dΓ = \frac{|\mathcal{M}|^2}{16(2π)^5 m_1} \int δ(m_μ - |\mathbf{p}_2| - |\mathbf{p}_2 + \mathbf{p}_4| - |\mathbf{p}_4|) \times$$

$$\frac{d^3\mathbf{p}_2 d^3\mathbf{p}_4}{|\mathbf{p_2}||\mathbf{p_4}||\mathbf{p_2} + \mathbf{p_4}|}$$

$$d^3\mathbf{p}_2 = |\mathbf{p}_2|^2 d|\mathbf{p}_2| \sin θ dϕ dθ$$

$$|\mathbf{p}_2 + \mathbf{p}_4|^2 = |\mathbf{p}_2|^2 + |\mathbf{p}_4|^2 + 2|\mathbf{p}_2||\mathbf{p}_4| \cos θ \equiv u^2$$

$$2u\,du = -2|\mathbf{p}_2||\mathbf{p}_4| \sin θ dθ$$

Define u here (to replace θ)

φ integral gives 2π

$$dΓ = \frac{|\mathcal{M}|^2}{16(2π)^4 m_1} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|} \frac{d|\mathbf{p}_2|}{|\mathbf{p}_2 + \mathbf{p}_4|} |\mathbf{p}_2| \int δ(m_μ - |\mathbf{p}_2| - |\mathbf{p}_2 + \mathbf{p}_4| - |\mathbf{p}_4|) \sin θ dθ$$

Let's work it out...

$$u = |\mathbf{p}_2 + \mathbf{p}_4|$$

$$2u\,du = -2|\mathbf{p}_2||\mathbf{p}_4|\sin\theta\,d\theta$$

$$d\Gamma = \frac{|\mathcal{M}|^2}{16(2\pi)^4 m_1} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|} \frac{d|\mathbf{p}_2|}{|\mathbf{p}_2 + \mathbf{p}_4|} |\mathbf{p}_2| \int \delta(m_\mu - |\mathbf{p}_2| - |\mathbf{p}_2 + \mathbf{p}_4| - |\mathbf{p}_4|)\sin\theta\,d\theta$$

$$d\Gamma = \frac{-|\mathcal{M}|^2}{16(2\pi)^4 m_1} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} d|\mathbf{p}_2| \int \delta(m_\mu - |\mathbf{p}_2| - u - |\mathbf{p}_4|)du$$

Limits of Integration:

$$u = \sqrt{|\mathbf{p}_2|^2 + |\mathbf{p}_4|^2 + 2|\mathbf{p}_2||\mathbf{p}_4|\cos\theta}$$

$$\theta = 0 \rightarrow u_+ = \sqrt{|\mathbf{p}_2|^2 + |\mathbf{p}_4|^2 + 2|\mathbf{p}_2||\mathbf{p}_4|} = \sqrt{(|\mathbf{p}_2| + |\mathbf{p}_4|)^2}$$

$$u_+ = |\mathbf{p}_2| + |\mathbf{p}_4|$$

$$\theta = \pi \rightarrow u_- = \sqrt{|\mathbf{p}_2|^2 + |\mathbf{p}_4|^2 - 2|\mathbf{p}_2||\mathbf{p}_4|} = \sqrt{(|\mathbf{p}_2| - |\mathbf{p}_4|)^2}$$

$$u_- = |(|\mathbf{p}_2| - |\mathbf{p}_4|)|$$

Let's work it out...

$$dΓ = \frac{|\mathcal{M}|^2}{16(2\pi)^4 m_1} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} d|\mathbf{p}_2| \int_{u_-}^{u^+} \delta(m_\mu - |\mathbf{p}_2| - u - |\mathbf{p}_4|) du$$

Removed minus sign for limit swap

$$u_+ = |\mathbf{p}_2| + |\mathbf{p}_4|$$
$$u_- = |(|\mathbf{p}_2| - |\mathbf{p}_4|)|$$

Does integral = 1 or 0? Does delta function contribute? Only if ...

$$u_- < (m_\mu - |\mathbf{p}_2| - |\mathbf{p}_4|) < u_+$$
$$|\mathbf{p}_2| - |\mathbf{p}_4| \leqslant u_-$$
$$|\mathbf{p}_4| - |\mathbf{p}_2| \leqslant u_-$$
$$|\mathbf{p}_2| - |\mathbf{p}_4| \leqslant m_\mu - |\mathbf{p}_4| - |\mathbf{p}_2|$$
$$|\mathbf{p}_4| - |\mathbf{p}_2| \leqslant m_\mu - |\mathbf{p}_4| - |\mathbf{p}_2|$$

$$u_- < (m_\mu - |\mathbf{p}_2| - |\mathbf{p}_4|) < u_+$$

$$|\mathbf{p}_2| - |\mathbf{p}_4| \leqslant u_-$$

$$|\mathbf{p}_4| - |\mathbf{p}_2| \leqslant u_-$$

So we get from the
u. inequality:

$$|\mathbf{p}_2| - |\mathbf{p}_4| \leqslant m_\mu - |\mathbf{p}_4| - |\mathbf{p}_2|$$

$$|\mathbf{p}_4| - |\mathbf{p}_2| \leqslant m_\mu - |\mathbf{p}_4| - |\mathbf{p}_2|$$

Does integral
= 1 or 0? Does delta
function contribute?
Only if ...

$$|\mathbf{p}_2| < m_\mu/2$$

$$|\mathbf{p}_4| < m_\mu/2$$

$$u_- < (m_\mu - |\mathbf{p}_2| - |\mathbf{p}_4|) < u_+$$

So we get from the u+ inequality:

$$|\mathbf{p}_2| + |\mathbf{p}_4| \geqslant u_+$$
$$m_\mu - |\mathbf{p}_4| - |\mathbf{p}_2| < |\mathbf{p}_2| + |\mathbf{p}_4|$$

Does integral = 1 or 0? Does delta function contribute? Only if ...

$$|\mathbf{p}_2| + |\mathbf{p}_4| \geqslant u_+$$
$$(|\mathbf{p}_2| + |\mathbf{p}_4|) < m_\mu/2$$

3

2

4

2 gets half of energy

$$|\mathbf{p}_2| < m_\mu/2$$
$$|\mathbf{p}_4| < m_\mu/2$$
$$(|\mathbf{p}_2| + |\mathbf{p}_4|) < m_\mu/2$$

3

2

4

4 gets half of energy

3

2

4

4 gets less than half of energy, 2+3 get a bit more
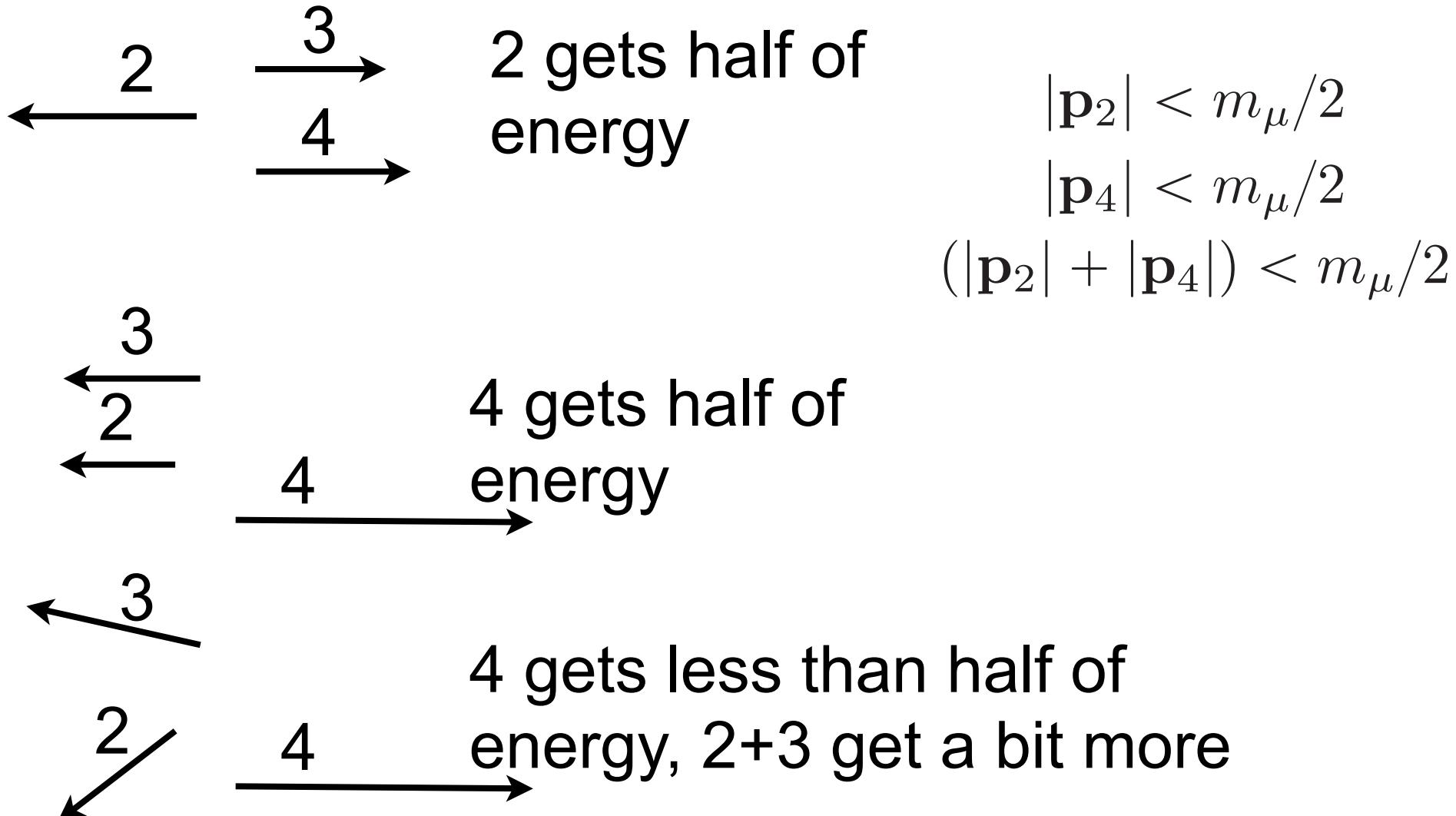
$$dΓ = \frac{|\mathcal{M}|^2}{16(2\pi)^4 m_1} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} d|\mathbf{p}_2| \int_{u_-}^{u^+} \delta(m_\mu - |\mathbf{p}_2| - u - |\mathbf{p}_4|)du$$

Becomes ...

$$dΓ = \frac{|\mathcal{M}|^2}{16(2\pi)^4 m_1} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} d|\mathbf{p}_2|$$

with ...

$$|\mathcal{M}|^2 = \frac{g_W^4 m_\mu^2}{M_W^4} |\mathbf{p}_2|(m_\mu - 2|\mathbf{p}_2|))$$

Now be careful to include |M|^2 in integral

$$Γ = \frac{g_W^4 m_\mu^2}{16(2\pi)^4 m_1 M_W^4} \int |\mathbf{p}_2|(m_\mu - 2|\mathbf{p}_2|)\frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} d|\mathbf{p}_2|$$

$$\Gamma = \frac{g_W^4 m_\mu^2}{16(2\pi)^4 m_1 M_W^4} \int |\mathbf{p}_2|(m_\mu - 2|\mathbf{p}_2|)\frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2}d|\mathbf{p}_2|$$

$$|\mathbf{p}_2| < m_\mu/2$$
$$|\mathbf{p}_4| < m_\mu/2 \qquad \text{So} \qquad \frac{m_\mu}{2} - |\mathbf{p}_4| < |\mathbf{p}_2| < \frac{m_\mu}{2}$$
$$(|\mathbf{p}_2| + |\mathbf{p}_4|) < m_\mu/2$$

$$\Gamma = \frac{g_W^4 m_\mu}{16(2\pi)^4 M_W^4} \int \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} \left[\frac{m_\mu x^2}{2} - \frac{2x^3}{3} \right]_{x=\frac{m_\mu}{2}-|\mathbf{p}_4|}^{x=\frac{m_\mu}{2}}$$

$$\Gamma = \frac{g_W^4 m_\mu}{16(2\pi)^4 M_W^4} \int_{|\mathbf{p}_4|=0}^{|\mathbf{p}_4|=\frac{m_\mu}{2}} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} \left(\frac{m_\mu|\mathbf{p}_4|^2}{2} - \frac{2}{3}|\mathbf{p}_4|^3 \right)$$

$$\Gamma = \frac{g_W^4 m_\mu}{16(2\pi)^4 M_W^4} \int_{|\mathbf{p}_4|=0}^{|\mathbf{p}_4|=\frac{m_\mu}{2}} \frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} \left(\frac{m_\mu |\mathbf{p}_4|^2}{2} - \frac{2}{3}|\mathbf{p}_4|^3 \right)$$

$$d^3\mathbf{p}_4 = 4\pi |\mathbf{p}_4|^2 d|\mathbf{p}_4|$$

For massless electron, $E_e = |\mathbf{p}_4|$

$$\Gamma = \frac{g_W^4 m_\mu}{64\pi^3 M_W^4} \int_{E=0}^{E=\frac{m_\mu}{2}} \left(\frac{m_\mu E^2}{2} - \frac{2}{3}E^3 \right) dE$$

$$\frac{d\Gamma}{dE} = \frac{g_W^4 m_\mu}{64\pi^3 M_W^4} \left(\frac{m_\mu E^2}{2} - \frac{2}{3}E^3 \right)$$

$$\frac{d\Gamma}{dE} = \frac{g_W^4 m_\mu}{64\pi^3 M_W^4}\left(\frac{m_\mu E^2}{2} - \frac{2}{3}E^3\right)$$
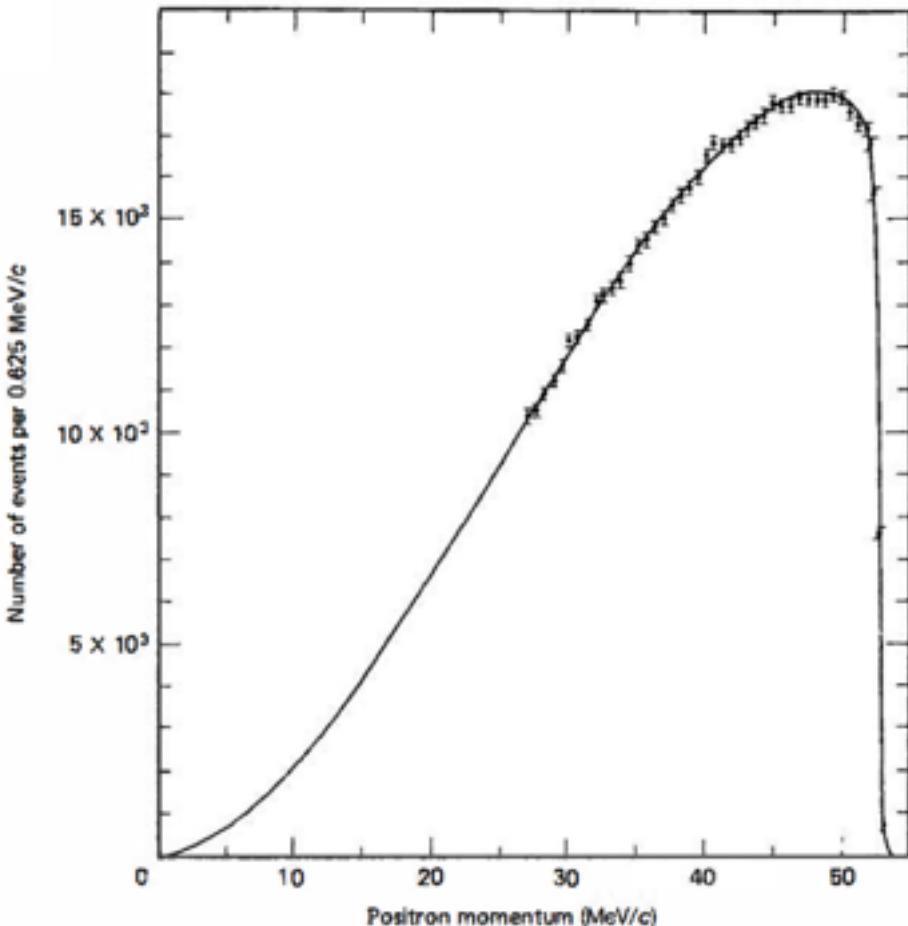
Fig. 9.1 Experimental spectrum of positrons in $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$. The solid line is the theoretically predicted spectrum based on Equation (9.33), corrected for electromagnetic effects. (Source: Bardon, M, et al. (1965) *Physical Review Letters*, 14, 449. For the latest high-precision data on muon decay go to the TWIST collaboration web site at TRIUMF, Vancouver, BC.)

Note: In the SM, there are no muon decays without neutrinos. But that might happen in BSM theories! What would we see if muons could occasionally decay directly to electrons? mu2e!

Total decay width

$$\Gamma = \frac{g_W^4 m_\mu}{64\pi^3 M_W^4} \int_{E=0}^{E=\frac{m_\mu}{2}} \left(\frac{m_\mu E^2}{2} - \frac{2}{3}E^3 \right) dE$$

$$\Gamma = \frac{g_W^4 m_\mu}{64\pi^3 M_W^4} \frac{m_\mu^4}{96}$$

$$\tau = \frac{1}{\Gamma} = \frac{6144\pi^3 M_W^4}{g_W^4 m_\mu^5}$$

Plugging in numbers... muon lifetime is 2.2 microseconds, but that is in muon rest frame. BIG time dilation factors

Weak fine structure constant/weak coupling is LARGER than EM coupling by ~x5! It is weak due to the massive nature of W/Z bosons

