Schrödinger equation

$$E = T + U = \frac{p^2}{2m} + U$$

$$\mathbf{p} \rightarrow -i\nabla, E \rightarrow i\frac{\partial}{dt}$$

$$E\Psi = T\Psi + U\Psi = \frac{p^2}{2m}\Psi + U\Psi$$

$$-\frac{1}{2m}\nabla^2\Psi + U\Psi = i\frac{\partial\Psi}{dt}$$

$$-\frac{1}{2m}\nabla^2\Psi = i\frac{\partial\Psi}{dt} \quad \text{(for free particle)}$$

Many problems with this! Most noticeably:
1) No discussion of spin
2) Energy is the classical definition
3) Equations are not Lorentz invariant (second order in position, first order in time!)

Let's build in Lorentz invariance

$$p^2 = p_\mu p^\mu = m^2$$

$$p_\mu \rightarrow i\partial_\mu = i\frac{\partial}{\partial x^\mu}$$

$$i^2\frac{\partial^2\Psi}{\partial t^2} - i^2\frac{\partial^2\Psi}{\partial x^2} - i^2\frac{\partial^2\Psi}{\partial y^2} - i^2\frac{\partial^2\Psi}{\partial z^2} = m^2\Psi$$

$$\boxed{\nabla^2\Psi - \frac{\partial^2\Psi}{\partial t^2} = m^2\Psi}$$

Klein-Gordon equation. One major problem left over - doesn't account for spin!

Dirac's crazy idea

$$(p^0)^2 - m^2 = 0 = (p^0 + m)(p^0 - m) = (E + m)(E - m) \text{ only for } \mathbf{p} = 0$$

What happens when **p** is not 0?

$$p^\mu p_\mu - m^2 = 0 = (\beta^k p_k + m)(\gamma^\lambda p_\lambda - m)$$

Need to find β and γ (each have 4 components, so 8 numbers in total)

$$\beta^k \gamma^\lambda p_k p_\lambda + m(\gamma^\lambda p_\lambda - \beta^k p_k) - m^2 = 0$$

Dirac's crazy idea rewritten a bit

$$\beta^k \gamma^\lambda p_k p_\lambda + m(\gamma^\lambda p_\lambda - \beta^k p_k) - m^2 = 0$$

Why can I rewrite the dummy indices like this? Let's write out all terms to check...

$$\beta^k \gamma^\lambda p_k p_\lambda + m(\gamma^k - \beta^k) p_k - m^2 = 0$$

$$\beta^k \gamma^\lambda p_k p_\lambda + m(\gamma^k - \beta^k) p_k - m^2 = 0 = p^\mu p_\mu - m^2$$

Note that right-hand side has no p_μ terms, which implies...

$$(\gamma^k - \beta^k) = 0 \rightarrow \gamma^k = \beta^k$$

$$\beta^k \gamma^\lambda p_k p_\lambda = \gamma^k \gamma^\lambda p_k p_\lambda = p^\mu p_\mu$$

Let's write this out

$$\gamma^k \gamma^\lambda p_k p_\lambda = p^\mu p_\mu$$

$$\gamma^k \; p_k \left(\gamma^0 p_0 - \gamma^1 p_1 - \gamma^2 p_2 - \gamma^3 p_3\right) = (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2$$

$$\left(\gamma^0 p_0 - \gamma^1 p_1 - \gamma^2 p_2 - \gamma^3 p_3\right)\left(\gamma^0 p_0 - \gamma^1 p_1 - \gamma^2 p_2 - \gamma^3 p_3\right) = (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2$$

$$\begin{aligned} p_0^2(\gamma^0\gamma^0) + p_1^2(\gamma^1\gamma^1) + p_2^2(\gamma^2\gamma^2) + p_3^2(\gamma^3\gamma^3) + \\ p_0\left(-\gamma^0\gamma^1 p_1 - \gamma^0\gamma^2 p_2 - \gamma^0\gamma^3 p_3\right) + \\ p_1\left(-\gamma^1\gamma^0 p_0 + \gamma^1\gamma^2 p_2 + \gamma^1\gamma^3 p_3\right) + \\ p_2\left(-\gamma^2\gamma^0 p_0 + \gamma^2\gamma^1 p_1 + \gamma^2\gamma^3 p_3\right) + \\ p_3\left(-\gamma^3\gamma^0 p_0 + \gamma^3\gamma^1 p_1 + \gamma^3\gamma^2 p_2\right) = \\ (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 \end{aligned}$$

Let's rewrite that a bit

$$p_0^2(\gamma^0\gamma^0) + p_1^2(\gamma^1\gamma^1) + p_2^2(\gamma^2\gamma^2) + p_3^2(\gamma^3\gamma^3) +$$
$$p_0\left(-\gamma^0\gamma^1 p_1 - \gamma^0\gamma^2 p_2 - \gamma^0\gamma^3 p_3\right) +$$
$$p_1\left(-\gamma^1\gamma^0 p_0 + \gamma^1\gamma^2 p_2 + \gamma^1\gamma^3 p_3\right) +$$
$$p_2\left(-\gamma^2\gamma^0 p_0 + \gamma^2\gamma^1 p_1 + \gamma^2\gamma^3 p_3\right) +$$
$$p_3\left(-\gamma^3\gamma^0 p_0 + \gamma^3\gamma^1 p_1 + \gamma^3\gamma^2 p_2\right) =$$
$$(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2$$

$$p_0^2(\gamma^0\gamma^0) + p_1^2(\gamma^1\gamma^1) + p_2^2(\gamma^2\gamma^2) + p_3^2(\gamma^3\gamma^3) +$$
$$p_0 p_1\left(-\gamma^0\gamma^1 - \gamma^1\gamma^0\right) +$$
$$p_0 p_2\left(-\gamma^0\gamma^2 - \gamma^2\gamma^0\right) +$$
$$p_0 p_3\left(-\gamma^0\gamma^3 - \gamma^3\gamma^0\right) +$$
$$p_1 p_2\left(\gamma^1\gamma^2 + \gamma^2\gamma^1\right) +$$
$$p_1 p_3\left(\gamma^1\gamma^3 + \gamma^3\gamma^1\right) +$$
$$p_2 p_3\left(\gamma^2\gamma^3 + \gamma^3\gamma^2\right) =$$
$$(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2$$

$$(\gamma^0)^2 = 1$$
$$(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$$
$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0, \mu \neq \nu$$

Our solution

Does not hold for ordinary numbers, but it works if they are **matrices**

$$(\gamma^0)^2 = 1$$

$$(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0, \mu \neq \nu$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$i = 1, 2, 3$$

Gamma matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$i = 1, 2, 3$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli spin matrices

Gamma matrices are really 4x4 matrices, easier to write in this form

Full Gamma matrix examples

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^y = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

Back to Dirac equation

$$\gamma^\mu p_\mu - m = 0, p_\mu \to i\partial_\mu$$

$$\boxed{i\gamma^\mu \partial_\mu \psi - m\psi = 0}$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Solution to Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

If wave function does not depend on position, then **p** = 0 (is that clear why?). Then equation becomes

$$i\gamma^0 \partial_0 \psi - m\psi = 0$$

$$i\gamma^0 \frac{\partial \psi}{dt} - m\psi = 0$$

$$i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial \psi}{\partial t} = m\psi$$

Solution to Dirac equation

$$i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial\psi}{\partial t} = m\psi$$

$$i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \partial\psi_A/\partial t \\ \partial\psi_B/\partial t \end{pmatrix} = m\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

Nicely factorizes like this since the gamma matrix is diagonal

$$\psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

Two-component spinors

Solution to Dirac equation

Electrons

Spin up

$$\psi_1 = e^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

Spin down

$$\psi_2 = e^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Positrons

Spin down

$$\psi_3 = e^{+imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Spin up

$$\psi_4 = e^{+imt}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Four solution to zero momentum Dirac equation

$$\partial\psi_A/\partial t = -im\psi_A \rightarrow \psi_A(t) = e^{-imt}\psi_A(0)$$

$$\partial\psi_B/\partial t = im\psi_B \rightarrow \psi_B(t) = e^{+imt}\psi_B(0)$$

Recall that we object was not moving, so energy of ψ_A = m, as expected. But energy of ψ_B = -m!

ψ_A represents particles such as electrons with mass m and positive energy, ψ_B represents anti-particles such as anti-electrons with mass m and negative energy

Let's try and find generic plane wave solutions

$$\psi(x) = ae^{-ik\cdot x}u(k)$$

u(k) is a bi-spinor

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

Write this way to be explicit:

$$\mathbf{k}\cdot\mathbf{r} = k_x r^x + k_y r^y + k_z r^z$$

Don't forget:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \neq \frac{\partial}{\partial x_\mu}$$

$$k\cdot x = k_0 t^t - \mathbf{k}\cdot\mathbf{r}$$

$$\partial_\mu \psi = \frac{\partial}{\partial t}\left[ae^{-i(k_0 t^t - \mathbf{k}\cdot\mathbf{r})}u(k)\right] - \nabla_\mu \left[ae^{-i(k_0 t^t - k_x r^x - k_y r^y - k_z r^y)}u(k)\right]$$

$$\partial_\mu \psi = -ik_0\psi - ik_x\psi - ik_y\psi - ik_z\psi = -ik_\mu\psi$$

Plugging that in

$$i\gamma^\mu(-ik_\mu\psi) - m\psi = 0$$

$$\gamma^\mu(k_\mu - m)\psi = (\gamma^\mu k_\mu - m)\, ae^{-ik\cdot x}u(k) = 0$$

$$(\gamma^\mu k_\mu - m)\, u(k) = 0$$

$$\gamma^\mu k_\mu = \gamma^0 k^0 - \gamma \cdot \mathbf{k} = k^0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \mathbf{k} \cdot \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} k^0 & -\vec{k}\cdot\vec{\sigma} \\ \vec{k}\cdot\vec{\sigma} & -k^0 \end{pmatrix}$$

So...

$$\begin{pmatrix} k^0 - m & -\vec{k}\cdot\vec{\sigma} \\ \vec{k}\cdot\vec{\sigma} & -k^0 - m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

Getting there

$$\begin{pmatrix} k^0 - m & -\vec{k}\cdot\vec{\sigma} \\ \vec{k}\cdot\vec{\sigma} & -k^0 - m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$\begin{pmatrix} (k^0 - m)u_A - (\vec{k}\cdot\vec{\sigma})u_B \\ (\vec{k}\cdot\vec{\sigma})u_A - (k^0 + m)u_B \end{pmatrix} = 0$$

$$(k^0 - m)u_A - (\vec{k}\cdot\vec{\sigma})u_B = 0$$

$$(\vec{k}\cdot\vec{\sigma})u_A - (k^0 + m)u_B = 0$$

$$(k^0 - m)u_A - (\vec{k}\cdot\vec{\sigma})u_B = 0 \rightarrow u_A = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 - m}u_B$$

$$(\vec{k}\cdot\vec{\sigma})u_A - (k^0 + m)u_B = 0 \rightarrow u_B = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 + m}u_A$$

Getting there

$$u_A = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 - m} u_B$$

$$u_B = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 + m} u_A$$

$$u_A = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 - m} \frac{\vec{k}\cdot\vec{\sigma}}{k^0 + m} u_A$$

$$u_A = \frac{(\vec{k}\cdot\vec{\sigma})^2}{(k^0)^2 - m^2} u_A$$

Getting there

$$\vec{k}\cdot\vec{\sigma} = k_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + k_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + k_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{k}\cdot\vec{\sigma} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix}$$

$$\left(\vec{k}\cdot\vec{\sigma}\right)^2 = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix}\begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix}$$

$$\left(\vec{k}\cdot\vec{\sigma}\right)^2 = \begin{pmatrix} k_z^2 + (k_x - ik_y)(k_x + ik_y) & k_z(k_x - ik_y) - k_z(k_x - ik_y) \\ k_z(k_x + ik_y) - k_z(k_x + ik_y) & (k_x + ik_y)(k_x - ik_y) + k_z^2 \end{pmatrix}$$

$$\left(\vec{k}\cdot\vec{\sigma}\right)^2 = \mathbf{k}^2 1$$

$$u_A = \frac{\vec{k}^2}{(k^0)^2 - m^2} u_A \longrightarrow (k^0)^2 - m^2 = \vec{k}^2$$

$$k^2 = m^2 \rightarrow k^\mu = \pm p^\mu$$

Let's get back to our solutions

$$u_A = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 - m}u_B = \frac{\vec{p}\cdot\vec{\sigma}}{E - m}u_B$$

$$u_B = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 + m}u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E + m}u_A$$