In principle (as Griffiths does) we could use the same formalism to calculate the lifetime of the neutron. We know of course that this will not be the full story, since the decay involves quarks, and not protons. Reminder that neutron lifetime (15 minutes!) is so long because of phase space considerations: masses of neutron and proton are so close together
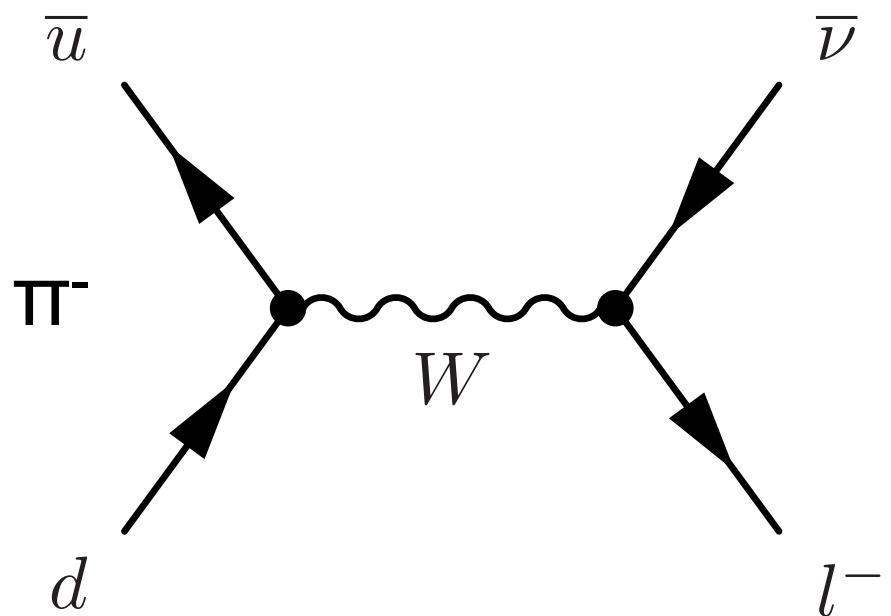
Decay of charged pion to leptons. Note that we skipped the discussion of form factors earlier. General problem: We don't know the overlap between the wave functions of the quark (if they are far apart in the pion bound state, then decay via this mode is less likely)
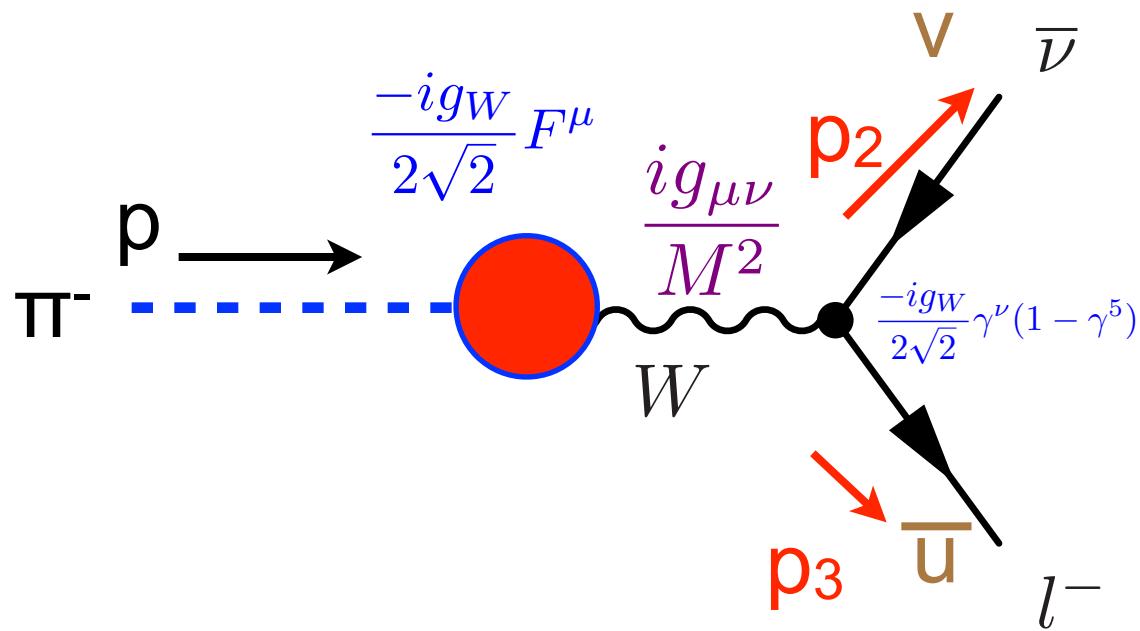
We know the coupling of the W to the leptons. Let's try and be as general as possible on the coupling of the pion to the W

$$\frac{-ig_W}{2\sqrt{2}}F^\mu$$

$$\frac{ig_{\mu\nu}}{M^2}$$

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)$$

p

π^-

W

v $\overline{\nu}$

p_2

p_3 $\overline{u}$ l^-

"Form factor" describing the blob must be a 4-vector (has to contract with γ_μ)

$$(2\pi)^4\delta^4(p - p_2 - p_3)$$

$$\frac{-ig_W}{2\sqrt{2}}f_\pi p^\mu$$

$$\frac{ig_{\mu\nu}}{M^2}$$

i

p₂

$\bar{\nu}$

v

p

π⁻

$\frac{-ig_W}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)$

$\frac{1}{(2\pi)^4}d^4p$

W

p=q

$\bar{u}$

p₃

l^-
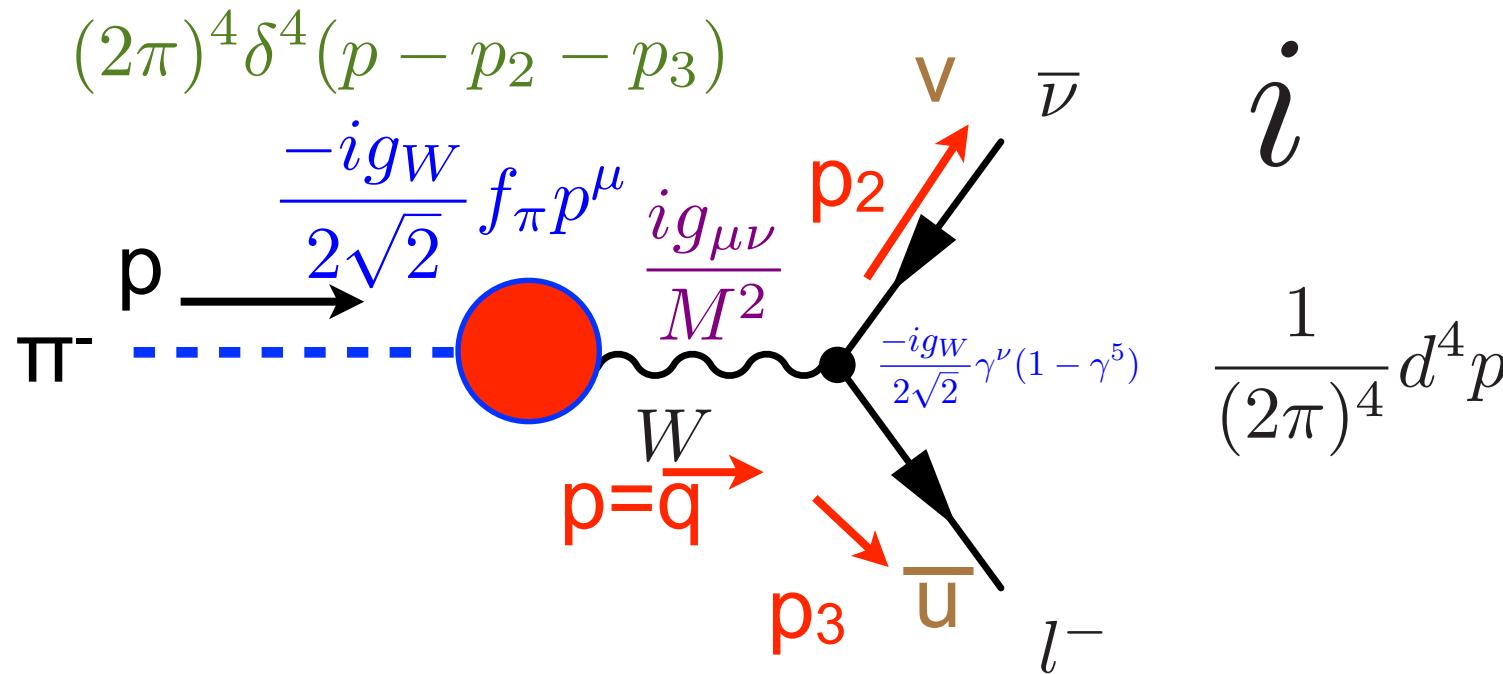
F must be be of form (F$^\mu$), but what must it be? Only quantity it can depend on is p, so it must be of form fp$^\mu$, where f is the pion decay constant (this is the unknown that we get instead of the wave function overlap)
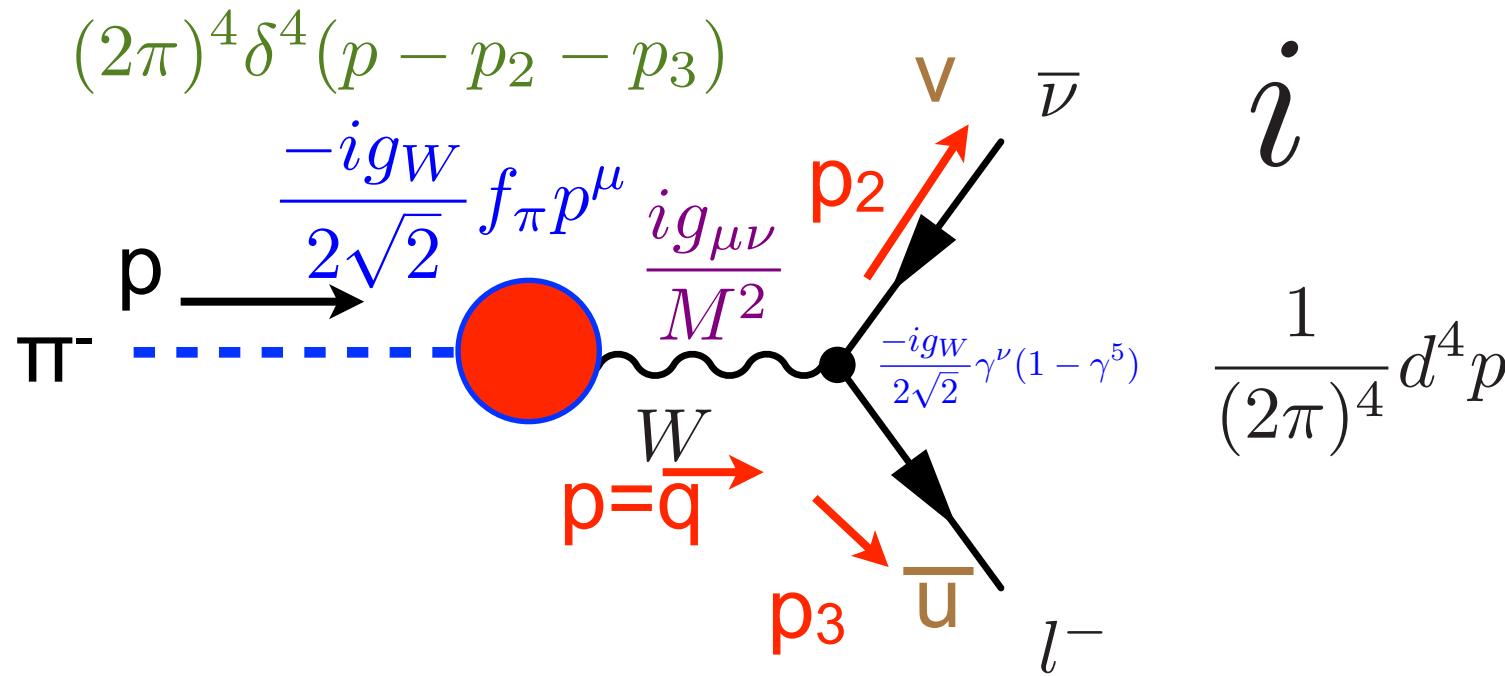
$$(2\pi)^4 \delta^4(p - p_2 - p_3)$$

$$\frac{-ig_W}{2\sqrt{2}} f_\pi p^\mu \quad \frac{ig_{\mu\nu}}{M^2}$$

p_2

v $\overline{\nu}$

i

p

π⁻

$\frac{-ig_W}{2\sqrt{2}} \gamma^\nu(1 - \gamma^5)$

W

$p = q$

$\frac{1}{(2\pi)^4} d^4 p$

p_3

$\overline{u}$

l^-

$$\mathcal{M} = \int i[\overline{u}(3)][-i\frac{g_W}{2\sqrt{2}}\gamma^\nu(1 - \gamma^5)][v(2)]\frac{ig_{\mu\nu}}{M^2}(-i\frac{g_W}{2\sqrt{2}}f_\pi p^\mu)(2\pi)^4(p - p_2 - p_3)\frac{d^4 p}{(2\pi)^4}$$

$$\mathcal{M} = \int i[\overline{u}(3)][-i\frac{g_W}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)][v(2)]\frac{ig_{\mu\nu}}{M^2}(-i\frac{g_W}{2\sqrt{2}}f_\pi p^\mu)(2\pi)^4(p-p_2-p_3)\frac{d^4p}{(2\pi)^4}$$

$$\mathcal{M} = \frac{f_\pi g_W^2}{8M_W^2}[\overline{u}(3)][\gamma^\nu(1-\gamma^5)][v(2)]p_\nu$$

Sum over outgoing spins
and apply Casimir tricks.
Also, we know that m$_2$ = 0

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 [\overline{u}(3)][\gamma^\nu(1-\gamma^5)][v(2)]p_\nu \times$$

$$\left[[\overline{u}(3)][\gamma^\mu(1-\gamma^5)][v(2)]p_\mu\right]^*$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8 M_W^2} \right)^2 [\overline{u}(3)][\gamma^\nu(1 - \gamma^5)][v(2)]p_\nu \times$$

$$\left[[\overline{u}(3)][\gamma^\mu(1 - \gamma^5)][v(2)]p_\mu \right]^*$$

$$\sum_{spins} [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^* = \text{Trace}[\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2(\not{p}_a + m_a)]$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8 M_W^2} \right)^2 p_\mu p_\nu Tr[\gamma^\nu(1 - \gamma^5)(\not{p}_2 - m_2)\overline{\gamma^\mu(1 - \gamma^5)}(\not{p}_3 + m_3)]$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8 M_W^2} \right)^2 p_\mu p_\nu Tr[\gamma^\nu(1 - \gamma^5)(\not{p}_2)\gamma^\mu(1 - \gamma^5)(\not{p}_3 + m_l)]$$

$$8(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} p_1 \cdot p_3 - i\epsilon^{\mu\nu\lambda\sigma} p_{1\lambda} p_{3\sigma})$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8M_W^2} \right)^2 p_\mu p_\nu Tr[\gamma^\nu (1 - \gamma^5)(\not{p}_2)\gamma^\mu (1 - \gamma^5)(\not{p}_3 + m_l)]$$

Thankfully, we calculated that trace already...

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8M_W^2} \right)^2 p_\mu p_\nu Tr[\gamma^\nu (1 - \gamma^5)(\not{p}_2)\gamma^\mu (1 - \gamma^5)(\not{p}_3 + m_l)]$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8M_W^2} \right)^2 8p_\mu p_\nu \left(p_2^\mu p_3^\nu + p_3^\mu p_2^\nu - g^{\mu\nu}(p_2 \cdot p_3) - i\epsilon^{\mu\nu\lambda\sigma} p_{2\lambda} p_{3\sigma} \right)$$

$$p = p_2 + p_3 \rightarrow$$

$$(p \cdot p_2) = p_2^2 + (p_2 \cdot p_3) = (p_2 \cdot p_3)$$

$$(p \cdot p_3) = p_3^2 + (p_2 \cdot p_3) = (p_2 \cdot p_3) + m_l^2$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8 M_W^2} \right)^2 8 p_\mu p_\nu \left(p_2^\mu p_3^\nu + p_3^\mu p_2^\nu - g^{\mu\nu}(p_2 \cdot p_3) - i \epsilon^{\mu\nu\lambda\sigma} p_{2\lambda} p_{3\sigma} \right)$$

$$(p \cdot p_2) = p_2^2 + (p_2 \cdot p_3) = (p_2 \cdot p_3)$$

$$(p \cdot p_3) = p_3^2 + (p_2 \cdot p_3) = (p_2 \cdot p_3) + m_l^2$$

$$p^2 = (p_2 + p_3)^2 = m_\pi^2 = p_2^2 + p_3^2 + 2(p_2 \cdot p_3) = m_l^2 + 2(p_2 \cdot p_3)$$

$$(p_2 \cdot p_3) = \frac{m_\pi^2 - m_l^2}{2} = (p \cdot p_2)$$

$$(p \cdot p_3) = \frac{m_\pi^2 + m_l^2}{2}$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8 M_W^2} \right)^2 8 \left((p \cdot p_2)(p \cdot p_3) + (p \cdot p_2)(p \cdot p_3) - p^2(p_2 \cdot p_3) - i \epsilon^{\mu\nu\lambda\sigma} p_\mu p_\nu p_{2\lambda} p_{3\sigma} \right)$$

Using earlier trick, epsilon terms will cancel...

$$\epsilon^{\mu\nu\lambda\sigma} p_\mu p_\nu p_{2\lambda} p_{3\sigma} = \epsilon^{01\lambda\sigma} p_0 p_1 p_{2\lambda} p_{3\sigma} + \epsilon^{10\lambda\sigma} p_0 p_1 p_{2\lambda} p_{3\sigma} + ...$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 8 \left((p \cdot p_2)(p \cdot p_3) + (p \cdot p_2)(p \cdot p_3) - p^2 (p_2 \cdot p_3)\right)$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 8 \left(\frac{m_\pi^2 - m_l^2}{2}\frac{m_\pi^2 + m_l^2}{2} + \frac{m_\pi^2 - m_l^2}{2}\frac{m_\pi^2 + m_l^2}{2} - m_\pi^2 \frac{m_\pi^2 - m_l^2}{2}\right)$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 8 \left(\frac{m_\pi^4 - m_l^4}{2} - \frac{m_\pi^4}{2} + \frac{m_\pi^2 m_l^2}{2}\right)$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{8M_W^2}\right)^2 8 \left(\frac{m_\pi^2 m_l^2 - m_l^4}{2}\right)$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{4M_W^2}\right)^2 \left(m_\pi^2 m_l^2 - m_l^4\right)$$

Recall for a matrix element that factorizes:

$$\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2$$

$$< |\mathcal{M}|^2 > = \left(\frac{f_\pi g_W^2}{4M_W^2}\right)^2 \left(m_\pi^2 m_l^2 - m_l^4\right)$$

And we even calculated the momentum way back then:

$$r = |\mathbf{p_2}| = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}$$

$$r = |\mathbf{p_2}| = \frac{1}{2m_\pi}\sqrt{m_\pi^4 + m_l^4 - 2m_\pi^2 m_l^2}$$

Here:

$$r = |\mathbf{p_2}| = \frac{1}{2m_\pi}(m_\pi^2 - m_l^2)$$

$$\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2 \qquad r = |\mathbf{p_2}| = \frac{1}{2m_\pi}(m_\pi^2 - m_l^2)$$

$$<|\mathcal{M}|^2> = \left(\frac{f_\pi g_W^2}{4M_W^2}\right)^2 \left(m_\pi^2 m_l^2 - m_l^4\right)$$

$$\Gamma = \frac{1}{8\pi m_\pi^2}\frac{1}{2m_\pi}(m_\pi^2 - m_l^2)\left(\frac{f_\pi g_W^2}{4M_W^2}\right)^2 \left(m_\pi^2 m_l^2 - m_l^4\right)$$

$$\Gamma = \frac{1}{\pi m_\pi^3}\left(\frac{f_\pi g_W^2}{16M_W^2}\right)^2 m_l^2(m_\pi^2 - m_l^2)^2$$

$$\frac{\Gamma(\pi^- \rightarrow e^- + \overline{\nu}_e)}{\Gamma(\pi^- \rightarrow \mu^- + \overline{\nu}_\mu)} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} = 10^{-4}$$

Agrees with experiment, but that is odd, because phase space arguments would predict the opposite! What is going on?

Note that if lepton mass is zero, decay rate is predicted to be zero! Why is that? Must have something to do with V-A coupling...

Pion is spin zero, so in its rest frame its decay products must have opposite spin

But anti-neutrino is **always** right-handed, which means the electron must be right-handed too
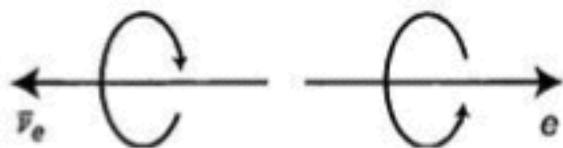
If electron had zero mass, it would only be left-handed. **Helicity suppression**

(Aside: What about decays to taus?)

In lepton sector, W couples only within a lepton generation (this is why electron number, muon number and tau number are conserved).

$$W^+ \rightarrow e^+ + \nu_e$$

$$W^+ \rightarrow \mu^+ + \nu_e$$

$$W^+ \rightarrow \tau^+ + \nu_e$$

In quark sector, might expect something analogous to hold
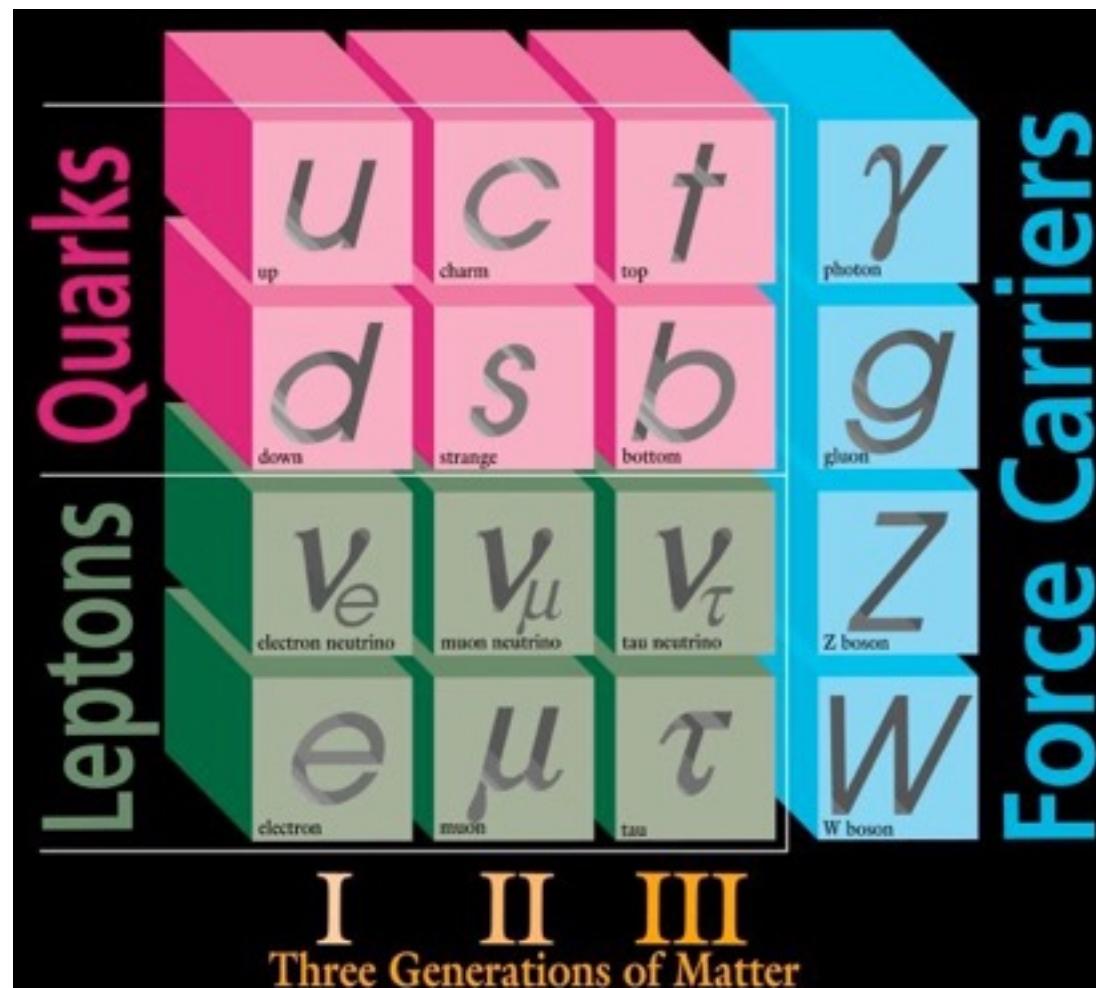
Coupling is predominantly but not entirely within a generation (there is no "conservation of 1st generation of quarks")

This is how 2nd and 3rd generations of quarks can decay to the lighter 1st generation quarks
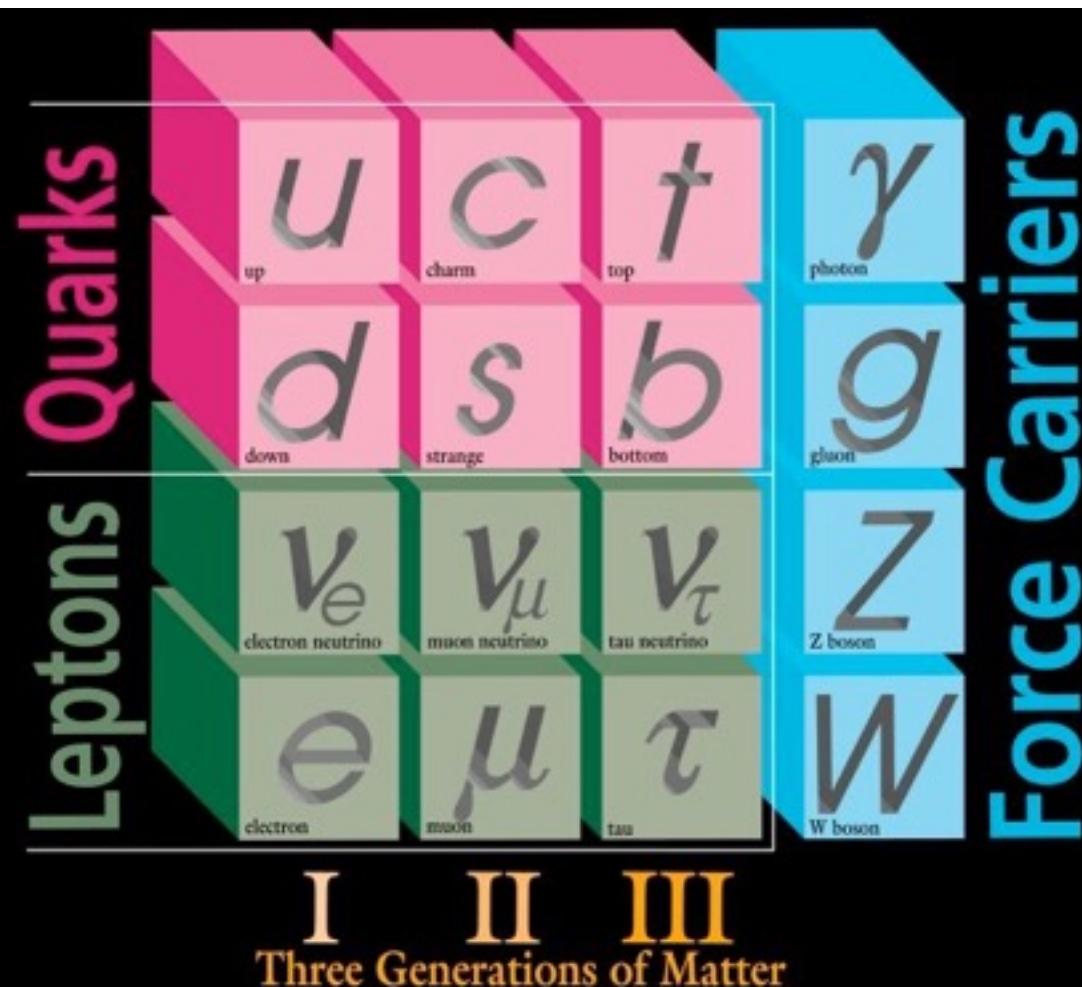
As we have discussed, the mass eigenstates that we know of are not the same eigenstates of the charged weak interaction, but instead are slightly rotated
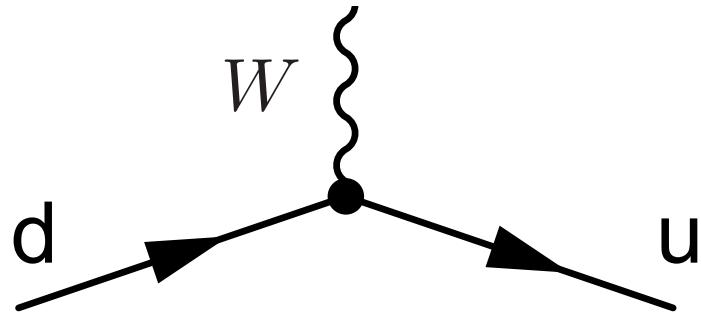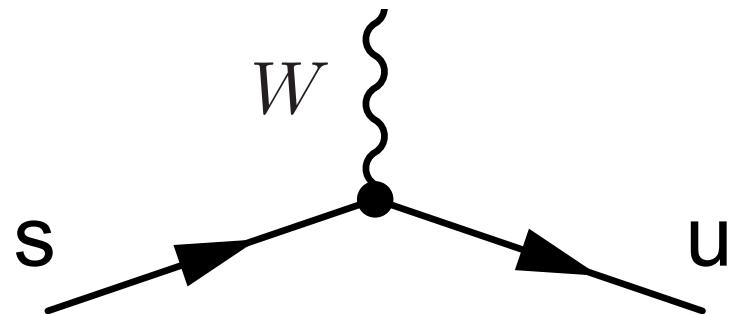
$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_c$$

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\sin\theta_c$$

θ_c (Cabbibo angle, after Nicola Cabibbo, who proposed it), is small (13°), so that strangeness-violating decay is weak
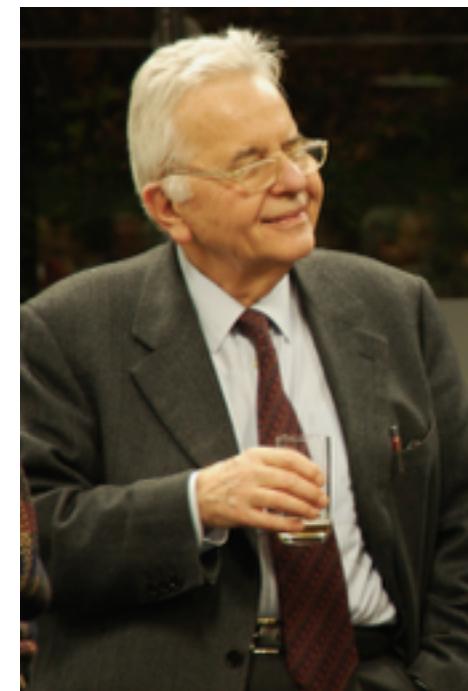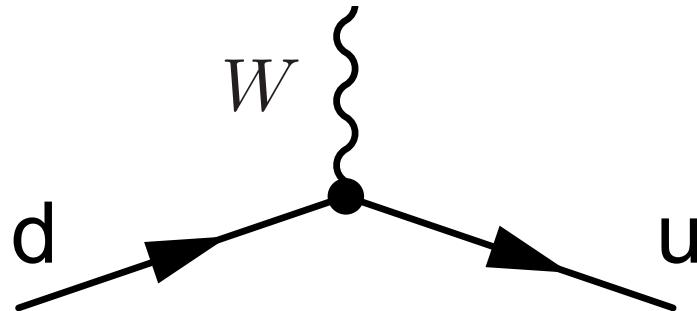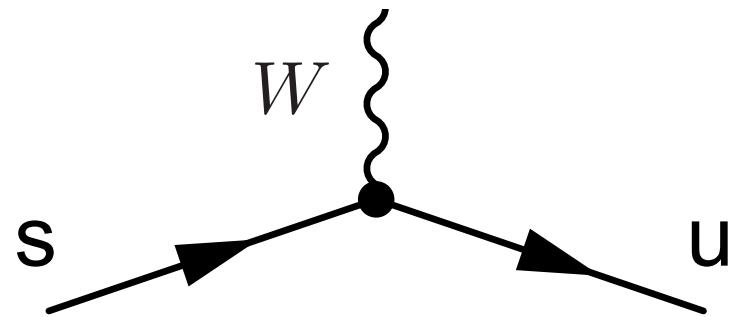
$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_c$$

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\sin\theta_c$$

$$\frac{\Gamma(K^- \to l- +\overline{\nu}_l)}{\Gamma(\pi^- \to l^- + \overline{\nu}_l)} = \frac{f_K^2}{f_\pi^2}\tan^2\theta_c\left(\frac{m_\pi}{m_K}\right)^3\frac{m_l^2(m_K^2-m_l^2)^2}{m_l^2(m_\pi^2-m_l^2)^2}$$

Let's assume form factors for kaon and pion are ~the same

$$\frac{\Gamma(K^- \to l- +\overline{\nu}_l)}{\Gamma(\pi^- \to l^- + \overline{\nu}_l)} = \tan^2\theta_c\left(\frac{m_\pi}{m_K}\right)^3\frac{m_l^2(m_K^2-m_l^2)^2}{m_l^2(m_\pi^2-m_l^2)^2}$$

Charged kaon mass ~0.494 GeV
Charged pion mass ~0.140 GeV
Electron mass ~ 0.0005 GeV
Muon mass ~0.106 GeV

$$\frac{\Gamma(K^- \rightarrow l- +\overline{\nu}_l)}{\Gamma(\pi^- \rightarrow l^- + \overline{\nu}_l)} = \tan^2 \theta_c \left(\frac{m_\pi}{m_K}\right)^3 \frac{(m_K^2 - m_l^2)^2}{(m_\pi^2 - m_l^2)^2}$$

Without Cabbibo angle:
electron ratio: 3.5
muon ratio: 17.6

With Cabbibo angle:
electron ratio: 0.19
muon ratio: 0.96

Observed (0.26/1.34)
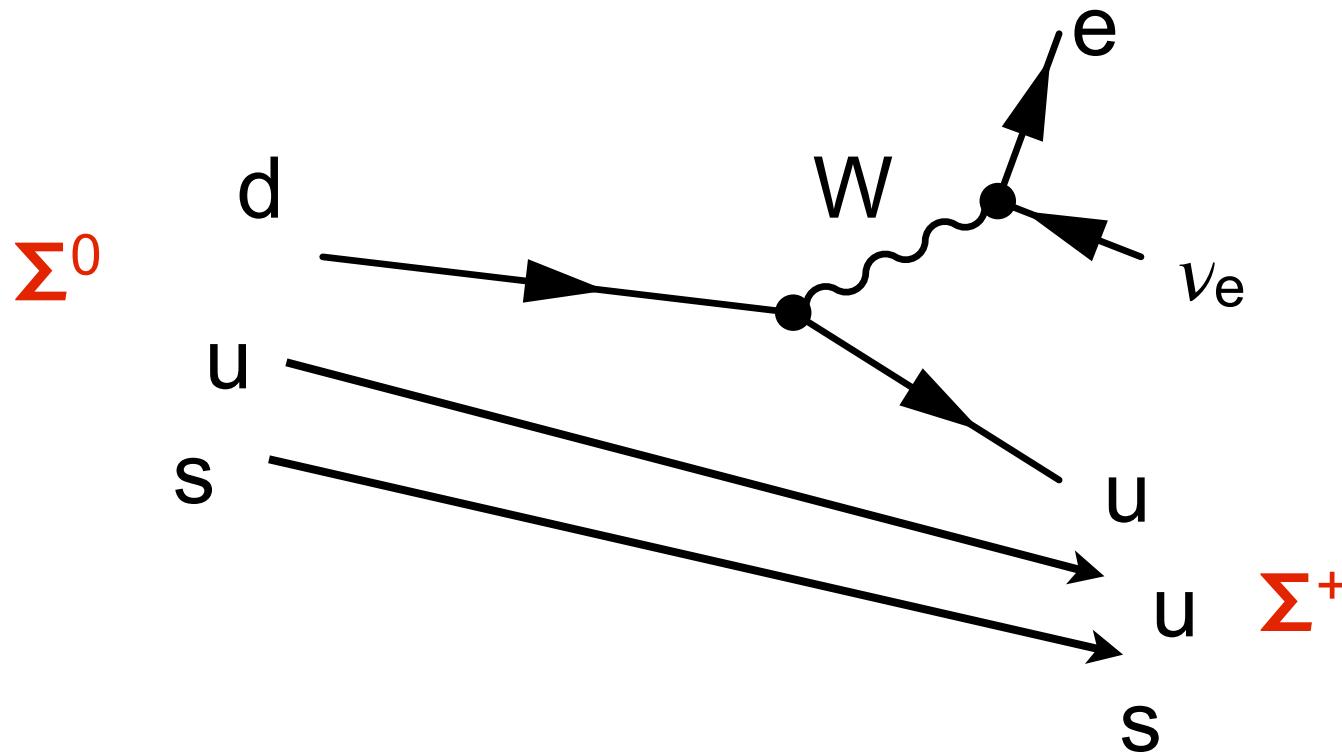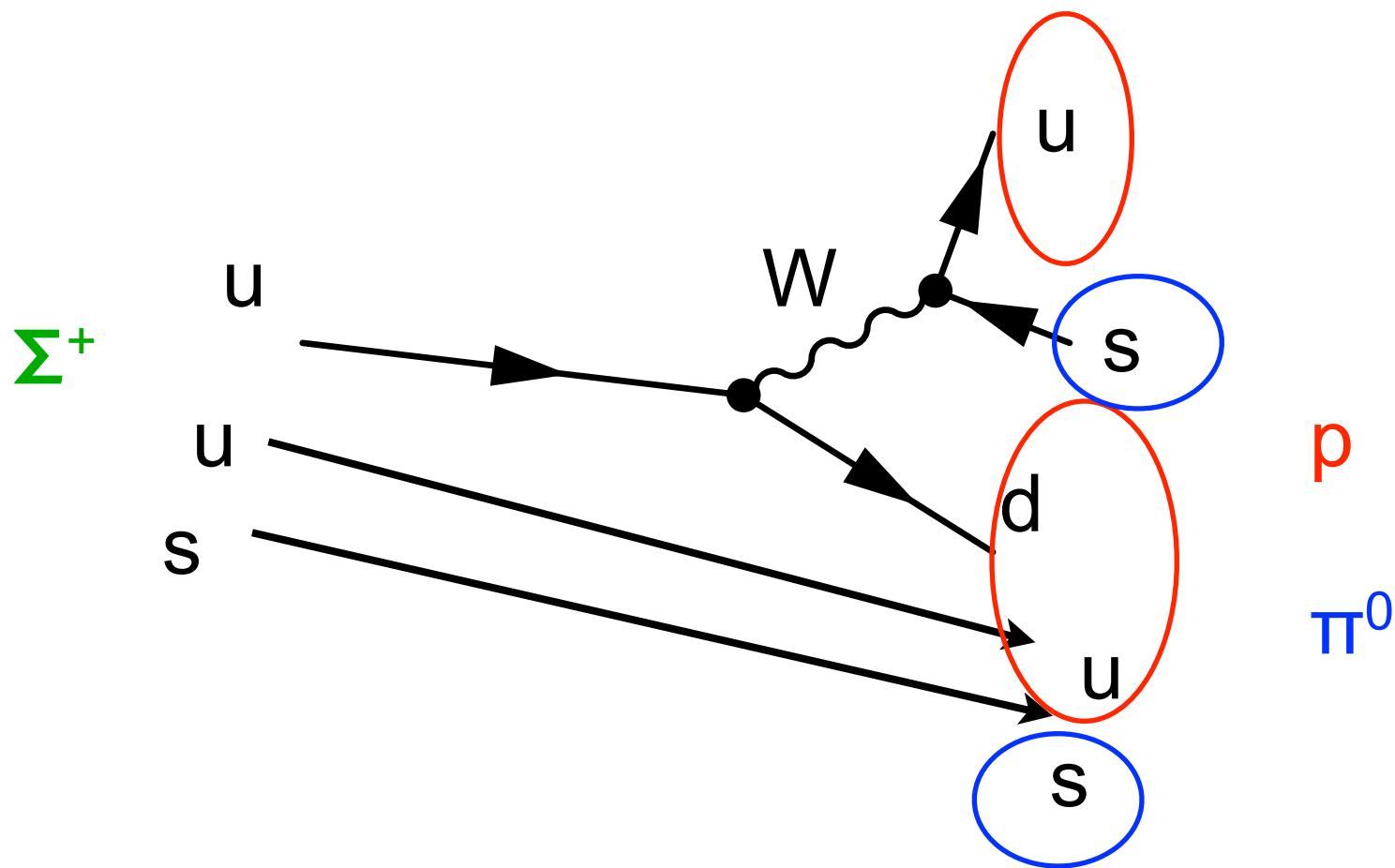
Kaon decays to muons observed to occur at a much much lower rate than observed. Why might that be?
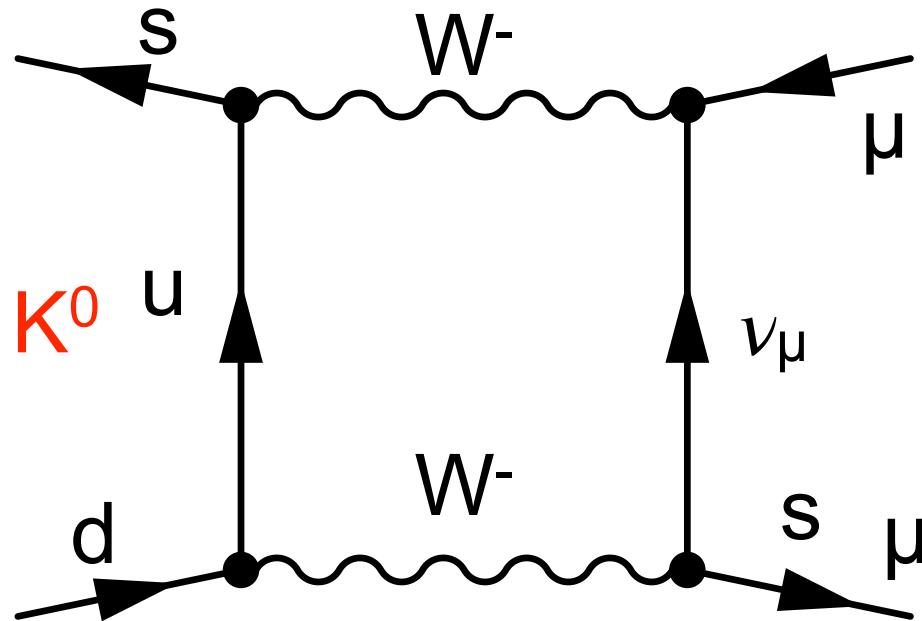
Glashow, Iliopoulous and Maiani proposed the "GIM" mechanism - a new 4th quark comes in with opposite sign and cancels the diagram!
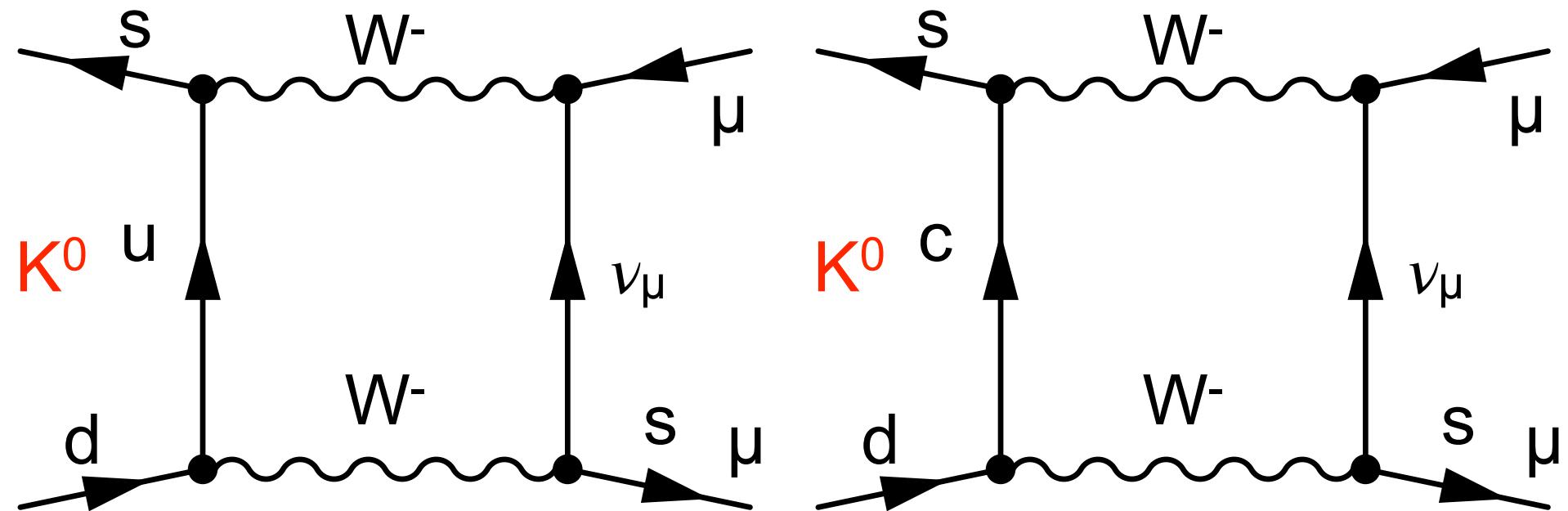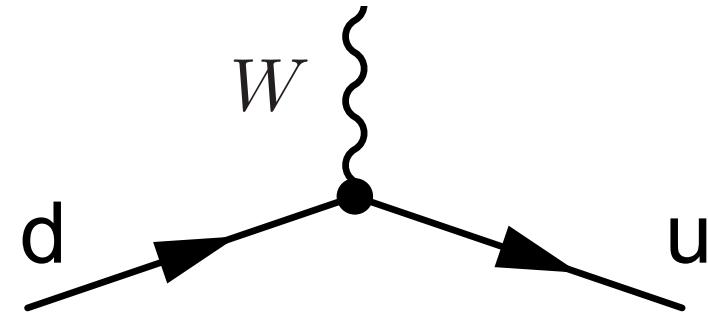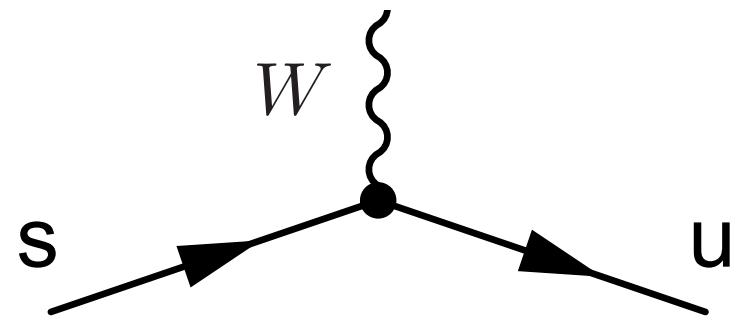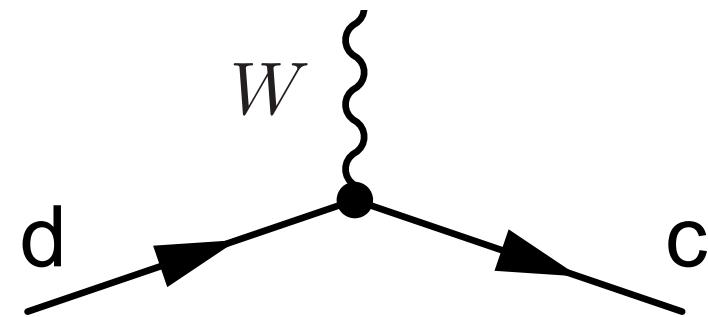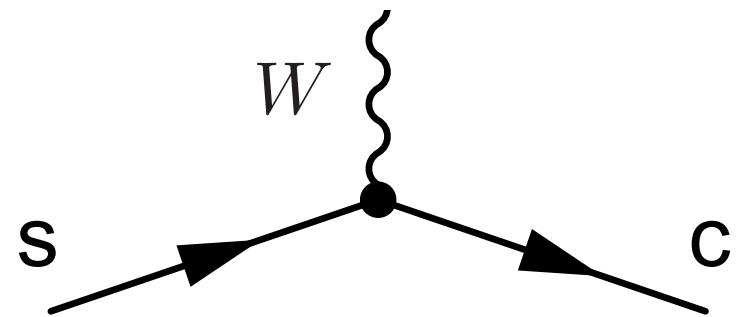
$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_c$$