Pick positive solution so we don't divide by zero when **p**=0

From before:

$$\vec{k}\cdot\vec{\sigma} = \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

Independent solution 1:

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_B = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 + m}u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E + m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_B = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 + m}u_A = \frac{1}{E + m}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$$

Let's get back to our solutions

$$u_A = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 - m} u_B = \frac{\vec{p}\cdot\vec{\sigma}}{E - m} u_B$$

$$u_B = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 + m} u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E + m} u_A$$

Pick positive solution so we don't divide by zero when **p**=0

From before:

$$\vec{k}\cdot\vec{\sigma} = \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

Independent solution 2:

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, u_B = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 + m} u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E + m} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, u_B = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 + m} u_A = \frac{1}{E + m} \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$

Let's get back to our solutions

$$u_A = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 - m} u_B = \frac{\vec{p}\cdot\vec{\sigma}}{E - m} u_B$$

$$u_B = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 + m} u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E + m} u_A$$

Pick negative solution so we don't divide by zero when **p**=0 (anti-particles)

From before:

$$\vec{k}\cdot\vec{\sigma} = \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

Independent solution 3:

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E - m} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_A = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 - m} u_B = \frac{1}{-E - m} \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$$

Let's get back to our solutions

$$u_A = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 - m} u_B = \frac{\vec{p}\cdot\vec{\sigma}}{E - m} u_B$$

$$u_B = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 + m} u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E + m} u_A$$

Pick negative solution so we don't divide by zero when **p**=0 (anti-particles)

From before:

$$\vec{k}\cdot\vec{\sigma} = \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

Independent solution 4:

$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E - m} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, u_A = \frac{\vec{k}\cdot\vec{\sigma}}{k^0 - m} u_B = \frac{1}{-E - m} \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$

On notation

As in Griffiths, we'll follow notation to use u for particles and v for anti-particles

Normalization of our spinors

Unnormalized 1)

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_B = \frac{\vec{k} \cdot \vec{\sigma}}{k^0 + m} u_A = \frac{1}{E + m} \begin{pmatrix} p_z \\ p_x + i p_y \end{pmatrix}$$

Unnormalized 2)

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, u_B = \frac{\vec{k} \cdot \vec{\sigma}}{k^0 + m} u_A = \frac{1}{E + m} \begin{pmatrix} p_x - i p_y \\ -p_z \end{pmatrix}$$

$$u(1) = N \begin{pmatrix} 1 \\ 0 \\ (p_z)/(E+m) \\ (p_x + i p_y)/(E+m) \end{pmatrix}$$

$$u^\dagger(1) u(1) = |N|^2 \left((1)(0) \left(\frac{p_z}{E+m} \right) \left(\frac{p_x - i p_y}{E+m} \right) \right) \begin{pmatrix} 1 \\ 0 \\ (p_z)/(E+m) \\ (p_x + i p_y)/(E+m) \end{pmatrix}$$

Normalization of our spinors

$$u^\dagger(1)u(1) = |N|^2 \left((1)(0) \left(\frac{p_z}{E+m} \right) \left(\frac{p_x - ip_y}{E+m} \right) \right) \begin{pmatrix} 1 \\ 0 \\ (p_z)/(E+m) \\ (p_x + ip_y)/(E+m) \end{pmatrix}$$

$$u^\dagger(1)u(1) = |N|^2 \left(1 + 0 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2 + p_y^2}{(E+m)^2} \right)$$

$$u^\dagger(1)u(1) = |N|^2 \left(1 + \frac{\mathbf{p^2}}{(E+m)^2} \right)$$

$$u^\dagger(1)u(1) = |N|^2 \left(\frac{\mathbf{p}^2 + (E+m)^2}{(E+m)^2} \right)$$

$$u^\dagger(1)u(1) = |N|^2 \left(\frac{E^2 - m^2 + E^2 + m^2 + 2Em}{(E+m)^2} \right)$$

$$u^\dagger(1)u(1) = |N|^2 \left(\frac{2E^2 + 2Em}{(E+m)^2} \right)$$

$$u^\dagger(1)u(1) = |N|^2 \frac{2E}{E+m}$$

What do we want the spinor to be normalized to? Convenient choice (following Griffiths) is 2E

Normalization of our spinors

$$u^\dagger(1)u(1) = |N|^2 \frac{2E}{E+m} = 2E$$

$$N = \sqrt{E+m}$$

I leave it to you to check that the same normalization holds true for u(2), v(1) and v(2)

System moving in x direction

$$\psi \rightarrow \psi' = S\psi$$

$$S = a_+ + a_-\gamma^0\gamma^1 = \begin{pmatrix} a_+ & 0 & 0 & 0 \\ 0 & a_+ & 0 & 0 \\ 0 & 0 & a_+ & 0 \\ 0 & 0 & 0 & a_+ \end{pmatrix} + a_- \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}$$

$$S = a_+ + a_-\gamma^0\gamma^1 = \begin{pmatrix} a_+ & 0 & 0 & 0 \\ 0 & a_+ & 0 & 0 \\ 0 & 0 & a_+ & 0 \\ 0 & 0 & 0 & a_+ \end{pmatrix} + a_- \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$S = a_+ + a_-\gamma^0\gamma^1 = \begin{pmatrix} a_+ & 0 & 0 & 0 \\ 0 & a_+ & 0 & 0 \\ 0 & 0 & a_+ & 0 \\ 0 & 0 & 0 & a_+ \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & a_- \\ 0 & 0 & a_- & 0 \\ 0 & a_- & 0 & 0 \\ a_- & 0 & 0 & 0 \end{pmatrix}$$

$$a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$$

$$\gamma = \left(1 - v^2\right)^{-1/2}$$

$$S = a_+ + a_-\gamma^0\gamma^1 = \begin{pmatrix} a_+ & 0 & 0 & a_- \\ 0 & a_+ & a_- & 0 \\ 0 & a_- & a_+ & 0 \\ a_- & 0 & 0 & a_+ \end{pmatrix}$$

Why is that? (Problem 7.11)

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

$$i\gamma^\mu \partial'_\mu \psi' - m\psi' = 0 \qquad \partial'_\mu = \frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\nu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x^{\mu'}} \partial_\nu$$

$$\psi' = S\psi$$

$$i\gamma^\mu \frac{\partial x^\nu}{\partial x^{\mu'}} \partial_\nu (S\psi) - m(S\psi) = 0$$

$$S^{-1}\left[i\gamma^\mu \frac{\partial x^\nu}{\partial x^{\mu'}} \partial_\nu (S\psi) - m(S\psi)\right] = 0$$

$$S^{-1}\left[i\gamma^\mu \frac{\partial x^\nu}{\partial x^{\mu'}} \partial_\nu (S\psi) - m(S\psi)\right] = 0 = i\gamma^\nu \partial_\nu \psi - m\psi$$

$$S^{-1}\gamma^\mu \frac{\partial x^\nu}{\partial x^{\mu'}} S = \gamma^\nu \qquad \boxed{S^{-1}\gamma^\mu S \frac{\partial x^\nu}{\partial x^{\mu'}} = \gamma^\nu}$$

Those partial derivatives

$$\frac{\partial x^0}{\partial x^{0'}} = \gamma$$

$$\frac{\partial x^0}{\partial x^{1'}} = \gamma v$$

$$\frac{\partial x^1}{\partial x^{0'}} = \gamma v$$

$$\frac{\partial x^1}{\partial x^{1'}} = \gamma$$

$$\frac{\partial x^2}{\partial x^{2'}} = 1$$

$$\frac{\partial x^3}{\partial x^{3'}} = 1$$

$$S^{-1}\gamma^\mu S\frac{\partial x^\nu}{\partial x^{\mu'}} = \gamma^\nu$$

Others are zero (these just come from the transformation laws)

$$\gamma^0 = S^{-1}\gamma^0 S\frac{\partial x^0}{\partial x^{0'}} + S^{-1}\gamma^1 S\frac{\partial x^0}{\partial x^{1'}} + S^{-1}\gamma^2 S\frac{\partial x^0}{\partial x^{2'}} + S^{-1}\gamma^3 S\frac{\partial x^0}{\partial x^{3'}}$$

$$\gamma^1 = S^{-1}\gamma^0 S\frac{\partial x^1}{\partial x^{0'}} + S^{-1}\gamma^1 S\frac{\partial x^1}{\partial x^{1'}} + S^{-1}\gamma^2 S\frac{\partial x^1}{\partial x^{2'}} + S^{-1}\gamma^3 S\frac{\partial x^1}{\partial x^{3'}}$$

$$\gamma^2 = S^{-1}\gamma^0 S\frac{\partial x^2}{\partial x^{0'}} + S^{-1}\gamma^1 S\frac{\partial x^2}{\partial x^{1'}} + S^{-1}\gamma^2 S\frac{\partial x^2}{\partial x^{2'}} + S^{-1}\gamma^3 S\frac{\partial x^2}{\partial x^{3'}}$$

$$\gamma^3 = S^{-1}\gamma^0 S\frac{\partial x^3}{\partial x^{0'}} + S^{-1}\gamma^1 S\frac{\partial x^3}{\partial x^{1'}} + S^{-1}\gamma^2 S\frac{\partial x^3}{\partial x^{2'}} + S^{-1}\gamma^3 S\frac{\partial x^3}{\partial x^{3'}}$$

Those partial derivatives

$$\frac{\partial x^0}{\partial x^{0'}} = \gamma$$

$$\frac{\partial x^0}{\partial x^{1'}} = \gamma v$$

$$\frac{\partial x^1}{\partial x^{0'}} = \gamma v$$

$$\frac{\partial x^1}{\partial x^{1'}} = \gamma$$

$$\frac{\partial x^2}{\partial x^{2'}} = 1$$

$$\frac{\partial x^3}{\partial x^{3'}} = 1$$

$$\gamma^0 = S^{-1}\gamma^0 S\gamma + S^{-1}\gamma^1 S\gamma v$$

$$\gamma^1 = S^{-1}\gamma^0 S\gamma v + S^{-1}\gamma^1 S\gamma$$

$$\gamma^2 = S^{-1}\gamma^2 S$$

$$\gamma^3 = S^{-1}\gamma^3 S$$

Now multiply by S

$$S\gamma^0 = \gamma^0 S\gamma + \gamma^1 S\gamma v$$

$$S\gamma^1 = \gamma^0 S\gamma v + \gamma^1 S\gamma$$

$$S\gamma^2 = \gamma^2 S$$

$$S\gamma^3 = \gamma^3 S$$

Does our solution satisfy these?

$$S\gamma^0 = \gamma^0 S\gamma + \gamma^1 S\gamma v$$
$$S\gamma^1 = \gamma^0 S\gamma v + \gamma^1 S\gamma$$
$$S\gamma^2 = \gamma^2 S$$
$$S\gamma^3 = \gamma^3 S$$

Useful relation: we've seen it, please remember !):

$$\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu, \mu \neq \nu$$

Ensures that last two equations hold (see why?)

$$S = a_+ + a_- \gamma^0 \gamma^1 = \begin{pmatrix} a_+ & 0 & 0 & 0 \\ 0 & a_+ & 0 & 0 \\ 0 & 0 & a_+ & 0 \\ 0 & 0 & 0 & a_+ \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & a_- \\ 0 & 0 & a_- & 0 \\ 0 & a_- & 0 & 0 \\ a_- & 0 & 0 & 0 \end{pmatrix}$$

$$S = a_+ + a_- \gamma^0 \gamma^1 = \begin{pmatrix} a_+ & 0 & 0 & a_- \\ 0 & a_+ & a_- & 0 \\ 0 & a_- & a_+ & 0 \\ a_- & 0 & 0 & a_+ \end{pmatrix}$$

First equation

$$S\gamma^0 = \gamma^0 S\gamma + \gamma^1 S\gamma v$$

$$\left(a_+ + a_-\gamma^0\gamma^1\right)\gamma^0 = \gamma^0\left(a_+ + a_-\gamma^0\gamma^1\right)\gamma + \gamma^1\left(a_+ + a_-\gamma^0\gamma^1\right)\gamma v$$

Another useful relation:

$$\gamma^\mu\gamma^\mu = 1, \mu = 0$$
$$\gamma^\mu\gamma^\mu = -1, \mu \neq 0$$

$$a_+\gamma^0 + a_-\gamma^0\gamma^1\gamma^0 = \gamma^0 a_+\gamma + \gamma^0 a_-\gamma^0\gamma^1\gamma + \gamma^1 a_+\gamma v + \gamma^1 a_-\gamma^0\gamma^1\gamma v$$

$$a_+\gamma^0 - a_-\gamma^0\gamma^0\gamma^1 = a_+\gamma\gamma^0 + a_-\gamma\gamma^0\gamma^0\gamma^1 + a_+\gamma v\gamma^1 - a_-\gamma v\gamma^1\gamma^1\gamma^0$$

$$a_+\gamma^0 - a_-\gamma^1 = a_+\gamma\gamma^0 + a_-\gamma\gamma^1 + a_+\gamma v\gamma^1 + a_-\gamma v\gamma^0$$

$$a_+\gamma^0 - a_-\gamma^1 = \left(a_+\gamma + a_-\gamma v\right)\gamma^0 + \left(a_-\gamma + a_+\gamma v\right)\gamma^1$$

Rewriting

$$a_+\gamma^0 - a_-\gamma^1 = (a_+\gamma + a_-\gamma v)\,\gamma^0 + (a_-\gamma + a_+\gamma v)\,\gamma^1$$

$$\gamma = \left(1 - v^2\right)^{-1/2} \rightarrow (\gamma)^2 = \frac{1}{1 - v^2}$$

$$1 - v^2 = \frac{1}{(\gamma)^2}$$

$$v^2 = 1 - \frac{1}{(\gamma)^2} = \frac{(\gamma)^2 - 1}{(\gamma)^2}$$

$$v = \frac{\sqrt{(\gamma + 1)(\gamma - 1)}}{\gamma}$$

Rewriting

$$a_+\gamma^0 - a_-\gamma^1 = (a_+\gamma + a_-\gamma v)\,\gamma^0 + (a_-\gamma + a_+\gamma v)\,\gamma^1$$

$$a_+\gamma + a_-\gamma v = \gamma\sqrt{\frac{1}{2}(\gamma+1)} - \gamma\sqrt{\frac{1}{2}(\gamma-1)}\,\frac{\sqrt{(\gamma+1)(\gamma-1)}}{\gamma}$$

$$a_+\gamma + a_-\gamma v = \gamma\sqrt{\frac{1}{2}(\gamma+1)} - \sqrt{\frac{1}{2}(\gamma-1)}\sqrt{(\gamma+1)(\gamma-1)}$$

$$a_+\gamma + a_-\gamma v = \gamma\sqrt{\frac{1}{2}(\gamma+1)} - (\gamma-1)\sqrt{\frac{1}{2}(\gamma+1)}$$

$$a_+\gamma + a_-\gamma v = \sqrt{\frac{1}{2}(\gamma+1)}\,(\gamma-\gamma+1)$$

$$a_+\gamma + a_-\gamma v = \sqrt{\frac{1}{2}(\gamma+1)} = a_+ \quad \checkmark$$

Now the second part

$$a_+\gamma^0 - a_-\gamma^1 = (a_+\gamma + a_-\gamma v)\,\gamma^0 + (a_-\gamma + a_+\gamma v)\,\gamma^1$$

$$a_-\gamma + a_+\gamma v = -\gamma\sqrt{\frac{1}{2}(\gamma-1)} + \gamma\sqrt{\frac{1}{2}(\gamma+1)}\,\frac{\sqrt{(\gamma+1)(\gamma-1)}}{\gamma}$$

$$a_-\gamma + a_+\gamma v = -\gamma\sqrt{\frac{1}{2}(\gamma-1)} + \sqrt{\frac{1}{2}(\gamma+1)}\sqrt{(\gamma+1)(\gamma-1)}$$

$$a_-\gamma + a_+\gamma v = -\gamma\sqrt{\frac{1}{2}(\gamma-1)} + (\gamma+1)\sqrt{\frac{1}{2}(\gamma-1)}$$

$$a_-\gamma + a_+\gamma v = \sqrt{\frac{1}{2}(\gamma-1)}\,(-\gamma+\gamma+1)$$

$$a_-\gamma + a_+\gamma v = \sqrt{\frac{1}{2}(\gamma-1)} = -a_- \quad ✓$$

Second equation

$$S\gamma^0 = \gamma^0 S\gamma + \gamma^1 S\gamma v$$

$$S\gamma^1 = \gamma^0 S\gamma v + \gamma^1 S\gamma$$

$$S\gamma^2 = \gamma^2 S$$

$$S\gamma^3 = \gamma^3 S$$

$$S = a_+ + a_-\gamma^0\gamma^1$$

$$S\gamma^1 = \gamma^0 S\gamma v + \gamma^1 S\gamma$$

$$\left(a_+ + a_-\gamma^0\gamma^1\right)\gamma^1 = \gamma v\gamma^0\left(a_+ + a_-\gamma^0\gamma^1\right) + \gamma\gamma^1\left(a_+ + a_-\gamma^0\gamma^1\right)$$

$$a_+\gamma^1 - a_-\gamma^0 = \gamma v a_+\gamma^0 + \gamma v a_-\gamma^1 + \gamma a_+\gamma^1 + \gamma a_-\gamma^0$$

$$a_+\gamma^1 - a_-\gamma^0 = \gamma^0(\gamma v a_+ + \gamma a_-) + \gamma^1(a_-\gamma v + a_+\gamma)$$

These are the exact conditions we already just worked out!

Phew! Is this an invariant?

$$\psi^\dagger \psi = (\psi_1^* \psi_2^* \psi_3^* \psi_4^*) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$$

NO!

$$\left(\psi^\dagger \psi\right)' = \left(\psi'^\dagger \psi'\right) = (S\psi)^\dagger S\psi = \psi^\dagger S^\dagger S \psi \neq \left(\psi^\dagger \psi\right)$$

Let's check this:

But define: $\overline{\psi} = \psi^\dagger \gamma^0 = ((\psi_1^*)\,(\psi_2^*)\,(-\psi_3^*)\,(-\psi_4^*))$

$$S^\dagger \gamma^0 S = \gamma^0$$

Part of your homework

$$\left(\overline{\psi}\psi\right)' = \left(\psi^\dagger \gamma^0 \psi\right)' = (\psi'^\dagger)\gamma^0 \psi'$$

$$\left(\overline{\psi}\psi\right)' = (S\psi)^\dagger \gamma^0 S\psi = \psi^\dagger S^\dagger \gamma^0 S \psi = \psi^\dagger \gamma^0 \psi = \left(\overline{\psi}\psi\right) \quad ✓$$

What about parity transformation?

Our earlier friend: $$S^{-1}\gamma^{\mu}S\frac{\partial x^{\nu}}{\partial x^{\mu'}} = \gamma^{\nu}$$

$$S = \gamma^0$$

$$x' = -x; y' = -y; z' = -z$$

$$t = t$$

Do we see why?

$$\frac{\partial x^0}{\partial x^{0'}} = 1$$

$$\frac{\partial x^1}{\partial x^{1'}} = -1$$

$$\frac{\partial x^2}{\partial x^{2'}} = -1$$

$$\frac{\partial x^3}{\partial x^{3'}} = -1$$

$$S^{-1}\gamma^0 S = \gamma^0 \rightarrow \gamma^0 S = S\gamma^0$$

$$-S^{-1}\gamma^1 S = \gamma^1 \rightarrow \gamma^1 S = -S\gamma^1$$

$$-S^{-1}\gamma^2 S = \gamma^2 \rightarrow \gamma^2 S = -S\gamma^2$$

$$-S^{-1}\gamma^3 S = \gamma^3 \rightarrow \gamma^3 S = -S\gamma^3$$

All other partial derivatives are zero

How does our spinor transform under P?

$$\psi \rightarrow \gamma^0 \psi$$

$$(\overline{\psi}\psi)' = (\psi^\dagger \gamma^0 \psi)' = (\psi^\dagger)' \gamma^0 \psi' = (\gamma^0 \psi)^\dagger \gamma^0 \gamma^0 \psi$$

$$(\overline{\psi}\psi)' = \psi^\dagger \gamma^{0^\dagger} \gamma^0 \gamma^0 \psi = (\psi^\dagger \gamma^0 \psi) = (\overline{\psi}\psi)$$

So it transforms like a scalar

Can we make a pseudoscalar?

Define: $\overline{\psi}\gamma^5\psi$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$(\overline{\psi}\gamma^5\psi)' = (\psi^\dagger\gamma^0\gamma^5\psi)' = \psi'^\dagger\gamma^{0\prime}\gamma^{5\prime}\psi'$$

$$(\overline{\psi}\gamma^5\psi)' = \psi^\dagger\gamma^{0\dagger}\gamma^0\gamma^5\gamma^0\psi$$

$$(\overline{\psi}\gamma^5\psi)' = \psi^\dagger\gamma^5\gamma^0\psi$$

$$\gamma^5\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = -i\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3 = +i\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3$$

$$\gamma^5\gamma^0 = -i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^5$$

$$(\overline{\psi}\gamma^5\psi)' = -\psi^\dagger\gamma^0\gamma^5\psi = -(\overline{\psi}\gamma^5\psi)$$

Pseudoscalar $\boxed{\overline{\psi}\gamma^5\psi}$

Similarly

Vector $\overline{\psi}\gamma^{\mu}\psi$

Pseudovector $\overline{\psi}\gamma^{\mu}\gamma^{5}\psi$

OK, let's get to QED. A useful table from Griffiths

Electrons	**Positrons**	
$\psi(x) = ae^{-(i/\hbar)p\cdot x}u^{(s)}(p)$	$\psi(x) = ae^{(i/\hbar)p\cdot x}v^{(s)}(p)$	(7.94)

where $s = 1, 2$ for the two spin states. The spinors $u^{(s)}$ and $v^{(s)}$ satisfy the momentum space Dirac equation(s):

$(\gamma^\mu p_\mu - mc)u = 0$	$(\gamma^\mu p_\mu + mc)v = 0$	(7.95)

their adjoints, $\bar{u} = u^\dagger\gamma^0, \bar{v} = v^\dagger\gamma^0$, satisfy

$\bar{u}(\gamma^\mu p_\mu - mc) = 0$	$\bar{v}(\gamma^\mu p_\mu + mc) = 0$	(7.96)