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\sin\theta_c$$

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)(-\sin\theta_c)$$

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_c$$

Terms cancel!
(Up until charm mass effects, so not identically)

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\sin\theta_c \qquad\qquad \frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_c$$
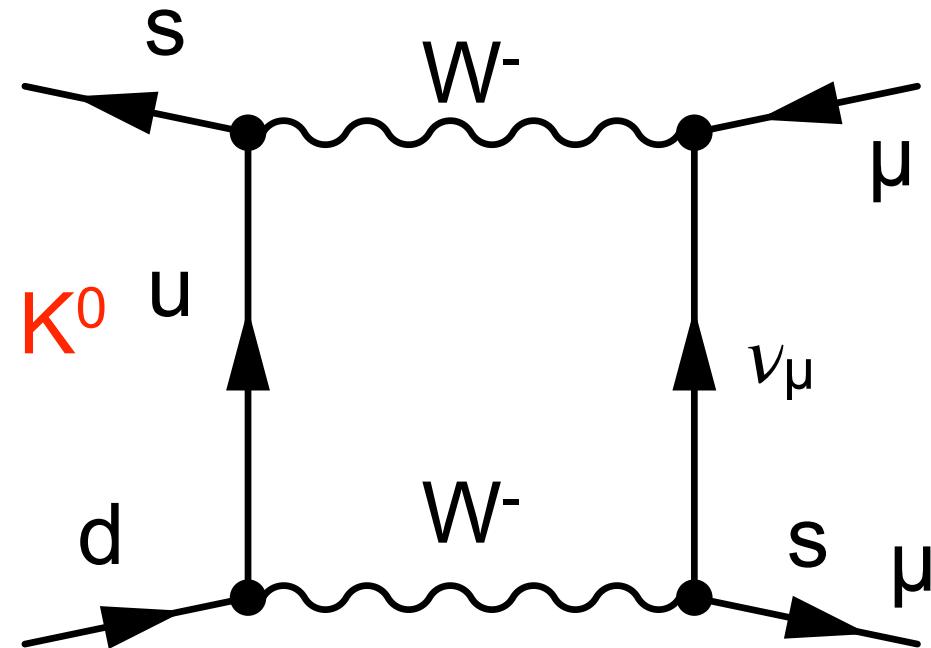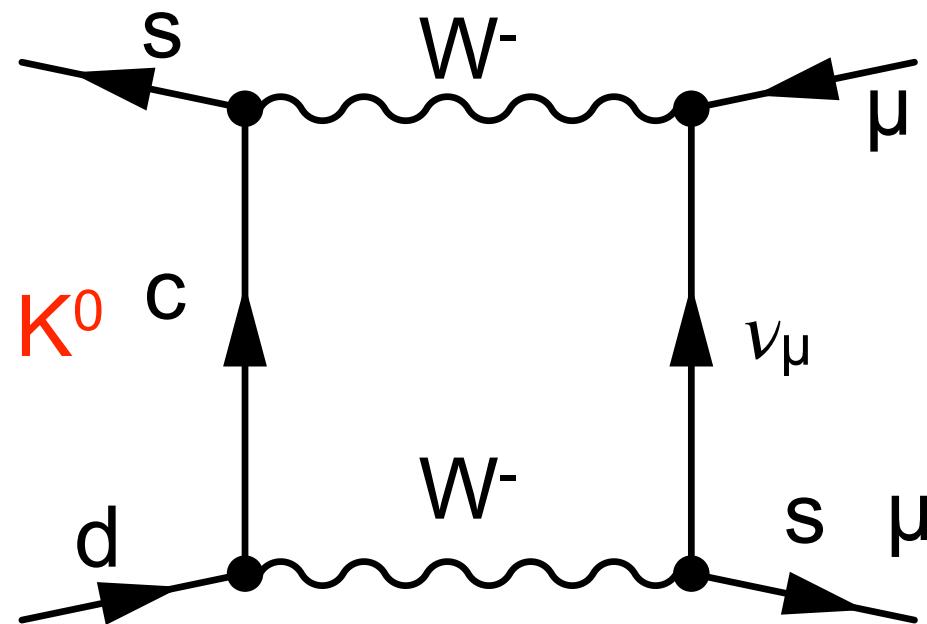
$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_c \qquad\qquad \frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)(-\sin\theta_c)$$

$$d' = d\cos\theta_c + s\sin\theta_c$$
$$s' = -d\sin\theta_c + s\cos\theta_c$$

W bosons couple to u and c quarks, but to linear combinations d' and s', not d and s

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

$$\begin{pmatrix} u \\ d' \end{pmatrix} \begin{pmatrix} c \\ s' \end{pmatrix} \quad \text{OR}$$

$$\begin{pmatrix} u \\ d\cos\theta_c + s\sin\theta_c \end{pmatrix} \begin{pmatrix} c \\ -d\sin\theta_c + s\cos\theta_c \end{pmatrix}$$

Kobayashi   Maskawa

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} 0.97434 & 0.22506 & 0.00357 \\ 0.22492 & 0.97351 & 0.0411 \\ 0.00875 & 0.0403 & 0.99915 \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

CKM matrix V has in general 3 "angles" and one phase (which allows for CP violating effects!) Lots of efforts to probe and test the CKM matrix and look for new physics (for example, is it unitary?)

CKM Fitter collaboration

http://ckmfitter.in2p3.fr/www/results/plots_summer18/ckm_res_summer18.html
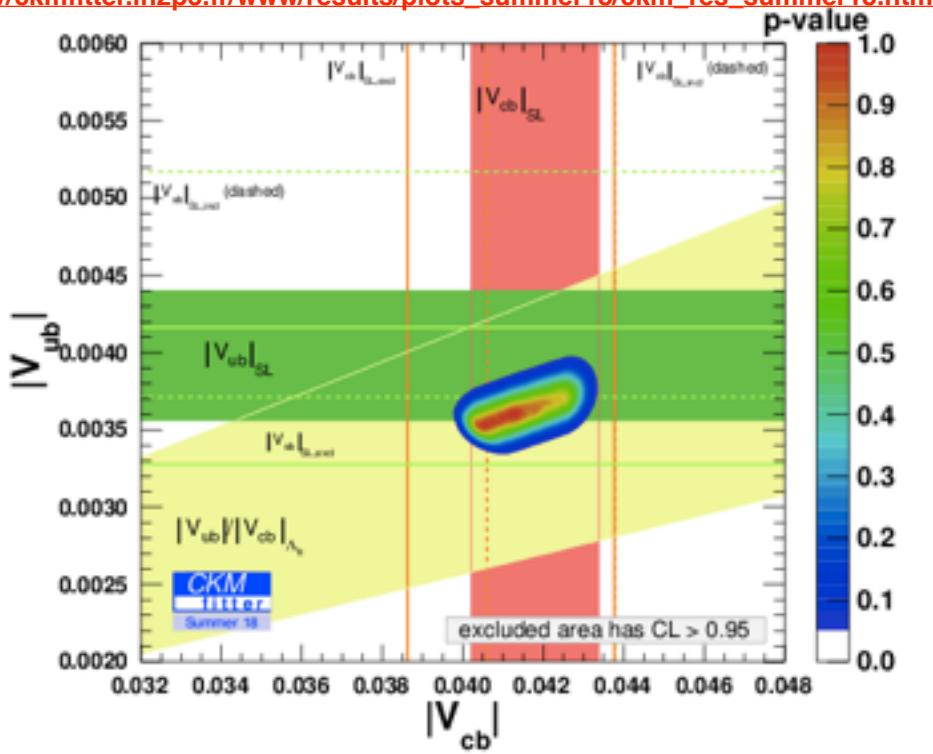
ATLAS+CMS V_{tb} measurements from single top quark production

1902.07158
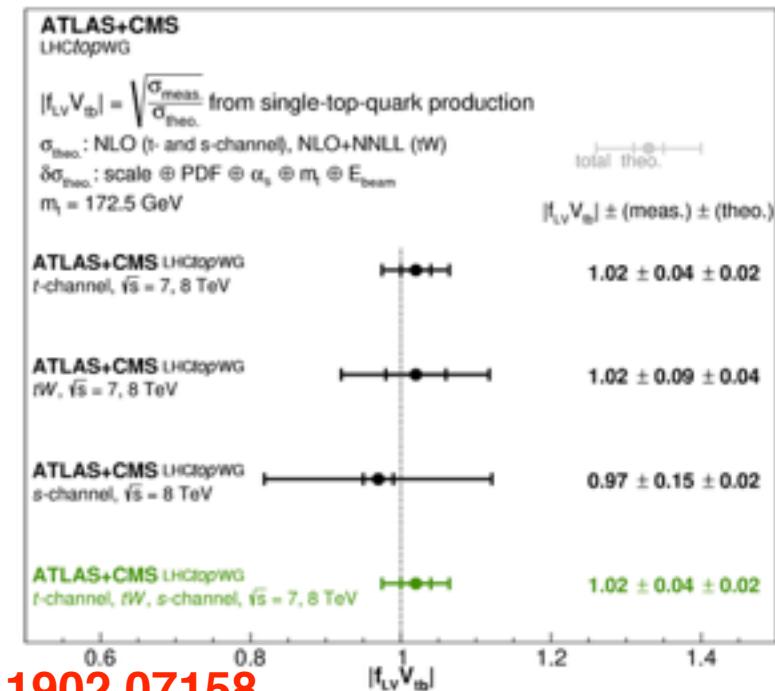
Electroweak unification (as we'll see) requires the existence of neutral weak processes
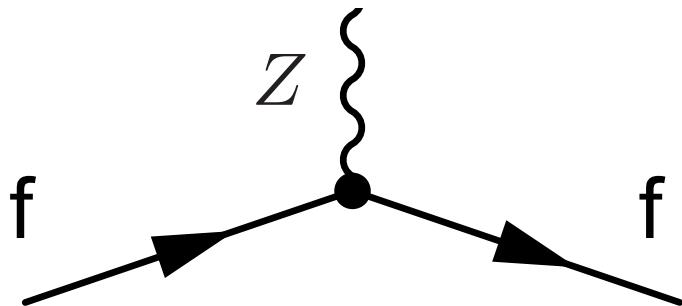
$$\frac{-ig_Z}{2}\gamma^\mu(c_V^f - c_A^f\gamma^5)$$

$$g_w = \frac{g_e}{\sin\theta_w}$$

Careful: 2 vs 2*sqrt(2)!

$$g_z = \frac{g_e}{\sin\theta_w \cos\theta_w}$$

$$\theta_w = 28.75^o$$

$$\sin^2\theta_w = 0.2314$$

f	c_V	c_A
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	$\frac{1}{2}$
e, μ, τ	$\frac{-1}{2} + 2\sin^2\theta_w$	$\frac{-1}{2}$
u, c, t	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_w$	$\frac{1}{2}$
d, s, b	$\frac{-1}{2} + \frac{2}{3}\sin^2\theta_w$	$\frac{-1}{2}$

$(2\pi)^4\delta^4(p_2 + q - p_4)$

e^-

p2 $\frac{-ig_Z}{2}\gamma^\sigma(c_v^e - c_a^e\gamma^5)$ p4 e^-

u $\overline{u}$

$\dot{i}$

$\frac{1}{(2\pi)^4}d^4q$

Z $\frac{ig_{\mu\sigma}}{M^2}$

q

p1

ν_μ u $\frac{-ig_Z}{2}\gamma^\mu(c_v^\nu - c_a^\nu\gamma^5)$ p3 $\overline{u}$ ν_μ

$(2\pi)^4\delta^4(p_1 - q - p_3)$

Careful about "nu" in numerator! Not an index

Assume small momentum transfer

$$\mathcal{M} = \int i[\overline{u}(3)][-i\frac{g_Z}{2}\gamma^\mu(c_V^\nu - c_A^\nu\gamma^5)][u(1)]\frac{ig_{\mu\sigma}}{M^2}[\overline{u}(4)][-i\frac{g_Z}{2}\gamma^\sigma(c_V^e - c_A^e\gamma^5)][u(2)]$$

$$(2\pi)^4\delta^4(p_1 - q - p_3)(2\pi)^4\delta^4(p_2 + q - p_4)\frac{d^4q}{(2\pi)^4}$$

Neutrino-electron scattering matrix element

$$\mathcal{M} = \int i[\bar{u}(3)][-i\frac{g_Z}{2}\gamma^\mu(c_V^\nu - c_A^\nu\gamma^5)][u(1)]\frac{ig_{\mu\sigma}}{M^2}[\bar{u}(4)][-i\frac{g_Z}{2}\gamma^\sigma(c_V^e - c_A^e\gamma^5)][u(2)]$$

$$(2\pi)^4\delta^4(p_1 - q - p_3)(2\pi)^4\delta^4(p_2 + q - p_4)\frac{d^4q}{(2\pi)^4}$$

Know that we can use earlier result (with appropriate substitution, and being careful about sqrt(2) difference in coupling) and ignoring electron mass

$$|\mathcal{M}|^2 = \frac{g_Z^4}{16M_Z^4}Tr[\gamma^\mu(C_V^\nu - C_A^\nu\gamma^5)(\not{p}_1)(\gamma^\sigma((C_V^\nu - C_A^\nu\gamma^5)(\not{p}_3)] \times$$

$$Tr[\gamma_\mu(C_V^e - C_A^e\gamma^5)(\not{p}_2)(\gamma_\sigma((C_V^e - C_A^e\gamma^5)(\not{p}_4)]$$

$$|\mathcal{M}|^2 = \frac{g_Z^4}{16M_Z^4} Tr[\gamma^\mu (C_V^\nu - C_A^\nu \gamma^5)(\not{p}_1)(\gamma^\sigma ((C_V^\nu - C_A^\nu \gamma^5)(\not{p}_3)] \times$$

$$Tr[\gamma_\mu (C_V^e - C_A^e \gamma^5)(\not{p}_2)(\gamma_\sigma ((C_V^e - C_A^e \gamma^5)(\not{p}_4)]$$