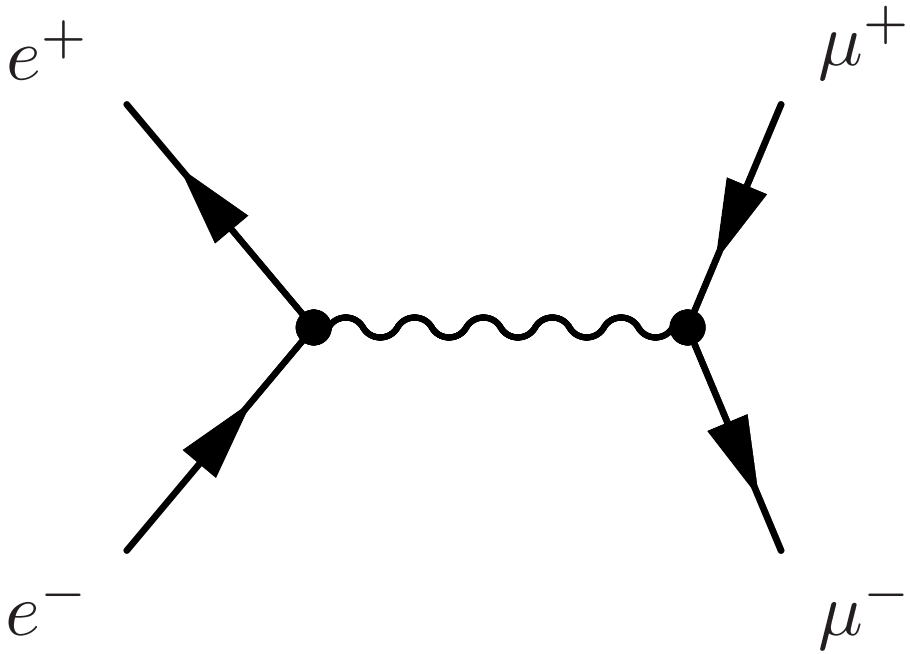
They are orthogonal,

$\bar{u}^{(1)}u^{(2)} = 0$	$\bar{v}^{(1)}v^{(2)} = 0$	(7.97)

normalized,

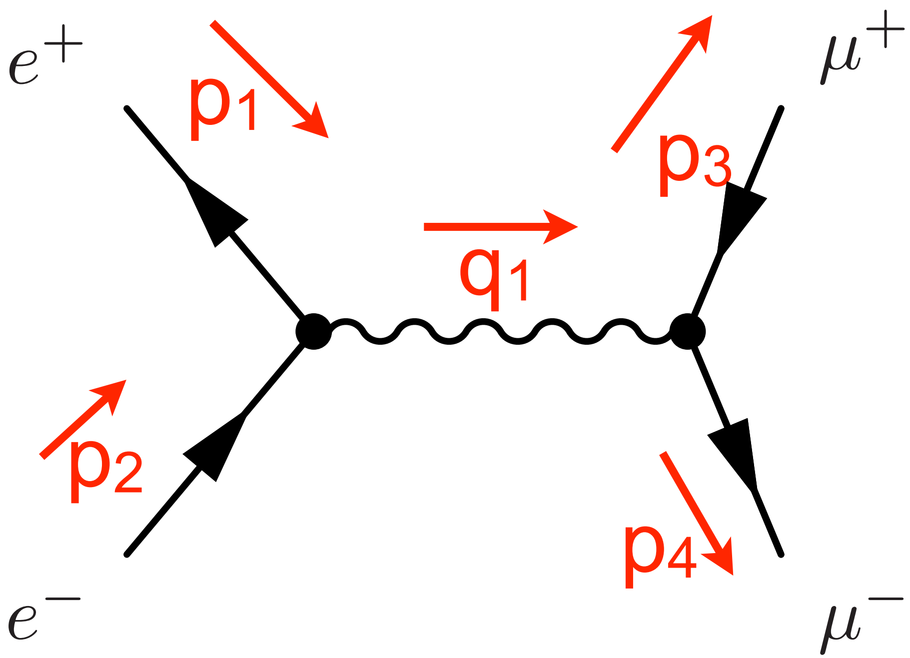
$\bar{u}u = 2mc$	$\bar{v}v = -2mc$	(7.98)

Example QED diagram



Here we have one of the diagrams contributing to electron-positron scattering into muon+antimuon pairs

Feynman rules for QED (1)

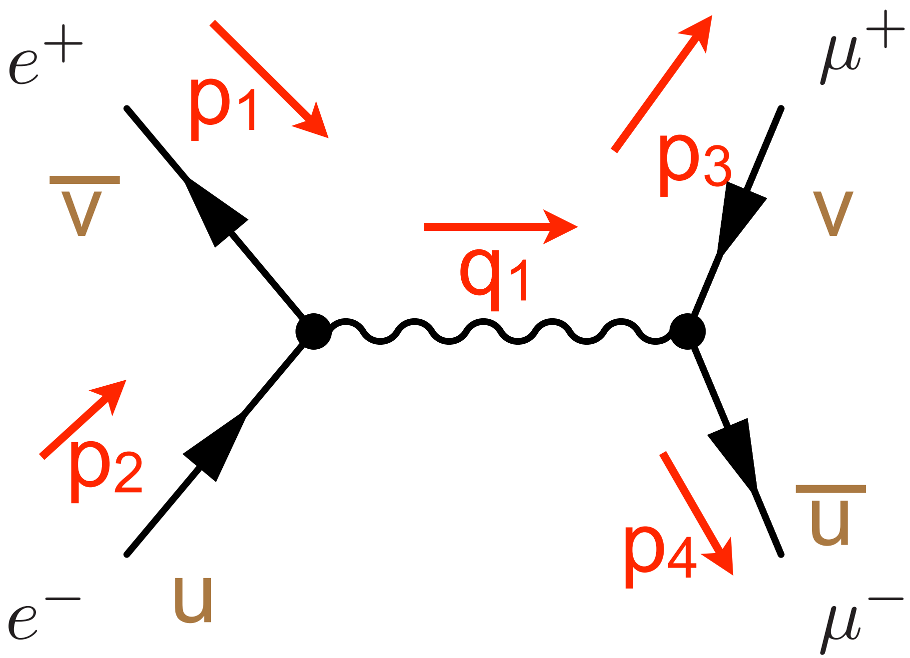


Label all incoming and outgoing lines with p_1, p_2, ... p_n

Internal lines can go either way

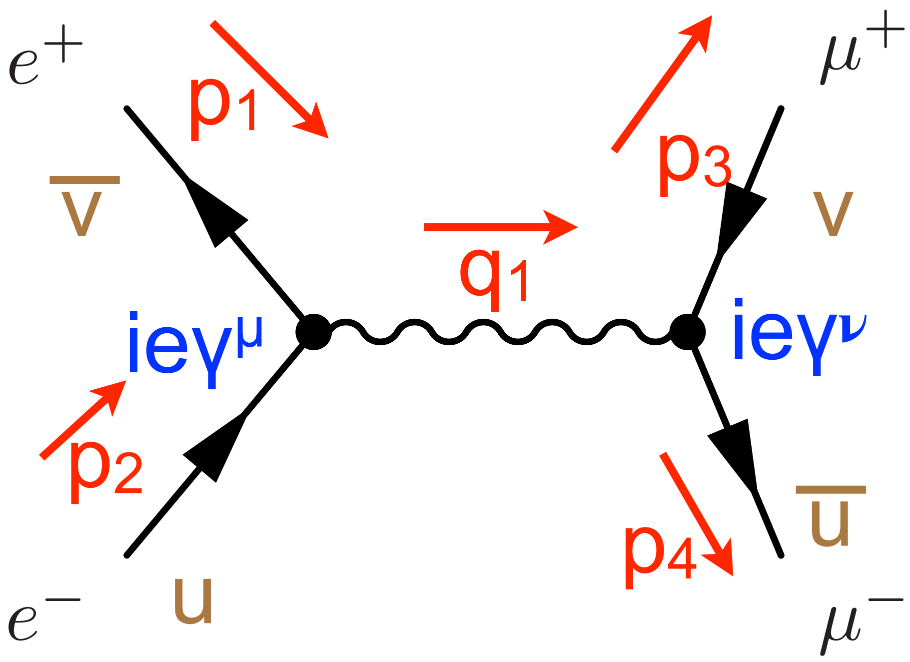
Use arrows to keep track of what is going in and out (of course we have arrows on the anti-particles, but that is different)

Feynman rules for QED (2)



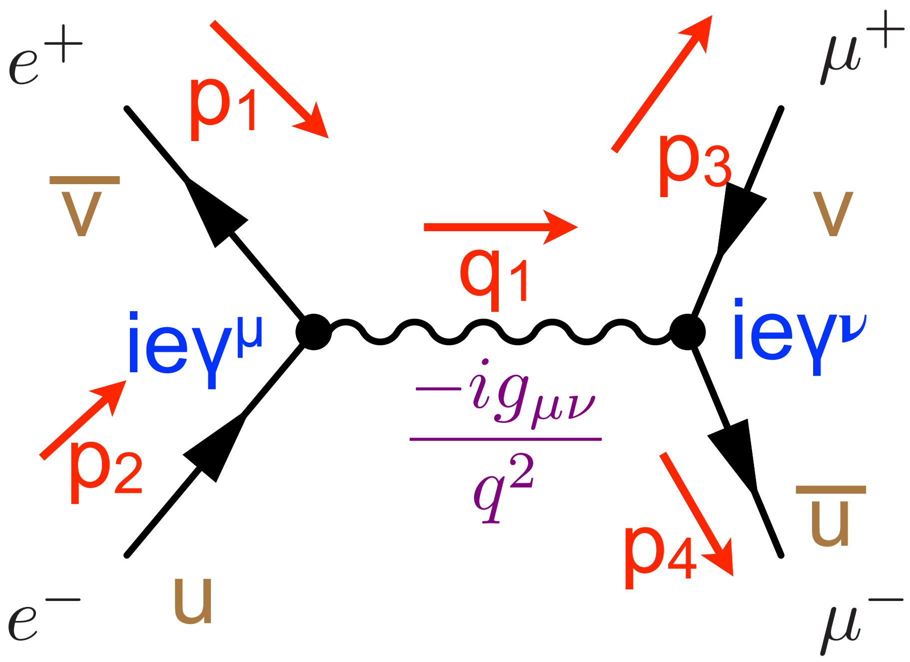
Incoming (outgoing) electrons/muons get a factor of u($\overline{u}$), outgoing (incoming) anti-electrons/anti-muons get a factor of v($\overline{v}$). Incoming (outgoing) photons get a factor of ε_μ (ε_μ^*)

Feynman rules for QED (3)



Add factors of $ie\gamma^\mu$ at each vertex

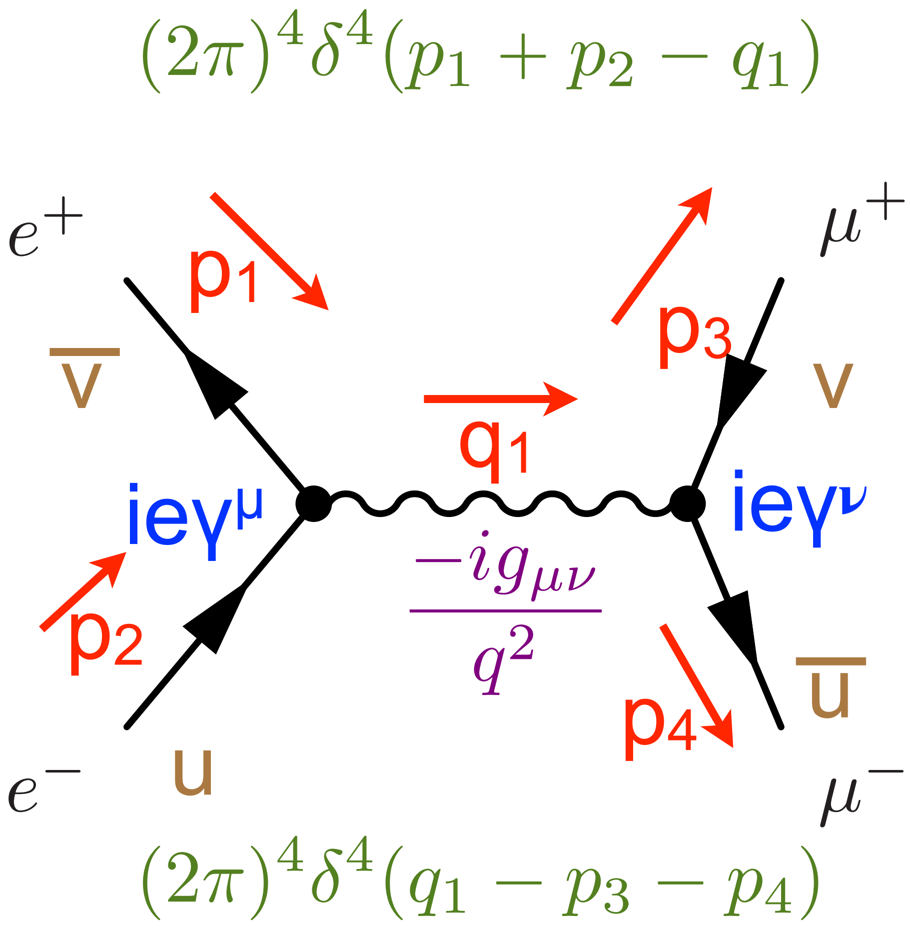
Feynman rules for QED (4)



For each internal photon line add a factor for the propagator

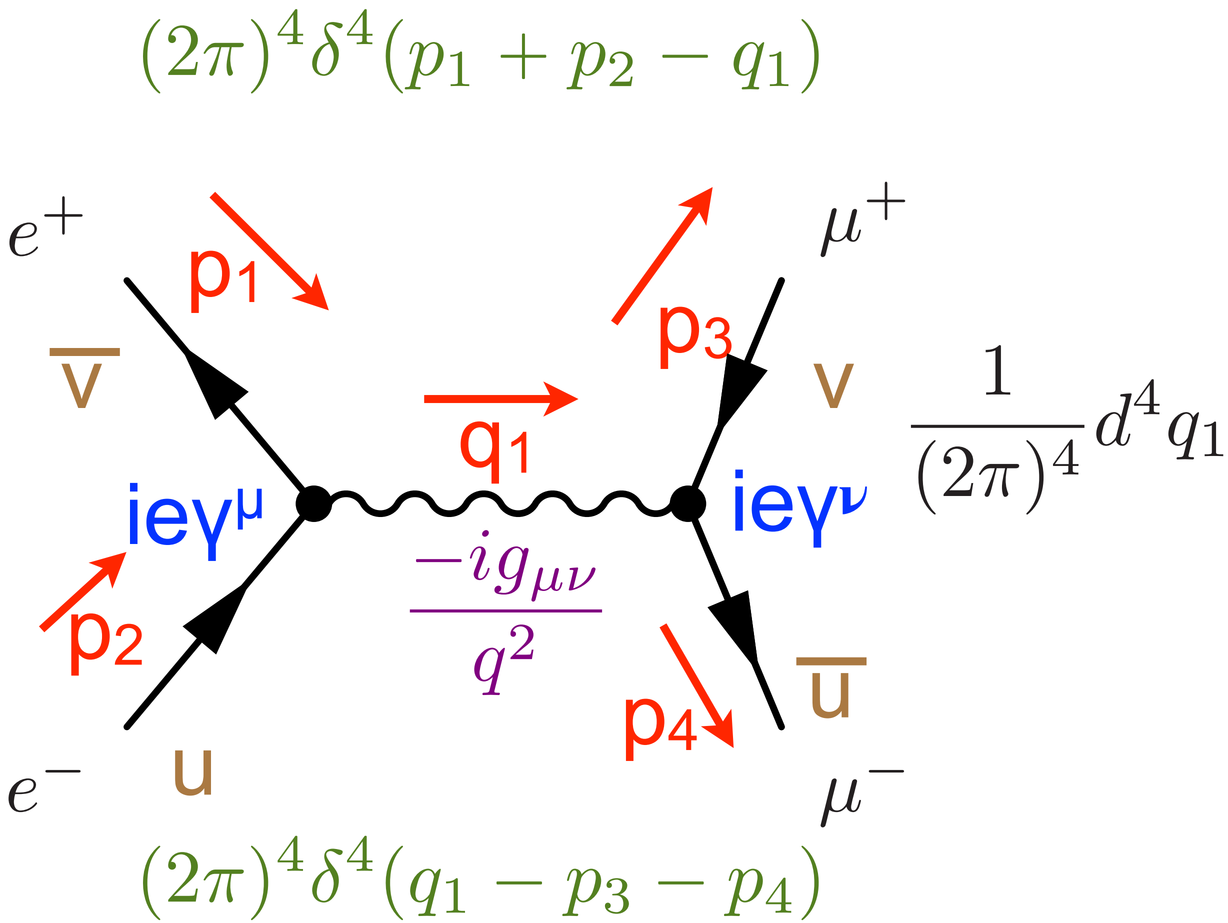
$$\frac{i\left(\gamma^\mu q_\mu + m\right)}{q^2 - m^2}$$ for internal electrons/positrons

Feynman rules for QED (5)



Impose conservation of energy and momentum at each vertex with 4d Dirac Delta function (with appropriate 2pi normalization)

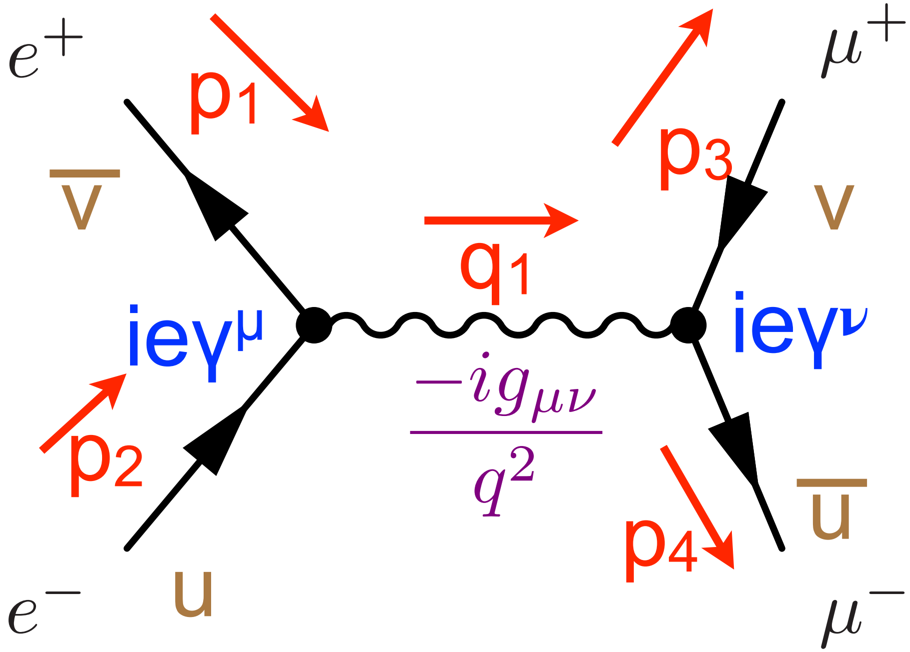
Feynman rules for QED (6)



Integrate over 4-momentum of internal lines with appropriate 2pi normalization factor

Feynman rules for QED (7)

$$(2\pi)^4\delta^4(p_1 + p_2 - q_1)$$



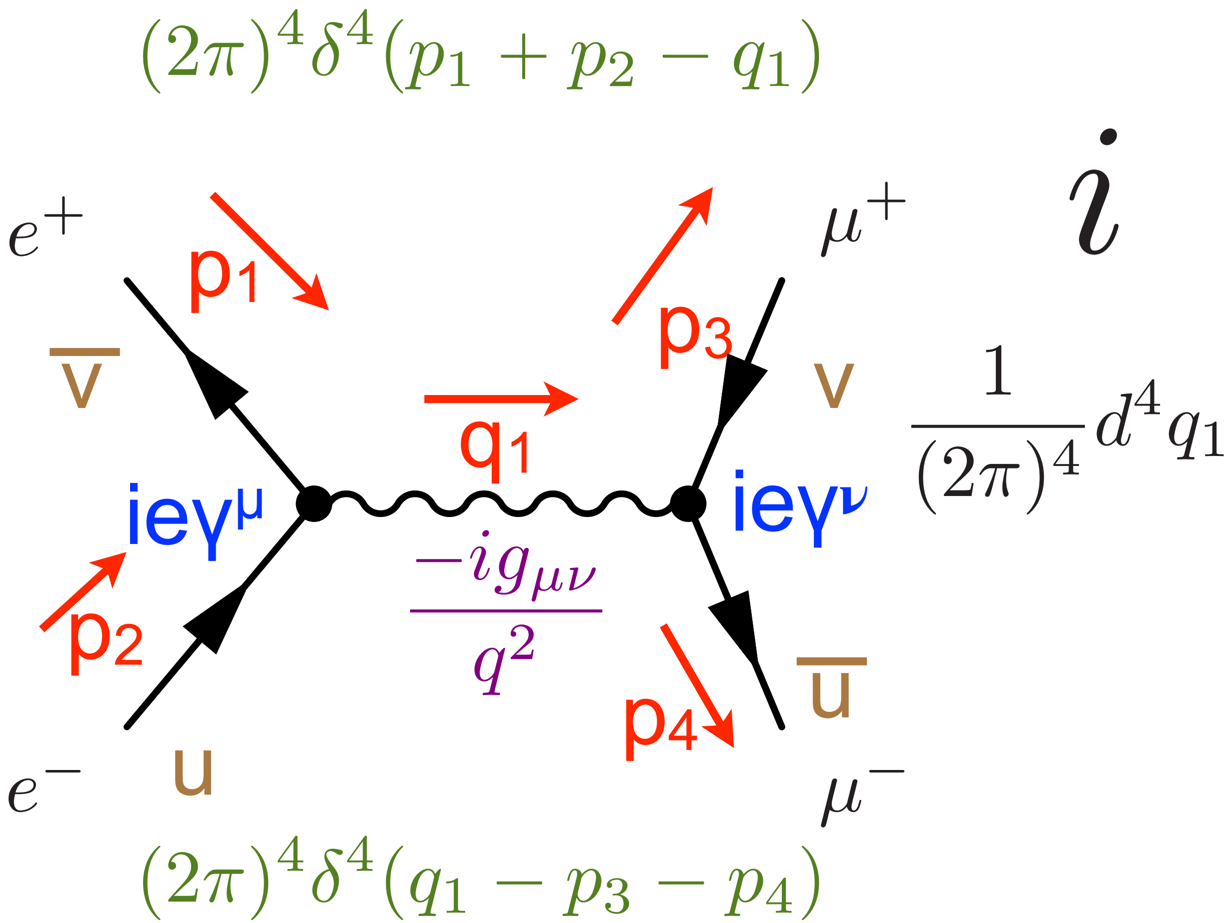
$$i$$

$$\frac{1}{(2\pi)^4}d^4q_1$$

$$(2\pi)^4\delta^4(q_1 - p_3 - p_4)$$

Cancel remaining delta function and add a factor of i, and you have the matrix element

Feynman rules for QED (8)



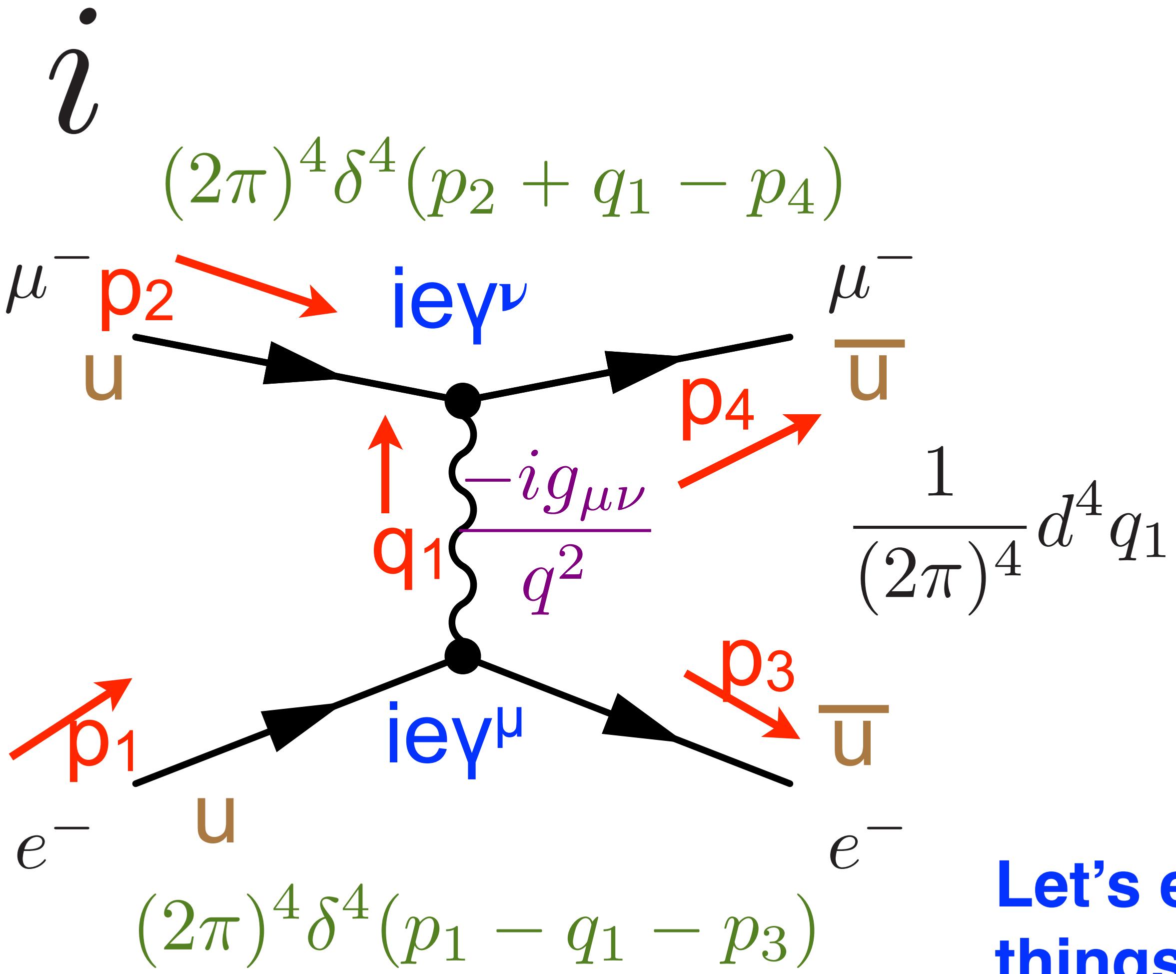
Add minus sign between diagrams differing only in exchange of two incoming or two outgoing fermions, or incoming fermion and outgoing anti-fermion (or vice versa)

One thing to be careful of

We'll call the E&M coupling "e" and not g_e so as not to get confused (Griffiths has "g"s all over the place!)