Let's focus on second trace

$$Tr[\gamma_\mu (C_V^e - C_A^e \gamma^5)(\not{p}_2)(\gamma_\sigma (C_V^e - C_A^e \gamma^5)(\not{p}_4)] =$$

$$Tr[\gamma_\mu C_V^e \gamma_\nu p_2^\nu \gamma_\sigma C_V^e \gamma_\alpha p_4^\alpha - \gamma_\mu C_V^e \gamma_\nu p_2^\nu \gamma_\sigma C_A^e \gamma^5 \gamma_\alpha p_4^\alpha -$$

$$\gamma_\mu C_A^e \gamma^5 \gamma_\nu p_2^\nu \gamma_\sigma C_V^e \gamma_\alpha p_4^\alpha + \gamma_\mu C_A^e \gamma^5 \gamma_\nu p_2^\nu \gamma_\sigma C_A^e \gamma^5 \gamma_\alpha p_4^\alpha]$$

$$Tr[\gamma_\mu (C_V^e - C_A^e \gamma^5)(\not{p}_2)(\gamma_\sigma (C_V^e - C_A^e \gamma^5)(\not{p}_4)] =$$

$$Tr[\gamma_\mu C_V^e \gamma_\nu p_2^\nu \gamma_\sigma C_V^e \gamma_\alpha p_4^\alpha + \gamma^5 \gamma_\mu C_V^e \gamma_\nu p_2^\nu \gamma_\sigma C_A^e \gamma_\alpha p_4^\alpha +$$

$$\gamma^5 \gamma_\mu C_A^e \gamma_\nu p_2^\nu \gamma_\sigma C_V^e \gamma_\alpha p_4^\alpha + \gamma_\mu C_A^e \gamma_\nu p_2^\nu \gamma_\sigma C_A^e \gamma_\alpha p_4^\alpha]$$

$$Tr[\gamma_\mu(C_V^e - C_A^e\gamma^5)(\not{p}_2)(\gamma_\sigma(C_V^e - C_A^e\gamma^5)(\not{p}_4)] =$$

$$Tr[\gamma_\mu C_V^e \gamma_\nu p_2^\nu \gamma_\sigma C_V^e \gamma_\alpha p_4^\alpha + \gamma^5\gamma_\mu C_V^e \gamma_\nu p_2^\nu \gamma_\sigma C_A^e \gamma_\alpha p_4^\alpha +$$

$$\gamma^5\gamma_\mu C_A^e \gamma_\nu p_2^\nu \gamma_\sigma C_V^e \gamma_\alpha p_4^\alpha + \gamma_\mu C_A^e \gamma_\nu p_2^\nu \gamma_\sigma C_A^e \gamma_\alpha p_4^\alpha]$$

$$Tr[\gamma_\mu(C_V^e - C_A^e\gamma^5)(\not{p}_2)(\gamma_\sigma(C_V^e - C_A^e\gamma^5)(\not{p}_4)] =$$

$$4(C_V^e)^2 p_2^\nu p_4^\alpha (g_{\mu\nu}g_{\sigma\alpha} - g_{\mu\sigma}g_{\nu\alpha} + g_{\mu\alpha}g_{\nu\sigma}) + 4iC_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha +$$

$$4iC_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha + 4(C_A^e)^2 p_2^\nu p_4^\alpha ((g_{\mu\nu}g_{\sigma\alpha} - g_{\mu\sigma}g_{nu\alpha} + g_{\mu\alpha}g_{\nu\sigma})]$$

$$Tr[\gamma_\mu(C_A^e - C_V^e\gamma^5)(\not{p}_2)(\gamma_\sigma(C_A^e - C_V^e\gamma^5)(\not{p}_4)] =$$

$$4((C_V^e)^2 + (C_A^e)^2)p_2^\nu p_4^\alpha (g_{\mu\nu}g_{\sigma\alpha} - g_{\mu\sigma}g_{\nu\alpha} + g_{\mu\alpha}g_{\nu\sigma}) + 8iC_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha$$

Neutrino-electron scattering matrix element

$$Tr[\gamma_\mu(C_A^e - C_V^e\gamma^5)(\not{p}_2)(\gamma_\sigma(C_A^e - C_V^e\gamma^5)(\not{p}_4)] =$$

$$4((C_V^e)^2 + (C_A^e)^2)p_2^\nu p_4^\alpha \left(g_{\mu\nu}g_{\sigma\alpha} - g_{\mu\sigma}g_{\nu\alpha} + g_{\mu\alpha}g_{\nu\sigma}\right) + 8iC_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha$$

$$= 4((C_V^e)^2 + (C_A^e)^2)(p_{2\mu}p_{4\sigma} - g_{\mu\sigma}(p_2 \cdot p_4) + p_{2\sigma}p_{4\mu}) + 8iC_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha$$

And then first trace is

$$= 4((C_V^\nu)^2 + (C_A^\nu)^2)(p_1^\mu p_3^\sigma - g^{\mu\sigma}(p_1 \cdot p_3) + p_1^\sigma p_3^\mu) + 8iC_V^\nu C_A^\nu \epsilon^{\mu\beta\sigma\lambda}p_{1\beta}p_{3\lambda}$$

Multiplying them and contracting gives

$$= 16((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2)\times$$

$$((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_2 \cdot p_4)(p_1 \cdot p_3) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4) + 4(p_1 \cdot p_3)(p_2 \cdot p_4) -$$

$$(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_2 \cdot p_3)(p_1 \cdot p_4) - (p_2 \cdot p_4)(p_1 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4))$$

$$-64C_V^e C_A^\nu C_V^\nu C_A^e \epsilon^{\mu\beta\sigma\lambda}p_{1\beta}p_{3\lambda}\epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha$$

$$+32iC_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha(p_1^\mu p_3^\sigma - g^{\mu\sigma}(p_1 \cdot p_3) + p_1^\sigma p_3^\mu)$$

$$+32iC_V^\nu C_A^\nu \epsilon^{\mu\beta\sigma\lambda}p_{1\beta}p_{3\lambda}(p_{2\mu}p_{4\sigma} - g_{\mu\sigma}(p_2 \cdot p_4) + p_{2\sigma}p_{4\mu})$$

Neutrino-electron scattering matrix element

$$= 16((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times$$

$$((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_2 \cdot p_4)(p_1 \cdot p_3) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4) + 4(p_1 \cdot p_3)(p_2 \cdot p_4) -$$

$$(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_2 \cdot p_3)(p_1 \cdot p_4) - (p_2 \cdot p_4)(p_1 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4))$$

$$-64 C_V^e C_A^\nu C_V^\nu C_A^e \epsilon^{\mu\beta\sigma\lambda} p_{1\beta} p_{3\lambda} \epsilon_{\mu\nu\sigma\alpha} p_2^\nu p_4^\alpha$$

$$+32i C_V^e C_A^e \epsilon_{\mu\nu\sigma\alpha} p_2^\nu p_4^\alpha (p_1^\mu p_3^\sigma - g^{\mu\sigma}(p_1 \cdot p_3) + p_1^\sigma p_3^\mu)$$

$$+32i C_V^\nu C_A^\nu \epsilon^{\mu\beta\sigma\lambda} p_{1\beta} p_{3\lambda} (p_{2\mu} p_{4\sigma} - g_{\mu\sigma}(p_2 \cdot p_4) + p_{2\sigma} p_{4\mu})$$

Pairs of 1st and 3rd terms cancel as we saw earlier (anti-symmetry)

g terms require μ=σ but then ε is zero

$$= 16((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times (2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$-64 C_V^e C_A^\nu C_V^\nu C_A^e \epsilon^{\mu\beta\sigma\lambda} p_{1\beta} p_{3\lambda} \epsilon_{\mu\nu\sigma\alpha} p_2^\nu p_4^\alpha$$

Recall:

$$\epsilon^{\mu\beta\sigma\lambda} \epsilon_{\mu\nu\sigma\alpha} = (-\epsilon^{\mu\sigma\beta\lambda} - \epsilon_{\mu\sigma\nu\alpha}) = \epsilon^{\mu\sigma\beta\lambda} \epsilon_{\mu\sigma\nu\alpha} = -2(\delta_\nu^\beta \delta_\alpha^\lambda - \delta_\alpha^\beta \delta_\nu^\lambda)$$

$$= 16((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times (2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$-64 C_V^e C_A^\nu C_V^\nu C_A^e \epsilon^{\mu\beta\sigma\lambda} p_{1\beta} p_{3\lambda} \epsilon_{\mu\nu\sigma\alpha} p_2^\nu p_4^\alpha$$

$$\epsilon^{\mu\beta\sigma\lambda} \epsilon_{\mu\nu\sigma\alpha} = (-\epsilon^{\mu\sigma\beta\lambda} - \epsilon_{\mu\sigma\nu\alpha}) = \epsilon^{\mu\sigma\beta\lambda} \epsilon_{\mu\sigma\nu\alpha} = -2(\delta_\nu^\beta \delta_\alpha^\lambda - \delta_\alpha^\beta \delta_\nu^\lambda)$$

$$= 16((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times (2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$-128 C_V^e C_A^\nu C_V^\nu C_A^e p_{1\beta} p_{3\lambda} (p_4^\beta p_2^\lambda - p_2^\beta p_4^\lambda)$$

$$= 32((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$+128 C_V^e C_A^\nu C_V^\nu C_A^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$= 32((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$+128 C_V^e C_A^\nu C_V^\nu C_A^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

Divide by two to average over electron spins

$$= 16((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$+64 C_V^e C_A^\nu C_V^\nu C_A^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

Add in missing terms from matrix element

$$= \frac{g_Z^4}{M_Z^4}((C_V^\nu)^2 + (C_A^\nu)^2)((C_V^e)^2 + (C_A^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$+\frac{4g_Z^4}{M_Z^4} C_V^e C_A^\nu C_V^\nu C_A^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

Let's plug in C terms for neutrino (both 1/2)

$$= \frac{g_Z^4}{2M_Z^4}((C_V^e)^2 + (C_A^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$+\frac{g_Z^4}{M_Z^4} C_V^e C_A^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$= \frac{g_Z^4}{2M_Z^4}((C_V^e)^2 + (C_A^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$+ \frac{g_Z^4}{M_Z^4} C_V^e C_A^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

Assume massless electron. Pick center of mass frame where $E_1 = E_2 = E_3 = E_4 = E$ and $|\mathbf{p}_i| = E$

$$(p_1 \cdot p_2) = E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 = E^2 + E^2 = 2E^2$$

$$(p_3 \cdot p_4) = E_3 E_4 - \mathbf{p}_3 \cdot \mathbf{p}_4 = E^2 + E^2 = 2E^2$$

$$(p_1 \cdot p_4) = E_1 E_4 - \mathbf{p}_1 \cdot \mathbf{p}_4 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

$$(p_2 \cdot p_3) = E_2 E_3 - \mathbf{p}_2 \cdot \mathbf{p}_3 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

$$= \frac{g_Z^4}{2M_Z^4}((C_A^e)^2 + (C_V^e)^2) \times ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$+ \frac{g_Z^4}{M_Z^4} C_A^e C_V^e ((p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))$$

$$(p_1 \cdot p_2) = E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 = E^2 + E^2 = 2E^2$$

$$(p_3 \cdot p_4) = E_3 E_4 - \mathbf{p}_3 \cdot \mathbf{p}_4 = E^2 + E^2 = 2E^2$$

$$(p_1 \cdot p_4) = E_1 E_4 - \mathbf{p}_1 \cdot \mathbf{p}_4 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

$$(p_2 \cdot p_3) = E_2 E_3 - \mathbf{p}_2 \cdot \mathbf{p}_3 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

$$= \frac{g_Z^4}{2M_Z^4}((C_A^e)^2 + (C_V^e)^2) \times (4E^4 + E^4(1 + \cos\theta)^2)$$

$$+ \frac{g_Z^4}{M_Z^4} C_A^e C_V^e (4E^4 - E^4(1 + \cos\theta)^2)$$

$$= \frac{g_Z^4}{2M_Z^4}((C_A^e)^2 + (C_V^e)^2) \times (4E^4 + E^4(1 + \cos\theta)^2)$$

$$+ \frac{g_Z^4}{M_Z^4}C_A^e C_V^e (4E^4 - E^4(1 + \cos\theta)^2)$$

Use half angle formula $\cos\frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}}$

$$= \frac{g_Z^4 E^4}{2M_Z^4}((C_A^e)^2 + (C_V^e)^2) \times (4 + 4\cos^4\frac{\theta}{2}) + \frac{g_Z^4 E^4}{M_Z^4}C_A^e C_V^e (4 - 4\cos^4\frac{\theta}{2})$$

$$< |\mathcal{M}|^2 > = \frac{g_Z^4 E^4}{M_Z^4} ((C_A^e)^2 + (C_V^e)^2)(2 + 2\cos^4 \frac{\theta}{2})$$

$$+ \frac{g_Z^4 E^4}{M_Z^4} C_A^e C_V^e (4 - 4\cos^4 \frac{\theta}{2})$$

$$< |\mathcal{M}|^2 > = \frac{2g_Z^4 E^4}{M_Z^4} ((C_A^e)^2 + (C_V^e)^2 + 2C_A^e C_V^e) + \cos^4 \frac{\theta}{2} ((C_A^e)^2 + (C_V^e)^2 - 2C_A^e C_V^e)$$

$$< |\mathcal{M}|^2 > = \frac{2g_Z^4 E^4}{M_Z^4} ((C_A^e + C_V^e)^2) + \cos^4 \frac{\theta}{2} (C_A^e - C_V^e)^2)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|M|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{1}{4E^2} \frac{2g_Z^4 E^4}{M_Z^4} ((C_A^e + C_V^e)^2) + \cos^4 \frac{\theta}{2} (C_A^e - C_V^e)^2)$$

Differential xsection to xsection

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{1}{4E^2}\frac{2g_Z^4 E^4}{M_Z^4}((C_A^e + C_V^e)^2) + \cos^4\frac{\theta}{2}(C_A^e - C_V^e)^2)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{128\pi^2}\frac{g_Z^4 E^2}{M_Z^4}((C_A^e + C_V^e)^2) + \cos^4\frac{\theta}{2}(C_A^e - C_V^e)^2)$$

Φ →

$$\sigma = \frac{2\pi}{128\pi^2}\frac{g_Z^4 E^2}{M_Z^4}\int_0^\pi [((C_A^e + C_V^e)^2) + \cos^4\frac{\theta}{2}(C_A^e - C_V^e)^2)]\sin\theta\, d\theta$$

First piece of integral (sin theta) gives 2

$$\int_0^\pi \cos^4\frac{\theta}{2}\sin\theta\, d\theta = \frac{1}{4}\int_0^\pi (1 + \cos\theta)^2\sin\theta\, d\theta$$

$$= \frac{1}{4}\int_0^\pi [\overset{2}{\sin}\theta + 2\cos\theta\sin\theta + \cos^2\theta\sin\theta]d\theta$$

Differential xsection to xsection

$$\frac{1}{4}\int_0^\pi [\sin\theta + 2\cos\theta\sin\theta + \cos^2\theta\sin\theta]d\theta = \frac{1}{4}[2 + \int_0^\pi 2\cos\theta\sin\theta d\theta + \cos^2\theta\sin\theta d\theta]$$

$$u = \cos\theta, du = -\sin\theta d\theta$$

$$\frac{1}{4}[2 + \int_0^\pi (-2u - u^2)du]$$

$$\frac{1}{2} + \frac{1}{4}[-\cos^2\theta]_{\theta=0}^{\theta=\pi} + \frac{1}{4}[-\cos^3\theta/3]_{\theta=0}^{\theta=\pi} = \frac{1}{2} + \frac{1}{4}\frac{1}{3}(2) = \frac{2}{3}$$

$$\sigma = \frac{2\pi}{128\pi^2}\frac{g_Z^4 E^2}{M_Z^4}(2(C_A^e + C_V^e)^2) + \frac{2}{3}(C_A^e - C_V^e)^2)$$

$$\sigma = \frac{1}{64\pi}\frac{g_Z^4 E^2}{M_Z^4}\frac{8}{3}[(C_A^e)^2 + (C_V^e)^2 + (C_V^e)(C_A^e)]$$

$$\boxed{\sigma = \frac{1}{24\pi}\frac{g_Z^4 E^2}{M_Z^4}[(C_A^e)^2 + (C_V^e)^2 + (C_V^e)(C_A^e)]}$$

Z boson not discovered until 1983! Why? Most of the interesting processes are masked by electromagnetism. Need lots of energy to reach masses where you are near Z pole (near Z resonance)...

$$(2\pi)^4 \delta^4(p_3 + p_4 - q)$$

e^+ p2

$\overline{v}$

$\dfrac{-i(g_{\mu\nu} - q_\mu q_\nu/M^2)}{q^2 - M^2 + iM\Gamma}$

$\dfrac{-ig_Z}{2}\gamma^\mu(c_v^e - c_a^e\gamma^5)$

p4   f

$\overline{u}$

$\dfrac{-ig_Z}{2}\gamma^\nu(c_v^f - c_a^f\gamma^5)$

$\dfrac{1}{(2\pi)^4}d^4q$

q Z p3

p1

$e^- u$ $\dfrac{-ig_Z}{2}\gamma^\mu(c_v^\nu - c_a^\nu\gamma^5)$

v

f

$$(2\pi)^4 \delta^4(p_1 + p_2 - q)$$

i
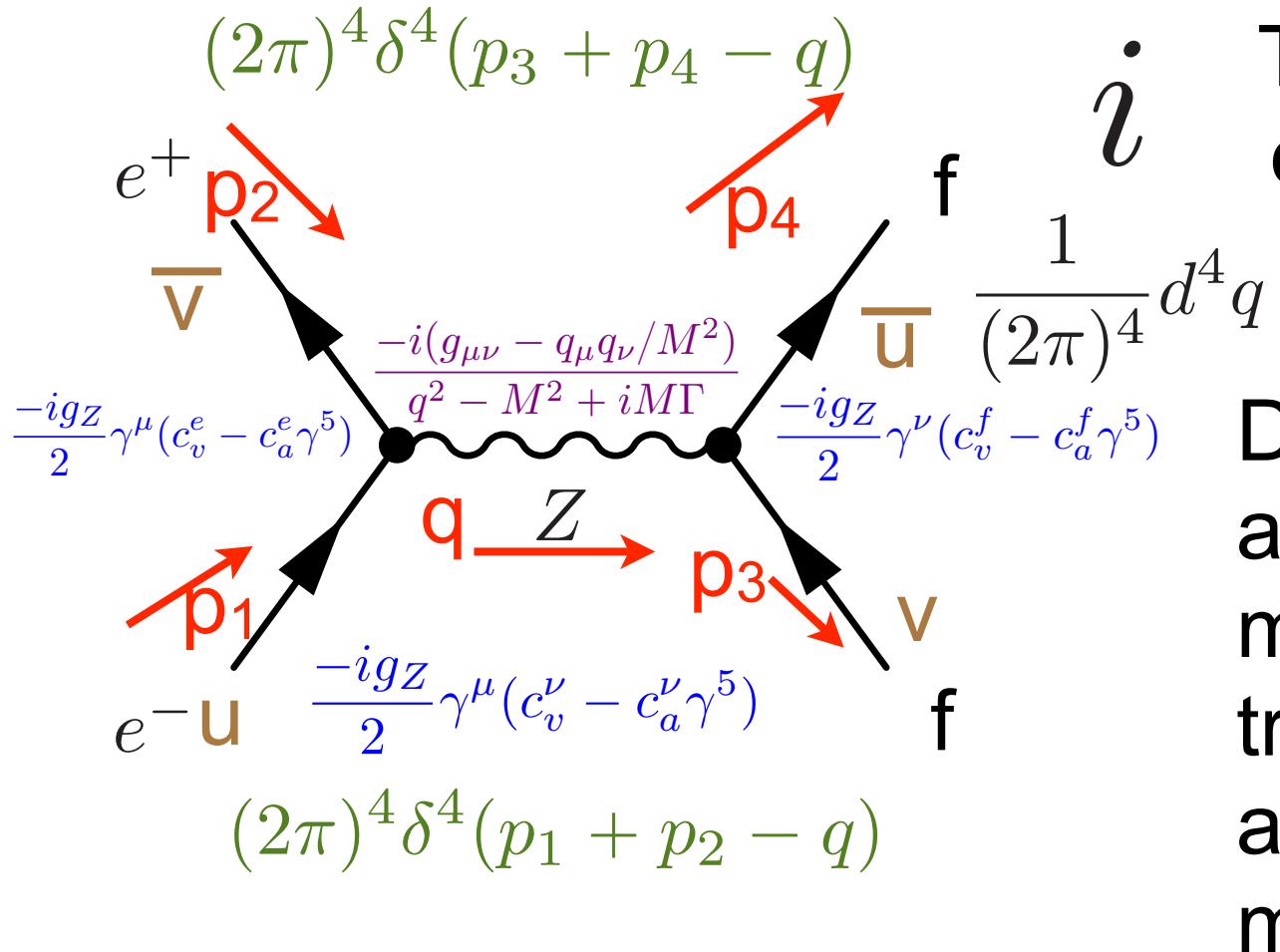
The LEP collider! electron-positron collisions