Electron-muon scattering example

$$i\int [\overline{u}^{(s3)}(p3)(ie\gamma^{\mu})u^{(s1)}(p1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}^{(s4)}(p4)(ie\gamma^{\nu})u^{(s2)}(p2)](2\pi)^4\delta^4(p_1-q_1-p_3)(2\pi)^4\delta^4(p_2+q_1-p_4)\frac{d^4q_1}{(2\pi)^4}$$



Let's examine the order of things. What order matters? What order doesn't matter?

Electron-muon scattering example

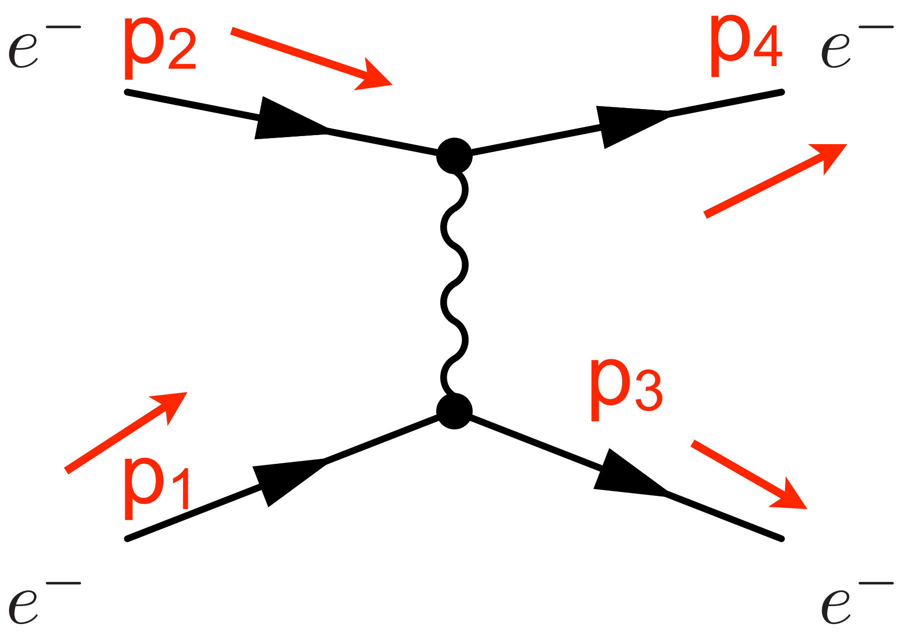
$$i\int [\overline{u}^{(s3)}(p3)(ie\gamma^{\mu})u^{(s1)}(p1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}^{(s4)}(p4)(ie\gamma^{\nu})u^{(s2)}(p2)](2\pi)^4\delta^4(p_1-q_1-p_3)(2\pi)^4\delta^4(p_2+q_1-p_4)\frac{d^4q_1}{(2\pi)^4}$$

Matrix element (let's check why together) =

$$\mathcal{M} = -\frac{e^2}{(p_1-p_3)^2}[\overline{u}^{(s3)}(p3)\gamma^{\mu}u^{(s1)}(p1)][\overline{u}^{(s4)}(p4)\gamma_{\mu}u^{(s2)}(p2)]$$

Doesn't look very insightful or useful, but this is a number that you can calculate given some initial-state and final-state kinematics

Electron-Electron scattering



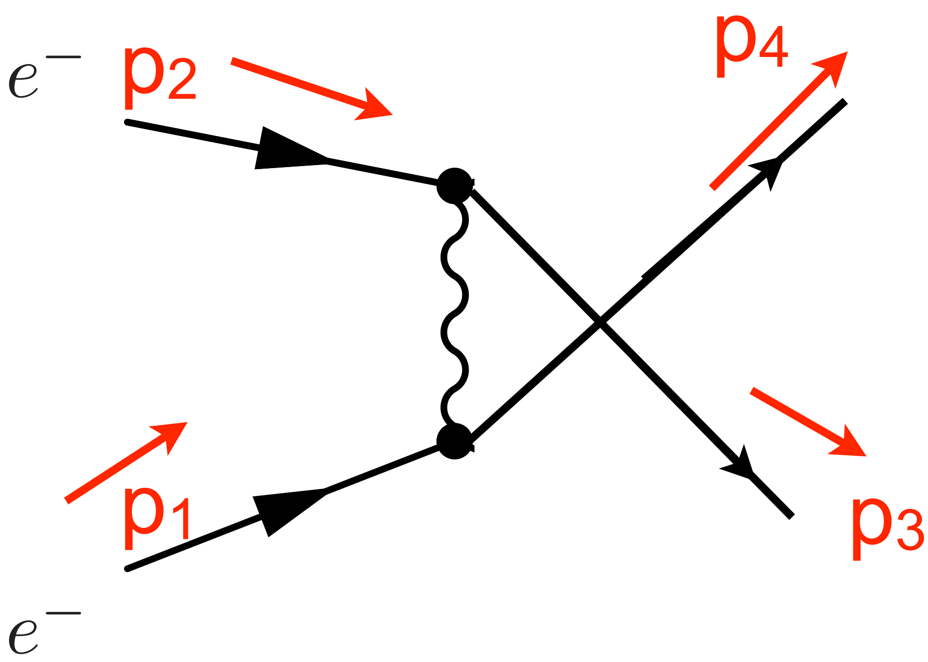
Matrix element (let's check together) =

$$\mathcal{M} = -\frac{e^2}{(p_1 - p_3)^2}[\overline{u}^{(s3)}(p3)\gamma^\mu u^{(s1)}(p1)][\overline{u}^{(s4)}(p4)\gamma_\mu u^{(s2)}(p2)]$$

$$-\frac{e^2}{(p_1 - p_3)^2}[\overline{u}(3)\gamma^\mu u(1)][\overline{u}(4)\gamma_\mu u(2)]$$

Slightly simpler notation

Electron-Electron scattering, second diagram



Twisted/crossed diagram! Rule 8 says they get a relative minus sign

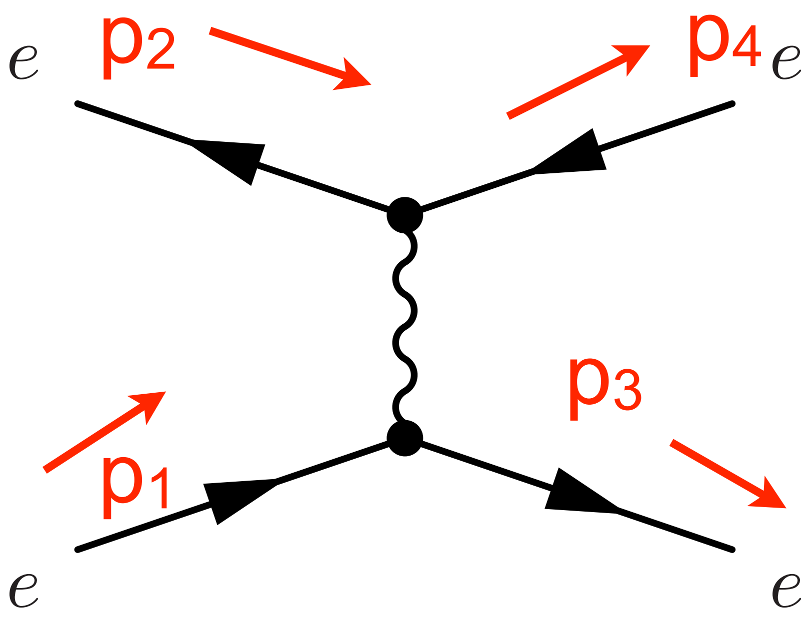
Total matrix element =

$$-\frac{e^2}{(p_1-p_3)^2}[\overline{u}(3)\gamma^\mu u(1)][\overline{u}(4)\gamma_\mu u(2)]+$$

$$\frac{e^2}{(p_1-p_4)^2}[\overline{u}(4)\gamma^\mu u(1)][\overline{u}(3)\gamma_\mu u(2)]+$$

Slightly simpler notation

Electron-Positron scattering, first diagram

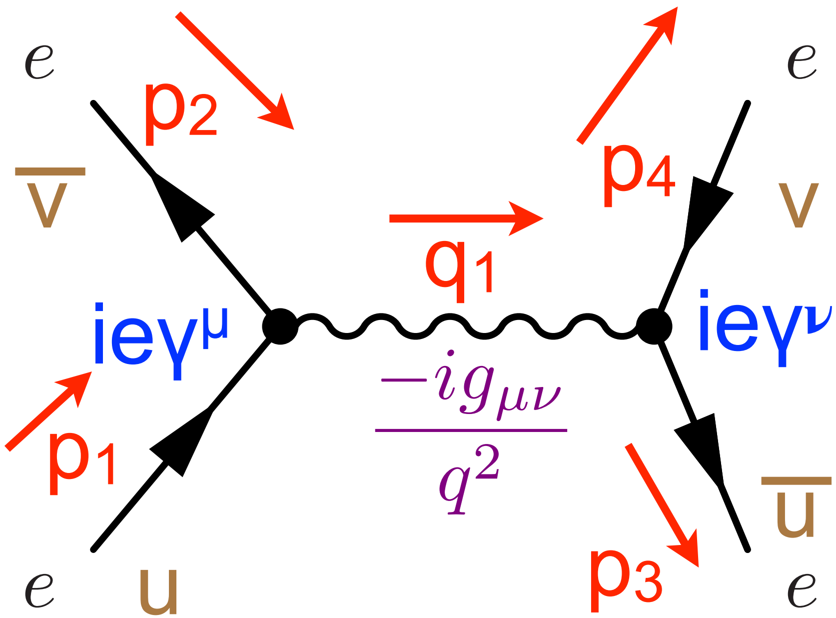


Note that we must follow fermion lines backwards to get the matrix algebra right

$$-\frac{e^2}{(p_1 - p_3)^2}[\overline{u}(3)\gamma^\mu u(1)][\overline{v}(2)\gamma_\mu v(4)]$$