Do NOT assume small momentum transfer! But assume fermion mass = 0

$$\mathcal{M} = \int i[\overline{u}(4)][-i\frac{g_Z}{2}\gamma^\nu(c_V^f - c_A^f\gamma^5)][v(3)]\left(\frac{-ig_{\mu\nu} - q_\mu q_\nu/M^2}{q^2 - M^2 + iM\Gamma}\right)[\overline{v}(2)][-i\frac{g_Z}{2}\gamma^\mu(c_V^e - c_A^e\gamma^5)][u(1)]$$

$$(2\pi)^4\delta^4(p_1 + p_2 - q)(2\pi)^4\delta^4(p_3 + p_4 - q)\frac{d^4q}{(2\pi)^4}$$

New term in propagator

$$\mathcal{M} = \int i[\overline{u}(4)][-i\frac{g_Z}{2}\gamma^\nu(c_V^f - c_A^f\gamma^5)][v(3)]\left(\frac{-ig_{\mu\nu} - q_\mu q_\nu/M^2}{q^2 - M^2 + iM\Gamma}\right)[\overline{v}(2)][-i\frac{g_Z}{2}\gamma^\mu(c_V^e - c_A^e\gamma^5)][u(1)]$$

$$(2\pi)^4\delta^4(p_1 + p_2 - q)(2\pi)^4\delta^4(p_3 + p_4 - q)\frac{d^4q}{(2\pi)^4}$$

$$[\overline{u}(4)][\gamma^\nu(c_A^f - c_V^f\gamma^5)][v(3)]q_\mu q_\nu = [\overline{u}(4)][\slashed{q}(c_A^f - c_V^f\gamma^5)][v(3)]q_\mu =$$

$$[\overline{u}(4)][(\slashed{p}_3 + \slashed{p}_4)(c_A^f - c_V^f\gamma^5)][v(3)]q_\mu$$

BUT $[\overline{u}(4)]\slashed{p}_4 = 0$ for m = 0

Why? Recall Dirac equation…

$$\overline{u}(\gamma^\mu k_\mu - m) = 0$$
$$\overline{v}(\gamma^\mu k_\mu + m) = 0$$

Plug in m=0

$$\mathcal{M} = \int i[\overline{u}(4)][-i\frac{g_Z}{2}\gamma^\nu(c_V^f - c_A^f\gamma^5)][v(3)] \left(\frac{-ig_{\mu\nu} - q_\mu q_\nu/M^2}{q^2 - M^2 + iM\Gamma} \right) [\overline{v}(2)][-i\frac{g_Z}{2}\gamma^\mu(c_V^e - c_A^e\gamma^5)][u(1)]$$

$$(2\pi)^4\delta^4(p_1 + p_2 - q)(2\pi)^4\delta^4(p_3 + p_4 - q)\frac{d^4q}{(2\pi)^4}$$

$$[\overline{u}(4)][\gamma^\nu(c_A^f - c_V^f\gamma^5)][v(3)]q_\mu q_\nu = [\overline{u}(4)][\slashed{q}(c_A^f - c_V^f\gamma^5)][v(3)]q_\mu =$$

$$[\overline{u}(4)][(\slashed{p}_4 + \slashed{p}_3)(c_A^f - c_V^f\gamma^5)][v(3)]q_\mu = [\overline{u}(4)][(c_A^f + c_V^f\gamma^5)]\slashed{p}_3[v(3)]q_\mu$$

$$\slashed{p}_3[v(3)] = \overline{u}(4)\slashed{p}_4 = 0 \quad \text{for m = 0}$$

$$\mathcal{M} = \int i[\overline{u}(4)][-i\frac{g_Z}{2}\gamma^\nu(c_V^f - c_A^f\gamma^5)][v(3)] \left(\frac{-ig_{\mu\nu}}{q^2 - M^2 + iM\Gamma} \right) [\overline{v}(2)][-i\frac{g_Z}{2}\gamma^\mu(c_V^e - c_A^e\gamma^5)][u(1)]$$

$$(2\pi)^4\delta^4(p_1 + p_2 - q)(2\pi)^4\delta^4(p_3 + p_4 - q)\frac{d^4q}{(2\pi)^4}$$

$$\mathcal{M} = -\frac{g_z^2}{4(q^2 - M^2 + iM\Gamma)}[\overline{u}(4)][\gamma_\mu(c_V^f - c_A^f\gamma^5)][v(3)][\overline{v}(2)][\gamma^\mu(c_V^e - c_A^e\gamma^5)][u(1)]$$

$$\mathcal{M} = -\frac{g_z^2}{4(q^2 - M^2 + iM\Gamma)}[\overline{u}(4)][\gamma_\mu(c_V^f - c_A^f\gamma^5)][v(3)][\overline{v}(2)][\gamma^\mu(c_V^e - c_A^e\gamma^5)][u(1)]$$

After averaging over spins

$$<|\mathcal{M}|^2> = \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times$$

$$Tr[\gamma_\mu(C_V^f - C_A^f\gamma^5)\not{p}_3\gamma_\nu(c_V^f - c_A^f\gamma^5)\not{p}_4] \times$$

$$Tr[\gamma^\mu(C_V^e - C_e^f\gamma^5)\not{p}_1\gamma^\nu(c_V^e - c_A^e\gamma^5)\not{p}_2]$$

Thankfully, we calculated this before...

$$Tr[\gamma_\mu(C_A^e - C_V^e\gamma^5)(\not{p}_2)(\gamma_\sigma(C_A^e - C_V^e\gamma^5)(\not{p}_4)] =$$

$$4((C_V^e)^2 + (C_A^e)^2)(p_{2\mu}p_{4\sigma} - g_{\mu\sigma}(p_2 \cdot p_4) + p_{2\sigma}p_{4\mu}) + 8iC_A^eC_V^e\epsilon_{\mu\nu\sigma\alpha}p_2^\nu p_4^\alpha$$

$$<|\mathcal{M}|^2> = \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times$$

$$Tr[\gamma_\mu(C_V^f - C_A^f\gamma^5)\not{p}_3\gamma_\nu(c_V^f - c_A^f\gamma^5)\not{p}_4] \times$$

$$Tr[\gamma^\mu(C_V^e - C_e^f\gamma^5)\not{p}_1\gamma^\nu(c_V^e - c_A^e\gamma^5)\not{p}_2]$$

$$<|\mathcal{M}|^2> = \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times$$

$$[4((C_V^f)^2 + (C_A^f)^2)(p_{3\mu}p_{4\nu} + p_{4\mu}p_{3\nu} - g_{\mu\nu}(p_3 \cdot p_4))$$

$$+8iC_V^f C_A^f \epsilon_{\mu\sigma\nu\alpha}p_3^\sigma p_4\alpha] \times$$

$$[4((C_V^e)^2 + (C_A^e)^2)(p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - g_{\mu\nu}(p_1 \cdot p_2))$$

$$+8iC_V^e C_A^e \epsilon^{\mu\lambda\nu\beta}p_{1\lambda}p_{2\beta}]$$

$$< |\mathcal{M}|^2 > = \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times$$

$$\left[4((C_V^f)^2 + (C_A^f)^2)(p_{3\mu}p_{4\nu} + p_{4\mu}p_{3\nu} - g_{\mu\nu}(p_3 \cdot p_4))\right.$$

$$\left. + 8iC_V^f C_A^f \epsilon_{\mu\sigma\nu\alpha}p_3^\sigma p_4\alpha\right] \times$$

$$\left[4((C_V^e)^2 + (C_A^e)^2)(p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - g_{\mu\nu}(p_1 \cdot p_2))\right.$$

$$\left. + 8iC_V^e C_A^e \epsilon^{\mu\lambda\nu\beta}p_{1\lambda}p_{2\beta}\right]$$

$$< |\mathcal{M}|^2 > = \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times [$$

$$16((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_2 \cdot p_3)(p_1 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4)$$

$$+ (p_1 \cdot p_4)(p_2 \cdot p_3) + (p_2 \cdot p_4)(p_1 \cdot p_3) - (p_1 \cdot p_2)(p_3 \cdot p_4)$$

$$- (p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) + 4(p_1 \cdot p_2)(p_3 \cdot p_4)]$$

$$+ 4((C_V^f)^2 + (C_A^f)^2)(8iC_V^f C_A^f \epsilon_{\mu\sigma\nu\alpha}p_3^\sigma p_4^\alpha)(p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - g_{\mu\nu}(p_1 \cdot p_2))$$

$$+ 4((C_V^e)^2 + (C_A^e)^2)(8iC_V^e C_A^e \epsilon^{\mu\lambda\nu\beta}p_{1\lambda}p_{2\beta}(p_{3\mu}p_{4\nu} + p_{4\mu}p_{3\nu} - g_{\mu\nu}(p_3 \cdot p_4))$$

$$- 64C_V^e C_A C_V^f C_A^f \epsilon_{\mu\sigma\nu\alpha}p_3^\sigma p_4^\alpha \epsilon^{\mu\lambda\nu\beta}p_{1\lambda}p_{2\beta}]$$

As before

$$< |\mathcal{M}|^2 > = \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times [$$

$$16((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2(p_2 \cdot p_3)(p_1 \cdot p_4)]$$

$$-64C_V^e C_A C_V^f C_A^f \epsilon_{\mu\sigma\nu\alpha} p_3^\sigma p_4^\alpha \epsilon^{\mu\lambda\nu\beta} p_{1\lambda} p_{2\beta}]$$

Recall:

$$\epsilon^{\mu\beta\sigma\lambda}\epsilon_{\mu\nu\sigma\alpha} = (-\epsilon^{\mu\sigma\beta\lambda} - \epsilon_{\mu\sigma\nu\alpha}) = \epsilon^{\mu\sigma\beta\lambda}\epsilon_{\mu\sigma\nu\alpha} = -2(\delta_\nu^\beta \delta_\alpha^\lambda - \delta_\alpha^\beta \delta_\nu^\lambda)$$

$$< |\mathcal{M}|^2 > = \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times [$$

$$16((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2(p_2 \cdot p_3)(p_1 \cdot p_4)]$$

$$+128C_V^e C_A C_V^f C_A^f ((p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3))]$$

Assume massless fermions and electrons. Pick center of mass frame where $E_1 = E_2 = E_3 = E_4 = E$ and $|\mathbf{p}_i| = E$
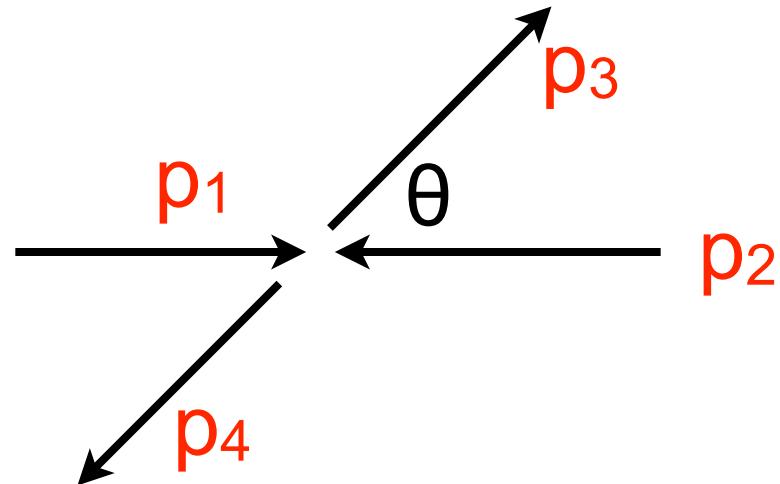
p_1

p_3

θ

p_2

p_4

$$(p_1 \cdot p_3) = E_1 E_3 - \mathbf{p}_1 \cdot \mathbf{p}_3 = E^2 - E^2 \cos\theta = E^2(1 - \cos\theta)$$

$$(p_2 \cdot p_4) = E_2 E_4 - \mathbf{p}_2 \cdot \mathbf{p}_4 = E^2 - E^2 \cos\theta = E^2(1 - \cos\theta)$$

$$(p_2 \cdot p_3) = E_2 E_3 - \mathbf{p}_2 \cdot \mathbf{p}_3 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

$$(p_1 \cdot p_4) = E_1 E_4 - \mathbf{p}_1 \cdot \mathbf{p}_4 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

q=p₁+p₂

and need to be a bit more careful about the magnitude of the imaginary number!

$$< |\mathcal{M}|^2 > = \frac{g_z^4}{64(q^2 - M^2 + iM\Gamma)^2} \times [$$

$$16((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2(p_2 \cdot p_3)(p_1 \cdot p_4)]$$

$$+128 C_V^e C_A C_V^f C_A^f ((p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3)]$$

$$(p_1 \cdot p_3) = E_1 E_3 - \mathbf{p}_1 \cdot \mathbf{p}_3 = E^2 - E^2 \cos\theta = E^2(1 - \cos\theta)$$

$$(p_2 \cdot p_4) = E_2 E_4 - \mathbf{p}_2 \cdot \mathbf{p}_4 = E^2 - E^2 \cos\theta = E^2(1 - \cos\theta)$$

$$(p_2 \cdot p_3) = E_2 E_3 - \mathbf{p}_2 \cdot \mathbf{p}_3 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

$$(p_1 \cdot p_4) = E_1 E_4 - \mathbf{p}_1 \cdot \mathbf{p}_4 = E^2 + E^2 \cos\theta = E^2(1 + \cos\theta)$$

$$q^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2(p_1 \cdot p_2) = 4E^2$$

$$< |\mathcal{M}|^2 > = \frac{g_z^4}{64(4E^2 - M^2)^2 + 64M^2\Gamma^2} \times [$$

$$16((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[2E^4(1 - \cos\theta)^2 + 2E^4(1 + \cos\theta)^2]$$

$$+128C_V^e C_A C_V^f C_A^f (E^4(1 - \cos\theta)^2 - E^4(1 + \cos\theta)^2)]$$

$$< |\mathcal{M}|^2 > = \frac{g_z^4 E^4}{2(4E^2 - M^2)^2 + 2M^2\Gamma^2} \times [$$

$$((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[(1 - \cos\theta)^2 + (1 + \cos\theta)^2]$$

$$+4C_V^e C_A C_V^f C_A^f ((1 - \cos\theta)^2 - (1 + \cos\theta)^2)]$$

$$< |\mathcal{M}|^2 > = \frac{g_z^4 E^4}{2(4E^2 - M^2)^2 + 2M^2\Gamma^2} \times [$$

$$((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[2 + 2\cos^2\theta]$$

$$+4C_V^e C_A C_V^f C_A^f (-4\cos\theta)$$

$$< |\mathcal{M}|^2 > = \frac{g_z^4 E^4}{2(4E^2 - M^2)^2 + 2M^2\Gamma^2} \times [$$

$$((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[2 + 2\cos^2\theta]$$

$$+4C_V^e C_A C_V^f C_A^f(-4\cos\theta)$$

$$< |\mathcal{M}|^2 > = \frac{g_z^4 E^4}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta\right]$$

$$< |\mathcal{M}|^2 > = \frac{g_z^4 E^4}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta \right]$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|M|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} \qquad \text{Equal}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2 E^2} \frac{g_z^4 E^4}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta \right]$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2 E^2} \frac{g_z^4 E^4}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta \right]$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2} \frac{g_z^4 E^2}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta \right]$$

Φ

$$\sigma = \frac{2\pi}{256\pi^2} \frac{g_z^4 E^2}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\int_{\theta=0}^{\theta=\pi} \left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta \right] \sin\theta d\theta$$

$$\sigma = \frac{1}{128\pi} \frac{g_z^4 E^2}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$

$$\int_{\theta=0}^{\theta=\pi} \left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2)[1 + \cos^2\theta] - 8C_V^e C_A C_V^f C_A^f \cos\theta \right] \sin\theta d\theta$$

$$\int_{\theta=0}^{\theta=\pi} \sin\theta d\theta = 2$$

$$\int_{\theta=0}^{\theta=\pi} \cos^2\theta \sin\theta d\theta = \frac{2}{3}$$

$$\int_{\theta=0}^{\theta=\pi} \cos\theta \sin\theta d\theta = 0$$

We just evaluated these a handful of slides ago

$$\boxed{\begin{array}{l} \sigma = \dfrac{1}{48\pi} \dfrac{g_z^4 E^2}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times \\[2mm] \left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2) \right] \end{array}}$$

Predictions for branching ratios to specific final states!

$$\sigma = \frac{1}{48\pi} \frac{g_z^4 E^2}{(4E^2 - M^2)^2 + M^2\Gamma^2} \times$$
$$\left[((C_V^f)^2 + (C_A^f)^2)((C_V^e)^2 + (C_A^e)^2) \right]$$

Z boson mass = 91.2 GeV

Z boson width = 2.5 GeV

If E not near M_Z, width term is negligible (at Z pole, it's needed - otherwise we end up with infinite cross section!) Reason width is there is that Z has a finite lifetime (so it's not a stable particle). EM (photon-mediated) contribution to the process is much bigger than neutral current process, except at Z pole
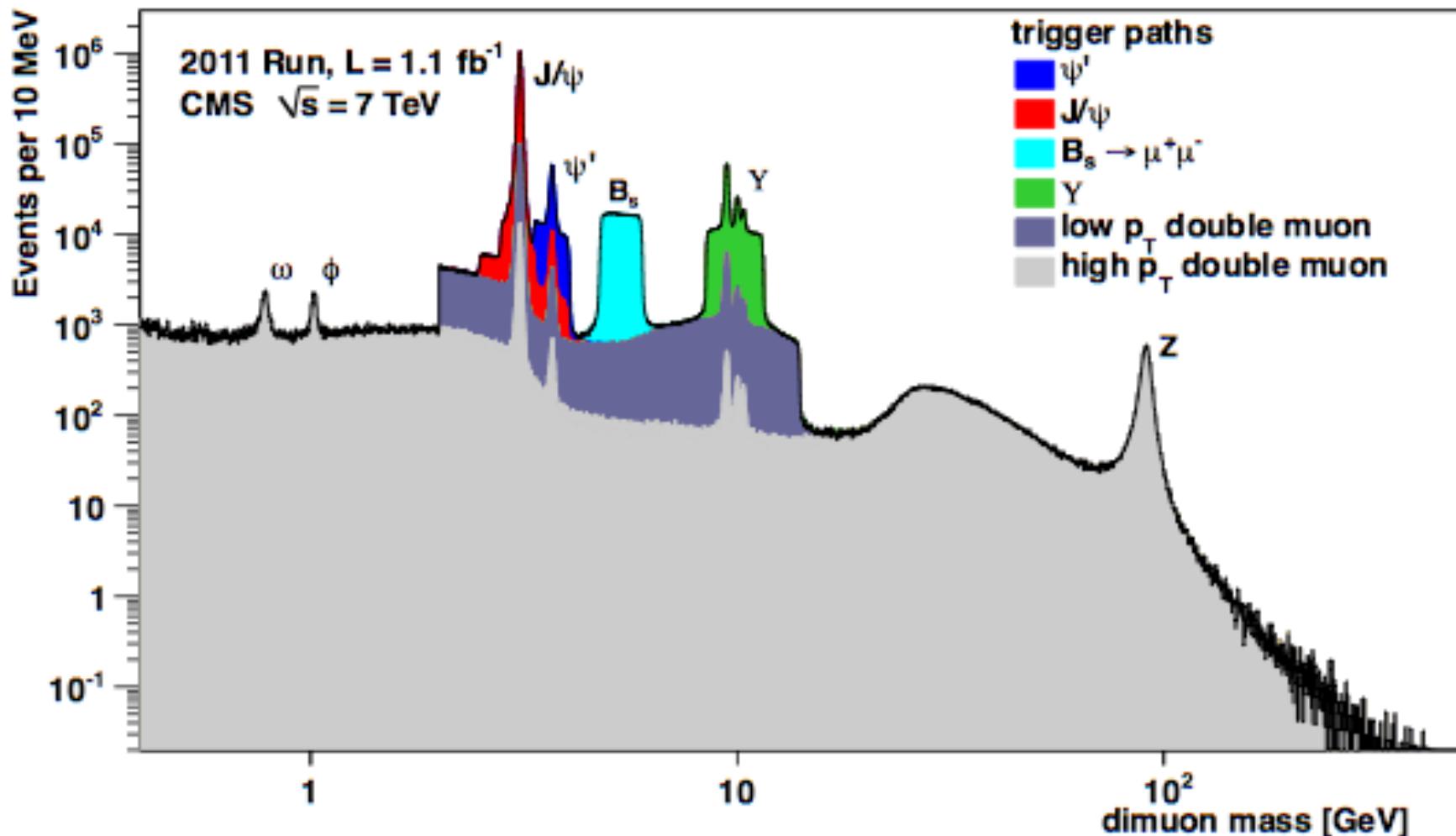
W^+ W^-

Z Z

W^+ W^-

W^+ W^-

W^+ W^-

Z
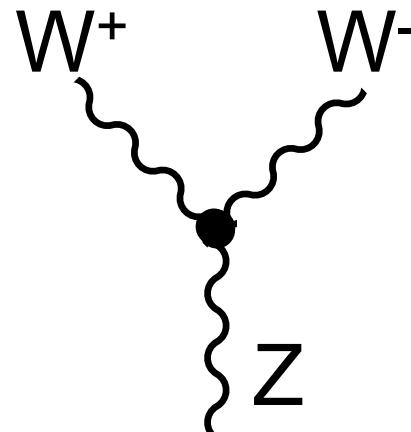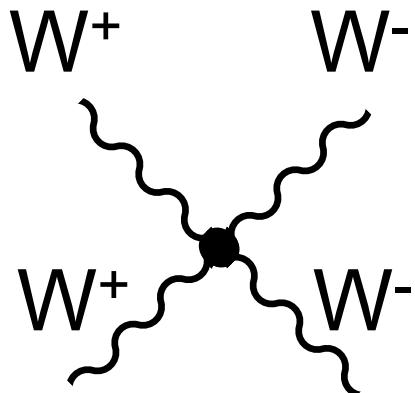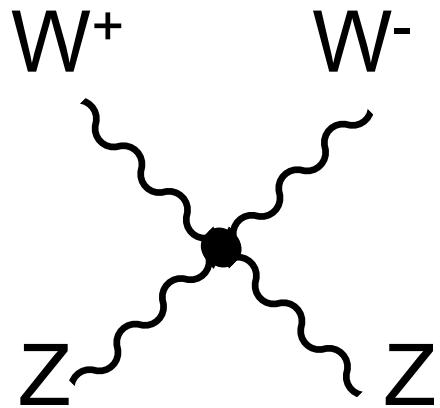
We won't study these vertices, but they exist, too!

W^+ W^-

γ Z

W^+ W^-

γ γ

W^+ W^-

γ
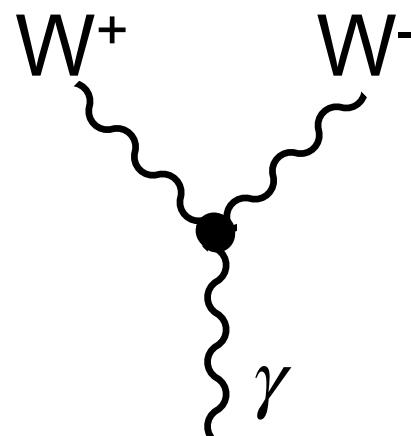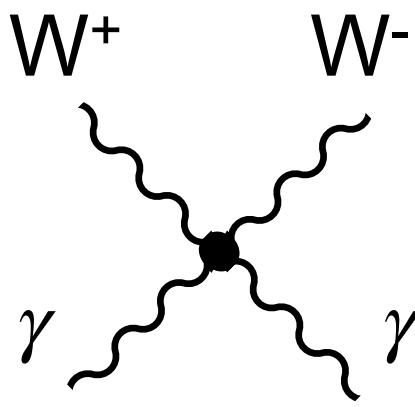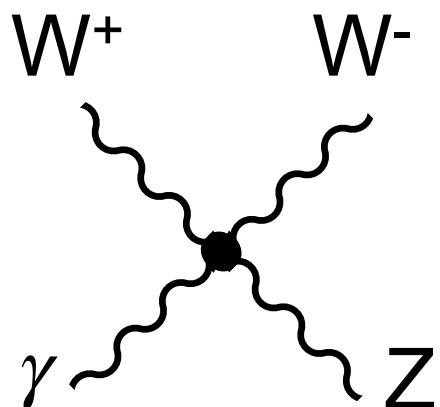
Long history of unifying forces in physics. Most famous example is electricity and magnetism (Maxwell). Einstein (SR) then showed that they are one and the same thing, differing only in reference frame. Wouldn't it be great to unify the weak force and E&M?

First problem - their strength is very different! Solution (Glashow): Give masses to W/Z bosons and not to photon

New problem - why are the masses so different? Solution: next chapter

Weak forces contain both axial and vector components, E&M contains a pure vector component

New solution: redefine our spinors

Remember the Dirac equation solutions:

Electrons

Spin up
$$\psi_1 = e^{-imt} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \psi_2 = e^{-imt} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$
Spin down

Positrons

Spin down
$$\psi_3 = e^{+imt} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \psi_4 = e^{+imt} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$
Spin up

Reminder that helicity h of a particle is the normalized dot product of its spin and direction of flight (h=+1 if spin is in same direction as motion and h=-1 if in opposite direction)

Also remember - helicity is NOT Lorentz invariant (can always change direction of motion in different reference frame unless m = 0)

Spin matrix

$$\hat{h} = \frac{\hat{\Sigma} \cdot \vec{p}}{|\vec{p}|} = \frac{1}{|\vec{p}|} \begin{pmatrix} \vec{\sigma} \cdot \vec{p} & 0 \\ 0 & \vec{\sigma} \cdot \vec{p} \end{pmatrix}$$

Let's find the eigenvalues of the helicity operator

$$\frac{1}{p}\begin{pmatrix} \sigma\cdot\vec{p} & 0 \\ 0 & \sigma\cdot\vec{p} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \lambda\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$(\sigma\cdot\hat{\mathbf{p}})u_A = p\lambda u_A$$

Way back when ...

$$(\sigma\cdot\hat{\mathbf{p}})u_B = p\lambda u_B$$

$$\vec{p}\cdot\vec{\sigma} = p_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$(\vec{p}\cdot\vec{\sigma})^2 = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$(\vec{p}\cdot\vec{\sigma})^2 = \begin{pmatrix} p_z^2 + (p_x - ip_y)(p_x + ip_y) & p_z(p_x - ip_y) - p_z(p_x - ip_y) \\ p_z(p_x + ip_y) - p_z(p_x + ip_y) & (p_x + ip_y)(p_x - ip_y) + p_z^2 \end{pmatrix}$$

$$(\vec{p}\cdot\vec{\sigma})^2 = \mathbf{p}^2 1$$

$$(\sigma \cdot \hat{\mathbf{p}})u_A = p\lambda u_A$$

$$(\sigma \cdot \hat{\mathbf{p}})u_B = p\lambda u_B$$

$$(\sigma \cdot \hat{\mathbf{p}})(\sigma \cdot \hat{\mathbf{p}})u_A = p\lambda(\sigma \cdot \hat{\mathbf{p}})u_A$$

$$(\sigma \cdot \hat{\mathbf{p}})(\sigma \cdot \hat{\mathbf{p}})u_B = p\lambda(\sigma \cdot \hat{\mathbf{p}})u_B$$

$$p^2 u_A = p^2 \lambda^2 u_A$$

$$p^2 u_B = p^2 \lambda^2 u_B$$

Helicity eigenvalue = +/- 1

Also way back when we had for spinor components

$$u_B = \frac{\vec{p} \cdot \vec{\sigma}}{E + m} u_A$$

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^5 u = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_A \\ \frac{\vec{p} \cdot \vec{\sigma}}{E + m} u_A \end{pmatrix}$$

u_A is a 2-component spinor

$$\gamma^5 u = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_A \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} u_A \end{pmatrix}$$

$$\gamma^5 u = \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m} u_A \\ u_A \end{pmatrix}$$

Try this:

$$\begin{pmatrix} \frac{(\vec{p}\cdot\vec{\sigma})}{E+m} & 0 \\ 0 & \frac{(\vec{p}\cdot\vec{\sigma})}{E-m} \end{pmatrix} u = \begin{pmatrix} \frac{(\vec{p}\cdot\vec{\sigma})}{E+m} & 0 \\ 0 & \frac{(\vec{p}\cdot\vec{\sigma})}{E-m} \end{pmatrix} \begin{pmatrix} u_A \\ \frac{(\vec{p}\cdot\vec{\sigma})}{E+m} u_A \end{pmatrix} = \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m} u_A \\ \frac{(\vec{p}\cdot\vec{\sigma})^2}{E^2-m^2} u_A \end{pmatrix} = \begin{pmatrix} \frac{(\vec{p}\cdot\vec{\sigma})}{E+m} u_A \\ \frac{p^2}{p^2} u_A \end{pmatrix}$$

$$\begin{pmatrix} \frac{(\vec{p}\cdot\vec{\sigma})}{E+m} & 0 \\ 0 & \frac{(\vec{p}\cdot\vec{\sigma})}{E-m} \end{pmatrix} u = \begin{pmatrix} \frac{(\vec{p}\cdot\vec{\sigma})}{E+m} u_A \\ u_A \end{pmatrix} = \gamma^5 u$$

What if the particle is massless?

$$\gamma^5 u = \begin{pmatrix} \frac{(\vec{p} \cdot \vec{\sigma})}{E} & 0 \\ 0 & \frac{(\vec{p} \cdot \vec{\sigma})}{E} \end{pmatrix}$$ If m small (or E>>m)

But in that case, we get back helicity operator!

$$E = |\vec{p}| \rightarrow \gamma^5 u = \begin{pmatrix} (\hat{p} \cdot \vec{\sigma}) & 0 \\ 0 & (\hat{p} \cdot \vec{\sigma}) \end{pmatrix}$$

Let's define another two operators

Left-/right-handed projection operators

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5)$$

What is $(\Sigma \cdot \hat{p})(P_\pm u)$ if E>>m?

$$(\Sigma \cdot \hat{p})(P_\pm u) = \frac{1}{2} \begin{pmatrix} (\sigma \cdot \hat{p}) & 0 \\ 0 & (\sigma \cdot \hat{p}) \end{pmatrix} (1 \pm \gamma^5)u$$

$$= \frac{1}{2} \left[\begin{pmatrix} (\sigma \cdot \hat{p}) & 0 \\ 0 & (\sigma \cdot \hat{p}) \end{pmatrix} u \pm \begin{pmatrix} (\sigma \cdot \hat{p}) & 0 \\ 0 & (\sigma \cdot \hat{p}) \end{pmatrix} \gamma^5 u \right]$$

$$= \frac{1}{2} \left[\gamma^5 u \pm (\gamma^5)^2 u \right] = \frac{1}{2} \left[\gamma^5 \pm 1 \right] u = \pm \frac{1}{2}(1 \pm \gamma^5)u = \pm P_\pm u$$

$$(P_+P_+)u = \frac{1}{2}(1+\gamma^5)\frac{1}{2}(1+\gamma^5)u = \frac{1}{4}(1+2\gamma^5+(\gamma^5)^2)u$$

$$(P_+P_+)u = \frac{1}{2}(1+\gamma^5)u = P_+u$$

$$(P_-P_-)u = \frac{1}{2}(1-\gamma^5)\frac{1}{2}(1-\gamma^5)u = \frac{1}{4}(1-2\gamma^5+(\gamma^5)^2)u$$

$$(P_-P_-)u = \frac{1}{2}(1-\gamma^5)u = P_-u$$

$$(P_+P_-)u = \frac{1}{2}(1+\gamma^5)\frac{1}{2}(1-\gamma^5)u = \frac{1}{4}(1-\gamma^5+\gamma^5-(\gamma^5)^2)u = 0$$

$$(P_-P_+)u = \frac{1}{2}(1-\gamma^5)\frac{1}{2}(1+\gamma^5)u = \frac{1}{4}(1+\gamma^5-\gamma^5-(\gamma^5)^2)u = 0$$

$$P_- + P_+ = \frac{1}{2}(1-\gamma^5) + \frac{1}{2}(1+\gamma^5) = 1$$

In massless limit, P₋ picks out the left-handed component of helicity and P₊ picks out the right-handed component of helicity

If we're not in the massless limit, can still define these operators and refer to them as the **chiral** components, even if helicity is not unique. Only in massless limit are the two equivalent

$$u_L(p) = \frac{1}{2}(1 - \gamma^5)u(p)$$

$$u_R(p) = \frac{1}{2}(1 + \gamma^5)u(p)$$

$$v_L(p) = \frac{1}{2}(1 + \gamma^5)v(p)$$

$$v_R(p) = \frac{1}{2}(1 - \gamma^5)v(p)$$

$$\overline{u}_L(p) = \left[\frac{1}{2}(1 - \gamma^5)u(p)\right]^{\dagger}\gamma^0 = u^{\dagger}(p)\frac{1}{2}(1 - \gamma^{5\dagger})\gamma^0$$

$$\overline{u}_L(p) = u^{\dagger}(p)\frac{1}{2}(1 - \gamma^5)\gamma^0 = u^{\dagger}(p)\frac{1}{2}(\gamma^0 - \gamma^5\gamma^0)$$

$$\overline{u}_L(p) = u^{\dagger}(p)\frac{1}{2}(\gamma^0 + \gamma^0\gamma^5) = u^{\dagger}(p)\gamma^0\frac{1}{2}(1 + \gamma^5)$$

$$\overline{u}_L(p) = \overline{u}(p)\frac{1}{2}(1 + \gamma^5)$$

Similarly...
$$\overline{u}_R(p) = \overline{u}(p)\frac{1}{2}(1 - \gamma^5)$$

$$\overline{v}_L(p) = \overline{v}(p)\frac{1}{2}(1 - \gamma^5)$$

$$\overline{v}_R(p) = \overline{v}(p)\frac{1}{2}(1 + \gamma^5)$$

Recall that we had this interaction….

W

e

ν_e
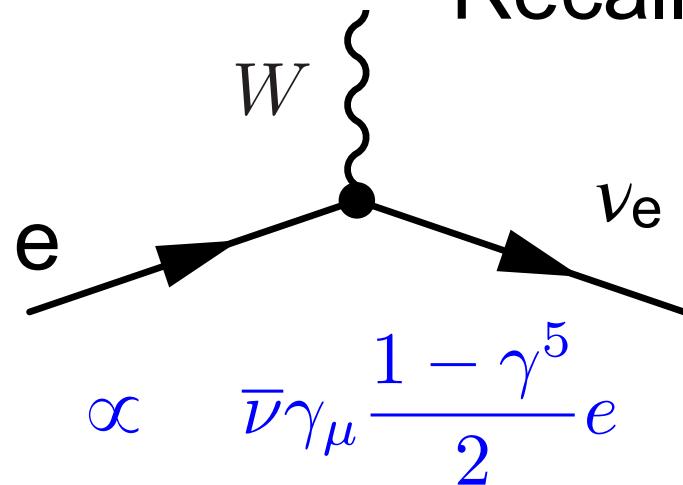
$$\propto \quad \overline{\nu}\gamma_\mu \frac{1-\gamma^5}{2} e$$

Call this the **weak current** j^+_μ

"Bar" is not anti-particle,
but adjoint!

$$\left(\frac{1-\gamma^5}{2}\right)^2 = \left(\frac{1-\gamma^5}{2}\right)$$

$$\gamma_\mu \left(\frac{1-\gamma^5}{2}\right) = \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu$$

$$\rightarrow \gamma_\mu \left(\frac{1-\gamma^5}{2}\right) = \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu \left(\frac{1-\gamma^5}{2}\right)$$

W

ν_e

e
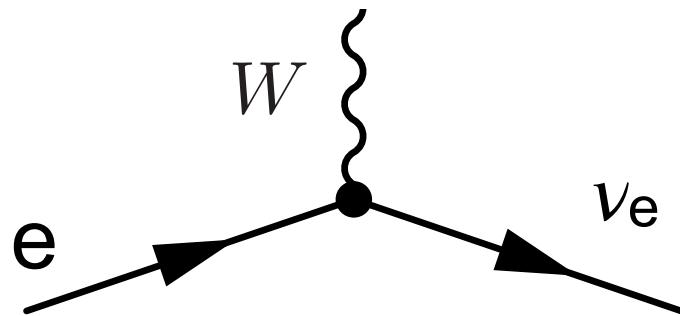
$$j_\mu^+ = \overline{\nu}\gamma_\mu \left(\frac{1-\gamma^5}{2}\right) e = \overline{\nu}\left(\frac{1+\gamma^5}{2}\right)\gamma_\mu\left(\frac{1-\gamma^5}{2}\right) e = \overline{\nu}_L\gamma_\mu e_L$$

So vertex couples left-handed electrons to left-handed neutrinos. This is now **purely** a vector coupling, just like in E&M
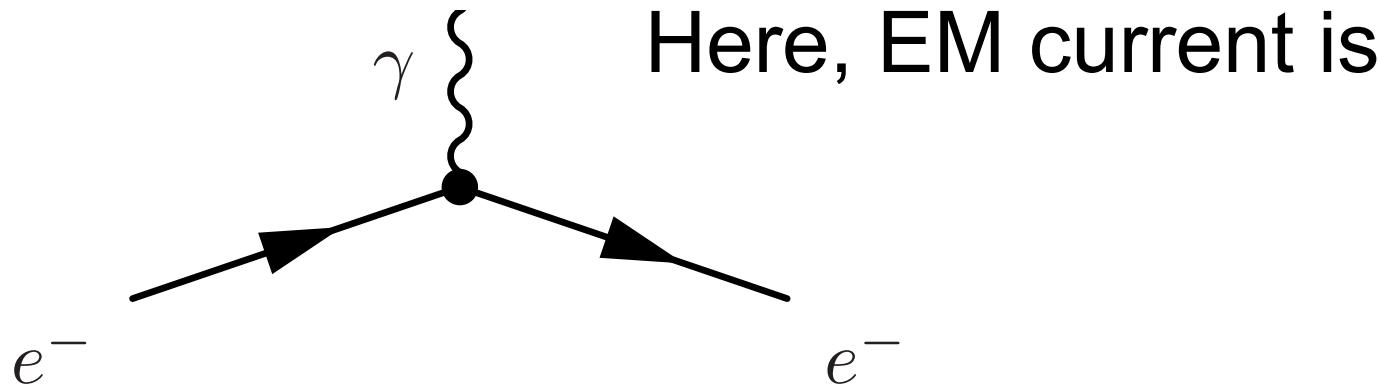
Here, EM current is

$$j_\mu^{EM} = -\bar{e}\gamma_\mu e = -(\bar{e}_L + \bar{e}_R)\gamma_\mu(e_L + e_R)$$

Since, as we saw:

$$\bar{e}_R\gamma_\mu e_L = \bar{e}_L\gamma_\mu e_R = 0$$

Is it clear why that is?

$$j_\mu^- = \overline{e}\gamma_\mu \left(\frac{1 - \gamma^5}{2}\right)\nu = \overline{e}_L\gamma_\mu\nu_L$$

ν_e

W

e
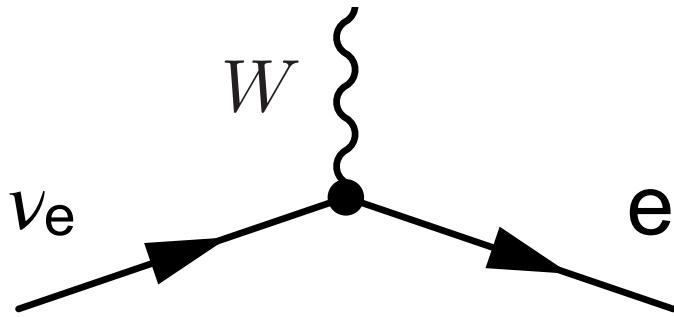
Define: $\quad \chi_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$

Conveniently then write:

$$j_\mu^\pm = \overline{\chi}_L\gamma_\mu\tau^\pm\chi_L$$

$$\tau^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Just like Pauli spin matrices!

$$\tau^\pm = \frac{1}{2}(\tau^1 \pm i\tau^2)$$
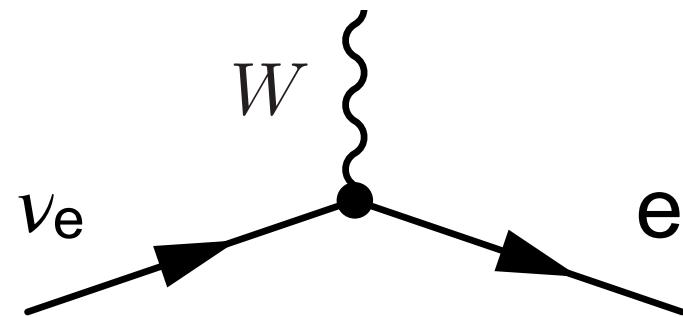
$$j_\mu^- = \overline{e}\gamma_\mu \left(\frac{1-\gamma^5}{2} \right) \nu = \overline{e}_L \gamma_\mu \nu_L$$

$$\chi_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$$

Conveniently then write:

$$j_\mu^\pm = \overline{\chi}_L \gamma_\mu \tau^\pm \chi_L$$

Reminds us perhaps
of isospin? Maybe current from
Z boson is then...

$$\tau^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$j_\mu^3 = \overline{\chi}_L \gamma_\mu \frac{1}{2} \tau^3 \chi_L = \frac{1}{2} \overline{\nu}_L \gamma_\mu \nu_L - \frac{1}{2} \overline{e}_L \gamma_\mu e_L$$

NOPE! Need neutral current to couple
to right-handed components, too...