Electron-Positron scattering, second diagram

$$(2\pi)^4\delta^4(p_1 + p_2 - q_1)$$



$$(2\pi)^4\delta^4(q_1 - p_3 - p_4)$$

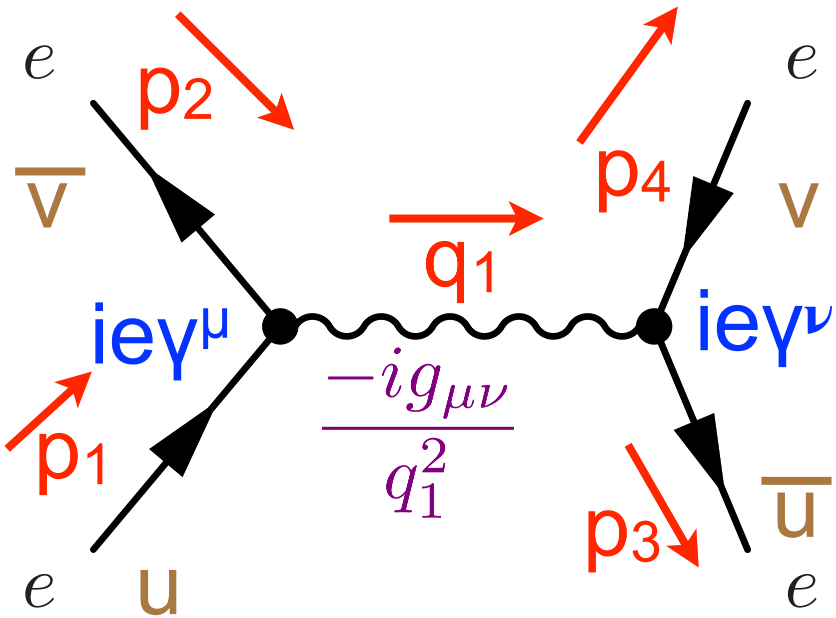
$$i$$

$$\frac{1}{(2\pi)^4}d^4q_1$$

Similar to when we wrote down the rules

Electron-Positron scattering, second diagram

$$(2\pi)^4\delta^4(p_1+p_2-q_1)$$



$$i$$

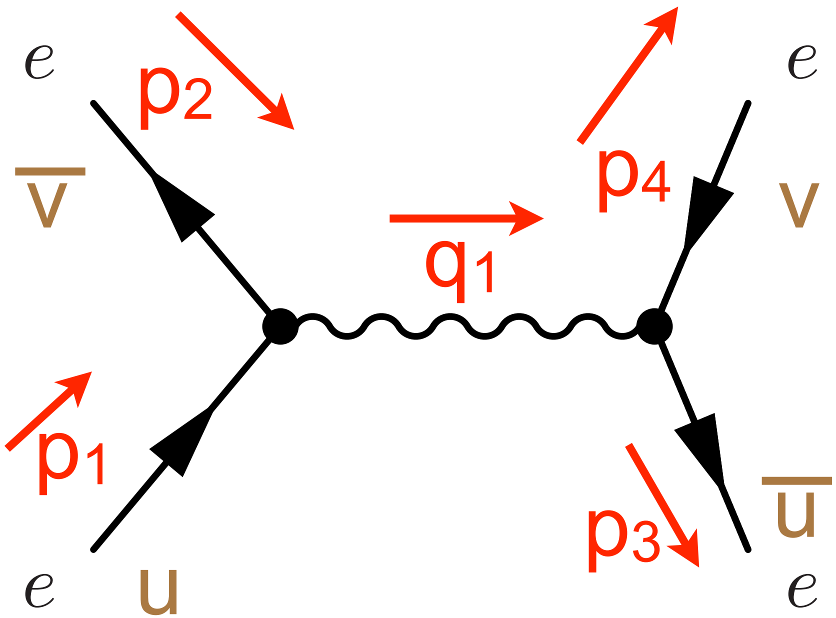
$$\frac{1}{(2\pi)^4}d^4q_1$$

$$(2\pi)^4\delta^4(q_1-p_3-p_4)$$

$$i\int[\overline{u}(3)(ie\gamma^\mu)v(4)]\frac{-ig_{\mu\nu}}{q_1^2}[\overline{v}(2)(ie\gamma^\nu)u(1)](2\pi)^4\delta^4(p_1+p_2-q_1)(2\pi)^4\delta^4(q_1-p_3-p_4)\frac{d^4q_1}{(2\pi)^4}$$

Electron-Positron scattering, second diagram

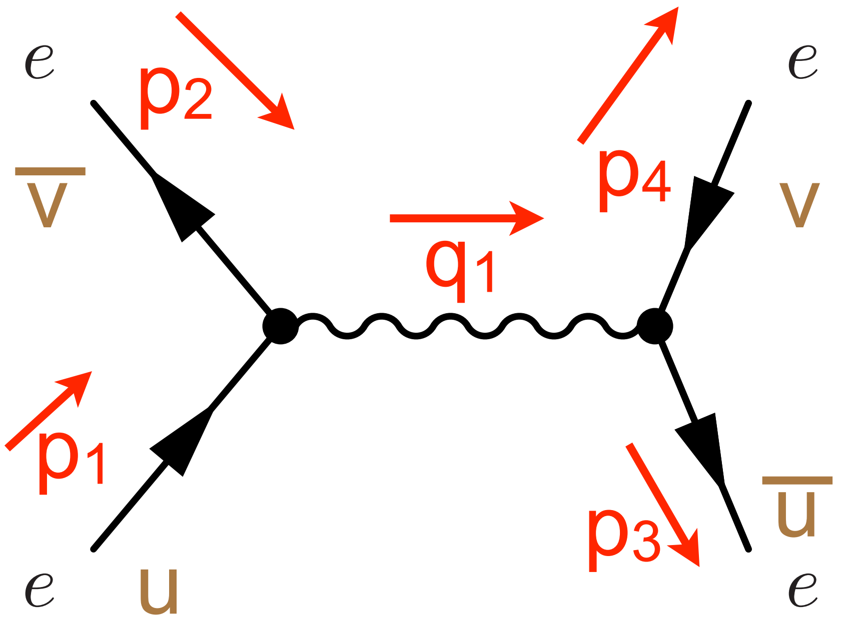
$$i\int[\overline{u}(3)(ie\gamma^{\mu})v(4)]\frac{-ig_{\mu\nu}}{q_1^2}[\overline{v}(2)(ie\gamma^{\nu})u(1)](2\pi)^4\delta^4(p_1+p_2-q_1)(2\pi)^4\delta^4(q_1-p_3-p_4)\frac{d^4q_1}{(2\pi)^4}$$



$$-(2\pi)^4e^2\int[\overline{u}(3)(\gamma^{\mu})v(4)]\frac{g_{\mu\nu}}{q_1^2}[\overline{v}(2)(\gamma^{\nu}u(1)]\delta^4(p_1+p_2-q_1)\delta^4(q_1-p_3-p_4)d^4q_1$$

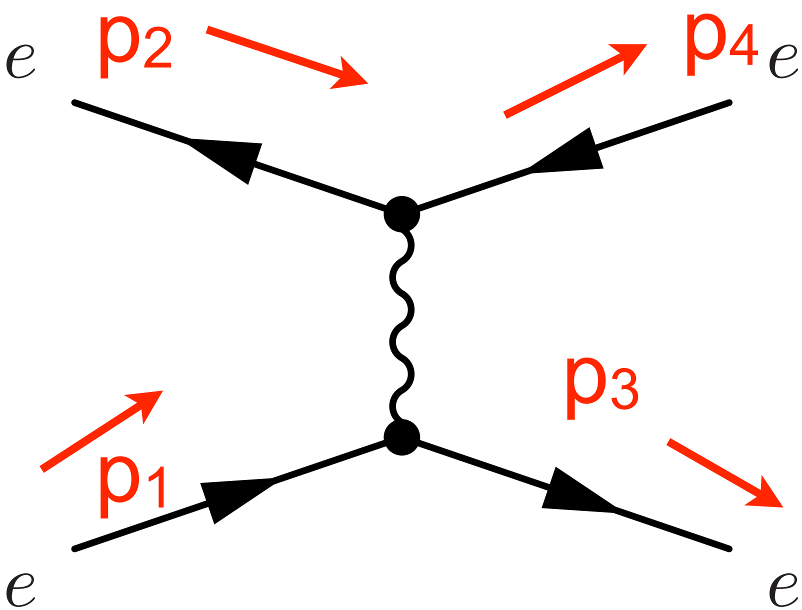
Electron-Positron scattering, second diagram

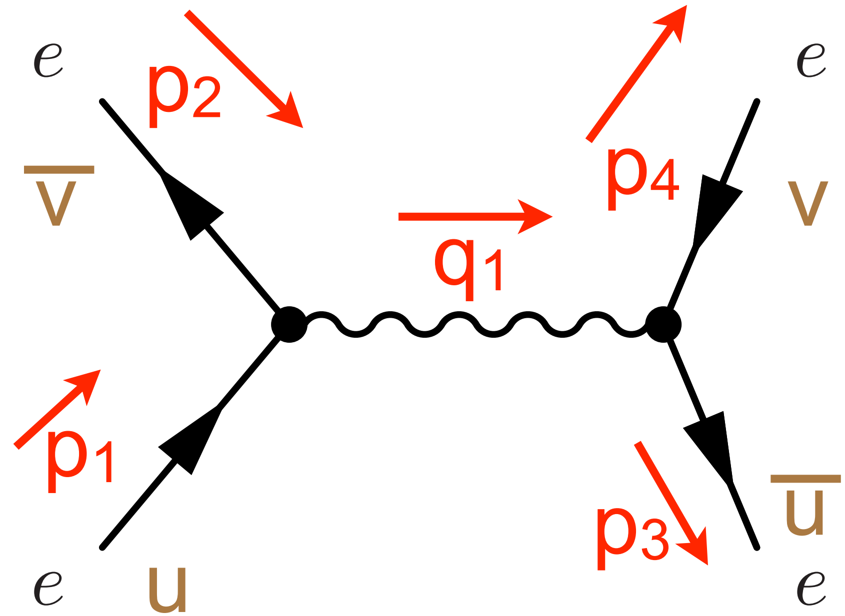
$$-(2\pi)^4 e^2 \int [\overline{u}(3)(\gamma^\mu)v(4)] \frac{g_{\mu\nu}}{q_1^2} [\overline{v}(2)(\gamma^\nu u(1)]\delta^4(p_1+p_2-q_1)\delta^4(q_1-p_3-p_4)d^4q_1$$



$$-\frac{(2\pi)^4 e^2}{(p_1+p_2)^2} \int [\overline{u}(3)(\gamma^\mu)v(4)] g_{\mu\nu} [\overline{v}(2)(\gamma^\nu u(1)]\delta^4(q_1-p_3-p_4)$$

Electron-Positron scattering, both diagrams





$$-\frac{e^2}{(p_1 - p_3)^2}[\overline{u}(3)\gamma^\mu u(1)][\overline{v}(2)\gamma_\mu v(4)]$$

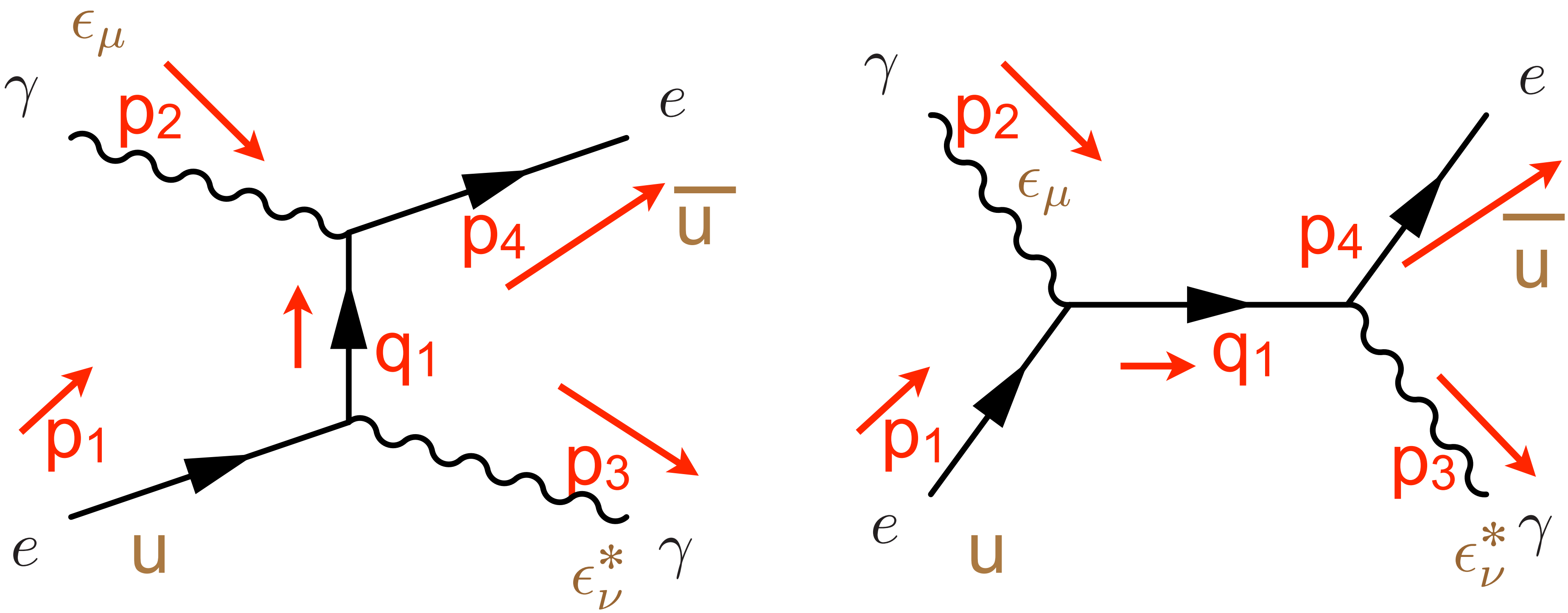
$$-\frac{e^2}{(p_1 + p_2)^2}[\overline{u}(3)(\gamma^\mu)v(4)][\overline{v}(2)\gamma_\mu u(1)]$$