Symmetry of our new, combined theory!

$$SU(2)_L \otimes U(1)$$

Let's check this definition of Y (weak hypercharge):

$$j_\mu^Y = 2j_\mu^{EM} - 2j_\mu^3 = -2\overline{e}_R\gamma_\mu e_R - \overline{e}_L\gamma_\mu e_L - \overline{\nu}_L\gamma_\mu\nu_L = -2\overline{e}_R\gamma_\mu e_R + \overline{\chi}_L\gamma_\mu\chi_L$$

In total we have three "weak isospin" currents

$$\mathbf{j}_\mu = \frac{1}{2}\overline{\chi}_L\gamma_\mu\tau\chi_L$$

And weak hypercharge current in addition

$$j_\mu^Y = -2\overline{e}_R\gamma_\mu\nu_R + \overline{\chi}_L\gamma_\mu\chi_L$$

Have three weak isospin currents coupling with the same strength to a "weak isotriplet" of vector bosons **W** (coupling g$_W$), and an isosinglet vector boson B coupled to the weak hypercharge current (coupling g'/2). Write as:

$$-i\left[g_W\mathbf{j}_\mu\mathbf{W}^\mu + \frac{g'}{2}j_\mu^y B^\mu\right]$$

$$\mathbf{j}_\mu \mathbf{W}^\mu = j_\mu^1 W^{\mu^1} + j_\mu^2 W^{\mu^2} + j_\mu^3 W^{\mu^3}$$

$$\mathbf{j}_\mu \mathbf{W}^\mu = \frac{1}{\sqrt{2}} j_\mu^+ W^{\mu+} + \frac{1}{\sqrt{2}} j_\mu^- W^{\mu-} + j_\mu^3 W^{\mu^3}$$

W bosons
observed

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \pm i W_\mu^2)$$

Plugging in to the interaction:

$$-i \left[\frac{g_W}{\sqrt{2}} j_\mu^+ W^{\mu+} + \frac{g_W}{\sqrt{2}} j_\mu^- W^{\mu-} + g_W j_\mu^3 W^{\mu^3} + \frac{g'}{2} j_\mu^y B^\mu \right]$$

W

e

ν_e

Back to when we first studied this, we get a term in matrix element from vertex of:
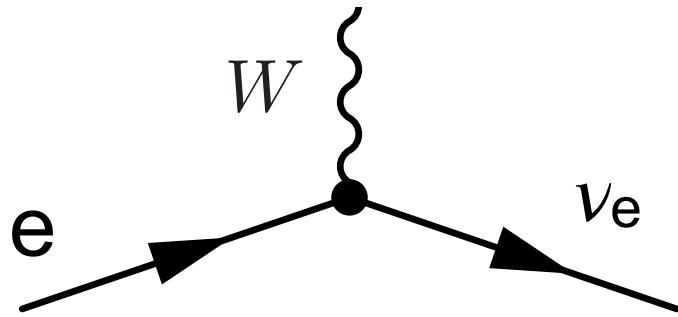
$$-ig_W \frac{1}{2\sqrt{2}} \gamma_\mu (1 - \gamma^5)$$

This is the term of interest (have W⁺)

Vertex factor agrees!

$$j_\mu^+ = \overline{\nu}_L \gamma_\mu e_L = \overline{\nu} \gamma_\mu \frac{(1 - \gamma^5)}{2} e$$

$$-i \left[\frac{g_W}{\sqrt{2}} j_\mu^+ W^{\mu+} + \frac{g_W}{\sqrt{2}} j_\mu^- W^{\mu-} + g_W j_\mu^3 W^{\mu^3} + \frac{g'}{2} j_\mu^y B^\mu \right]$$

They mix! One of them is massless (photon) and other is the massive Z boson

$$A_\mu = B_\mu \cos\theta_W + W_\mu^3 \sin\theta_W$$

$$Z_\mu = -B_\mu \sin\theta_W + W_\mu^3 \cos\theta_W$$

Rewrite:

$$(A_\mu \sin\theta_W + Z_\mu \cos\theta_W) = W_\mu^3 (\sin^2\theta_W + \cos^2\theta_W)$$

$$A_\mu \sin\theta_W + Z_\mu \cos\theta_W = W_\mu^3$$

$$(A_\mu \cos\theta_W - Z_\mu \sin\theta_W) = B_\mu (\cos^2\theta_W + \sin^2\theta_W)$$

$$A_\mu \cos\theta_W - Z_\mu \sin\theta_W = B_\mu$$

$$A_\mu \sin\theta_W + Z_\mu \cos\theta_W = W_\mu^3$$

$$A_\mu \cos\theta_W - Z_\mu \sin\theta_W = B_\mu$$

$$-i\left[\frac{g_W}{\sqrt{2}}j_\mu^+ W^{\mu+} + \frac{g_W}{\sqrt{2}}j_\mu^- W^{\mu-} + g_W j_\mu^3 W^{\mu 3} + \frac{g'}{2}j_\mu^y B^\mu\right]$$

Neutral pieces:

$$-i\left[g_W j_\mu^3\left(A^\mu \sin\theta_W + Z^\mu \cos\theta_W\right) + \frac{g'}{2}j_\mu^y\left(A^\mu \cos\theta_W - Z^\mu \sin\theta_W\right)\right]$$

$$-i\left[A^\mu\left(g_W j_\mu^3 \sin\theta_W + \frac{g'}{2}j_\mu^y \cos\theta_W\right) + Z^\mu\left(g_W j_\mu^3 \cos\theta_W - \frac{g'}{2}j_\mu^y \sin\theta_W\right)\right]$$

$$-i \left[A^{\mu} \left(g_W j_{\mu}^3 \sin \theta_W + \frac{g'}{2} j_{\mu}^y \cos \theta_W \right) + Z^{\mu} \left(g_W j_{\mu}^3 \cos \theta_W - \frac{g'}{2} j_{\mu}^y \sin \theta_W \right) \right]$$

$$j_{\mu}^{em} = j_{\mu}^3 + \frac{1}{2} j_{\mu}^Y$$

Do we remember why?

$$g_W \sin \theta_W = g' \cos \theta_W = g_e$$

Relates the electromagnetic and weak couplings!

$$-i\left[A^\mu\left(g_W j^3_\mu \sin\theta_W + \frac{g'}{2} j^y_\mu \cos\theta_W\right) + Z^\mu\left(g_W j^3_\mu \cos\theta_W - \frac{g'}{2} j^y_\mu \sin\theta_W\right)\right]$$

$$j^Y_\mu = 2j^{EM}_\mu - 2j^3_\mu$$

$$g_W \sin\theta_W = g' \cos\theta_W = g_e$$

$$g_W = g_e/\sin\theta_W$$

$$g' = g_e/\cos\theta_W$$
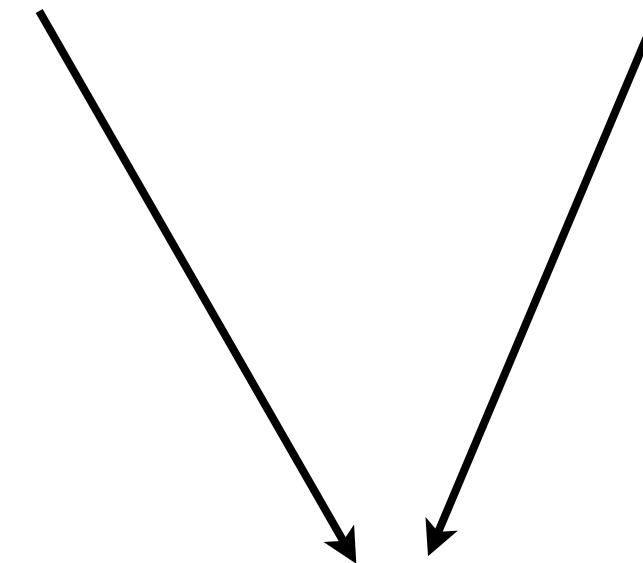
$$\left(g_W j^3_\mu \cos\theta_W - g' j^{EM}_\mu \sin\theta_W + g' j^3_\mu \sin\theta_W\right)$$

$$j^3_\mu\left(g_W \cos\theta_W + g' \sin\theta_W\right) - g' j^{EM}_\mu \sin\theta_W$$

$$j^3_\mu\left(g_e \frac{\cos\theta_W}{\sin\theta_W} + g_e \frac{\sin\theta_W}{\cos\theta_W}\right) - g_e j^{EM}_\mu \frac{\sin\theta_W}{\cos\theta_W}$$

$$j_\mu^3 \left(g_e \frac{\cos \theta_W}{\sin \theta_W} + g_e \frac{\sin \theta_W}{\cos \theta_W} \right) - g_e j_\mu^{EM} \frac{\sin \theta_W}{\cos \theta_W}$$

Define: $$g_Z = \frac{g_e}{\sin \theta_W \cos \theta_W}$$

$$g_Z \left[j_\mu^3 \left(\cos^2 \theta_W + \sin^2 \theta_W \right) - j_\mu^{EM} \sin^2 \theta_W \right]$$

$$g_Z \left[j_\mu^3 - j_\mu^{EM} \sin^2 \theta_W \right]$$

Couplings to
Z boson are
of the form:

$$-ig_Z \left[j_\mu^3 - j_\mu^{EM} \sin^2 \theta_W \right]$$

Recall:
$$j_\mu^3 = \overline{\chi}_L \gamma_\mu \frac{1}{2} \tau^3 \chi_L = \frac{1}{2} \overline{\nu}_L \gamma_\mu \nu_L - \frac{1}{2} \overline{e}_L \gamma_\mu e_L$$

$$\overline{\nu}_L \gamma_\mu \nu_L = \overline{\nu} \frac{1}{2}(1 + \gamma^5) \gamma_\mu \frac{1}{2}(1 - \gamma^5)\nu$$

$$\overline{\nu}_L \gamma_\mu \nu_L = \frac{1}{4} \overline{\nu}(1 + \gamma^5) \gamma_\mu (1 - \gamma^5)\nu$$

$$\overline{\nu}_L \gamma_\mu \nu_L = \frac{1}{4} \overline{\nu} \gamma_\mu (1 - \gamma^5)(1 - \gamma^5)\nu$$

$$\overline{\nu}_L \gamma_\mu \nu_L = \frac{1}{4} \overline{\nu} \gamma_\mu (1 - 2\gamma^5 + (\gamma^5)^2)\nu$$

$$\overline{\nu}_L \gamma_\mu \nu_L = \frac{1}{4} \overline{\nu} \gamma_\mu (1 - 2\gamma^5 + 1)\nu$$

$$\overline{\nu}_L \gamma_\mu \nu_L = \frac{1}{2} \overline{\nu} \gamma_\mu (1 - \gamma^5)\nu$$

So:

$$j_\mu^3 = \frac{1}{4} \left[\overline{\nu} \gamma_\mu (1 - \gamma^5)\nu - \overline{e} \gamma_\mu (1 - \gamma^5)e \right]$$

Couplings to Z boson are of the form:

$$-i g_Z \left[j_\mu^3 - j_\mu^{EM} \sin^2 \theta_W \right]$$

$$j_\mu^{EM} = -\overline{e} \gamma_\mu e \qquad j_\mu^3 = \frac{1}{4} \left[\overline{\nu} \gamma_\mu (1 - \gamma^5) \nu - \overline{e} \gamma_\mu (1 - \gamma^5) e \right]$$

$$-i \frac{g_Z}{4} \left[\overline{\nu} \gamma_\mu (1 - \gamma^5) \nu - \overline{e} \gamma_\mu (1 - \gamma^5) e \right] + i g_Z \sin^2 \theta_W \overline{e} \gamma_\mu e$$

Compare with Neutral weak vertex factor:

$$\frac{-i g_Z}{2} \gamma_\mu (c_V^f - c_A^f \gamma^5)$$

Couplings to Z boson are of the form:
$$\frac{-ig_Z}{4}\left[\overline{\nu}\gamma_\mu(1-\gamma^5)\nu - \overline{e}\gamma_\mu(1-\gamma^5)e\right] - i\sin^2\theta_W\overline{e}\gamma_\mu e$$

Compare with Neutral weak vertex factor:
$$\frac{-ig_Z}{2}\gamma_\mu(c_V^f - c_A^f\gamma^5)$$

For neutrinos:
$$\frac{-ig_Z}{2}\gamma_\mu(c_V^f - c_A^f\gamma^5) = -i\frac{g_Z}{4}\gamma_\mu(1-\gamma^5)$$

$c_V = 1/2, \; c_A = 1/2$

Couplings to Z boson are of the form:

$$\frac{-ig_Z}{4}\left[\overline{\nu}\gamma_\mu(1-\gamma^5)\nu - \overline{e}\gamma_\mu(1-\gamma^5)e\right] - i\sin^2\theta_W\overline{e}\gamma_\mu e$$

Compare with Neutral weak vertex factor:

$$\frac{-ig_Z}{2}\gamma_\mu(c_V^f - c_A^f\gamma^5)$$

For electrons: $\frac{-ig_Z}{2}\gamma_\mu(c_V^f - c_A^f\gamma^5) = +i\frac{g_Z}{4}\gamma_\mu(1-\gamma^5) - i\sin^2\theta_W\gamma_\mu$

$c_V = -1/2 + 2\sin^2\theta_W$, $c_A = -1/2$

Couplings to Z boson are of the form:

$$-ig_Z \left[j_\mu^3 - j_\mu^{EM} \sin^2 \theta_W \right] \quad \chi = \begin{pmatrix} \mathrm{u} \\ \mathrm{d} \end{pmatrix}$$

$$j_\mu^3 = \overline{\chi}_L \gamma_\mu \frac{1}{2} \tau^3 \chi_L = \frac{1}{2} (\overline{u} \, \overline{d})_L \gamma_\mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{d} \end{pmatrix}_L$$

$$j_\mu^3 = \frac{1}{2} (\overline{u} \, \overline{d})_L \gamma_\mu \begin{pmatrix} \mathrm{u} \\ \mathrm{-d} \end{pmatrix}_L$$

$$j_\mu^3 = \frac{1}{2} (\overline{u}_L \gamma_\mu u_L - \overline{d}_L \gamma_\mu d_L)$$

$$j_\mu^3 = \frac{1}{2} \left(\overline{u} \frac{(1+\gamma^5)}{2} \gamma_\mu \frac{(1-\gamma^5)}{2} u - \overline{d} \frac{(1+\gamma^5)}{2} \gamma_\mu \frac{(1-\gamma^5)}{2} d \right)$$

Couplings to Z boson are of the form:

$$-ig_Z \left[j_\mu^3 - j_\mu^{EM} \sin^2 \theta_W \right]$$

$$j_\mu^3 = \frac{1}{2} \left(\overline{u} \frac{(1+\gamma^5)}{2} \gamma_\mu \frac{(1+\gamma^5)}{2} u - \overline{d} \frac{(1-\gamma^5)}{2} \gamma_\mu \frac{(1-\gamma^5)}{2} d \right)$$

$$j_\mu^3 = \frac{1}{8} \left(\overline{u}(1+\gamma^5)\gamma_\mu(1-\gamma^5)u - \overline{d}(1+\gamma^5)\gamma_\mu(1-\gamma^5)d \right)$$

$$j_\mu^3 = \frac{1}{8} \left(\overline{u}\gamma_\mu(1-\gamma^5)(1-\gamma^5)u - \overline{d}\gamma_\mu(1-\gamma^5)(1-\gamma^5)d \right)$$

$$j_\mu^3 = \frac{1}{4} \left(\overline{u}\gamma_\mu(1-\gamma^5)u - \overline{d}\gamma_\mu(1-\gamma^5)d \right)$$

and

$$j_\mu^{EM} = \left(\frac{2}{3}\overline{u}\gamma_\mu u - \frac{1}{3}\overline{d}\gamma_\mu d \right)$$

Couplings to Z boson are of the form:

$$-ig_Z \left[j_\mu^3 - j_\mu^{EM} \sin^2 \theta_W \right]$$

$$j_\mu^3 = \frac{1}{4} \left(\overline{u} \gamma_\mu (1 - \gamma^5) u - \overline{d} \gamma_\mu (1 - \gamma^5) d \right)$$

$$j_\mu^{EM} = \left(\frac{2}{3} \overline{u} \gamma_\mu u - \frac{1}{3} \overline{d} \gamma_\mu d \right)$$

Compare with Neutral weak vertex factor:

$$\frac{-ig_Z}{2} \gamma_\mu (c_V^f - c_A^f \gamma^5)$$

For up quarks:

$c_V = 1/2 - (4/3)\sin^2\theta_w$

$c_A = 1/2$

For down quarks:

$c_V = -1/2 + (2/3)\sin^2\theta_w$

$c_A = -1/2$

f	c_V	c_A
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	$\frac{1}{2}$
e, μ, τ	$\frac{-1}{2} + 2\sin^2\theta_w$	$\frac{-1}{2}$
u, c, t	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_w$	$\frac{1}{2}$
d, s, b	$\frac{-1}{2} + \frac{2}{3}\sin^2\theta_w$	$\frac{-1}{2}$

You might be asking

The symmetries we examined are clearly not intact in the final results (they are 'broken') - why is the photon massless but the W and Z bosons are not? Are we ready for your (presumably) first introduction to field theory? :)