If we swap p2 (incoming positron) and p3 (outgoing electron) in first diagram, we get the second diagram! Need relative minus sign

Electron-Positron total matrix element

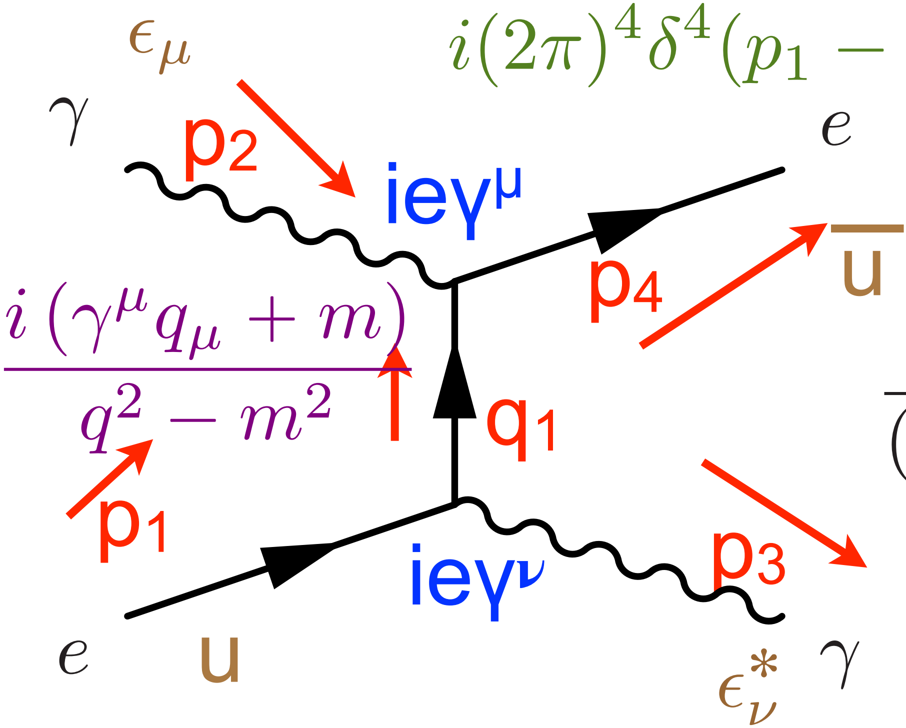
$$-\frac{e^2}{(p_1-p_3)^2}[\overline{u}(3)\gamma^\mu u(1)][\overline{v}(2)\gamma_\mu v(4)]+$$
$$+\frac{e^2}{(p_1+p_2)^2}[\overline{u}(3)(\gamma^\mu)v(4)][\overline{v}(2)\gamma_\mu u(1)]$$

Compton scattering



Let's start with first diagram

$$i(2\pi)^4\delta^4(p_1 - q_1 - p_3)(2\pi)^4\delta^4(p_2 + q_1 - p_4)$$

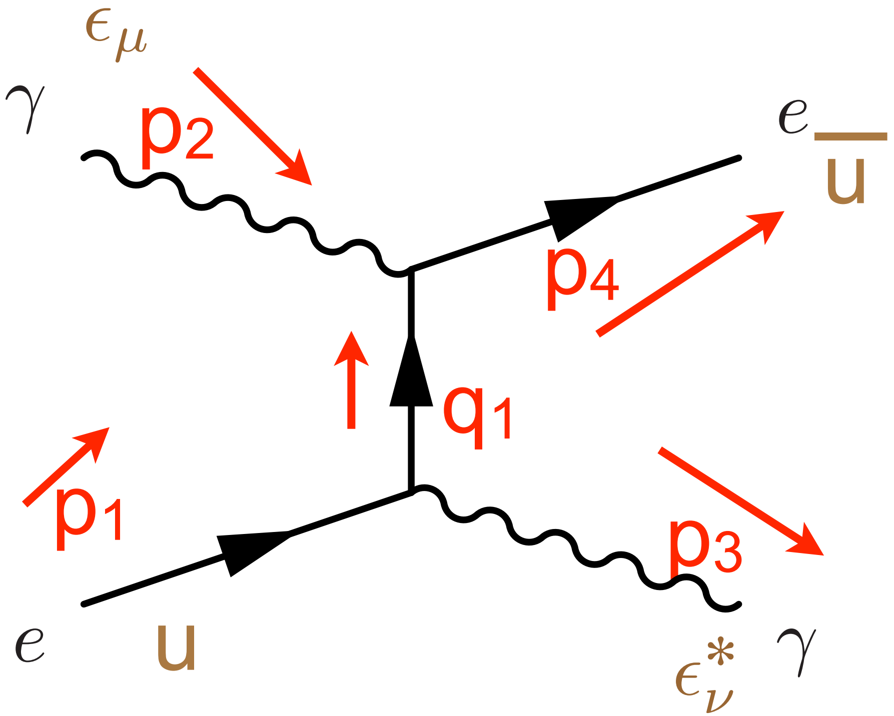


$$\frac{1}{(2\pi)^4} d^4 q_1$$

$$\int i(2\pi)^4\delta^4(p_1 - q_1 - p_3)(2\pi)^4\delta^4(p_2 + q_1 - p_4)\epsilon_\mu(2)\overline{u}(4)(ie\gamma^\mu)\frac{i(\gamma^\lambda q_\lambda + m)}{q^2 - m^2}(ie\gamma^\nu)u(1)\epsilon_\nu^*(3)\frac{d^4 q_1}{(2\pi)^4}$$

Let's start with first diagram

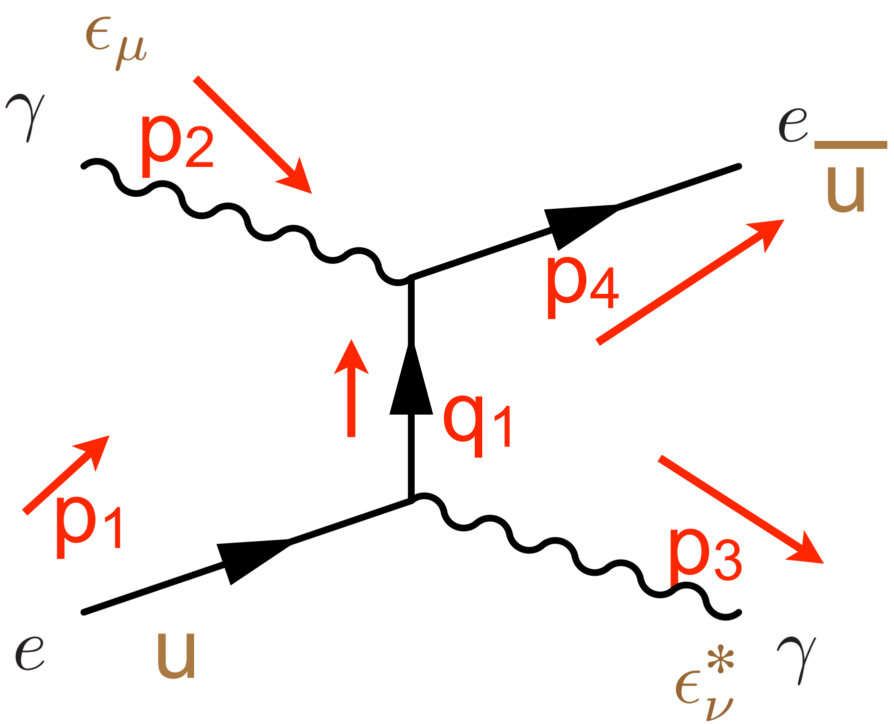
$$\int i(2\pi)^4\delta^4(p_1-q_1-p_3)(2\pi)^4\delta^4(p_2+q_1-p_4)\epsilon_\mu(2)\overline{u}(4)(ie\gamma^\mu)\frac{i(\gamma^\lambda q_\lambda+m)}{q^2-m^2}(ie\gamma^\nu)u(1)\epsilon^*_\nu(3)\frac{d^4q_1}{(2\pi)^4}$$



$$(2\pi)^4e^2\int\delta^4(p_1-q_1-p_3)\delta^4(p_2+q_1-p_4)\epsilon_\mu(2)\overline{u}(4)(\gamma^\mu)\frac{(\gamma^\lambda q_\lambda+m)}{q^2-m^2}(\gamma^\nu)u(1)\epsilon^*_\nu(3)d^4q_1$$

$$=\frac{(\gamma^\lambda(p_1-p_3)_\lambda+m)}{(p_1-p_3)^2-m^2}e^2(2\pi)^4\delta^4(p_2+q_1-p_4)\epsilon_\mu(2)v(4)\overline{u}(4)(\gamma^\mu)(\gamma^\nu)u(1)\epsilon^*_\nu(3)$$

So first diagram matrix element is



$$e^2 \frac{(\gamma^\lambda (p_1 - p_3)_\lambda + m)}{(p_1 - p_3)^2 - m^2} \epsilon_\mu(2)\overline{u}(4)(\gamma^\mu)(\gamma^\nu)u(1)\epsilon^*_\nu(3)$$

Let's introduce some (more!) notiation

"Slash" notation is very useful:

$$\not{x} = x^{\mu}\gamma_{\mu}$$

Rewriting first diagram matrix element ...

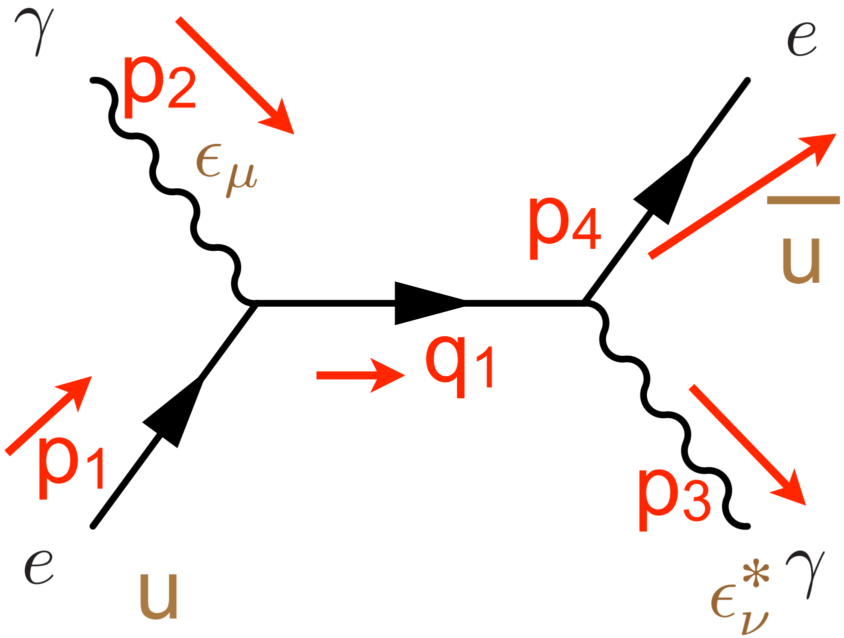
$$e^2 \frac{(\gamma^\lambda (p_1 - p_3)_\lambda + m)}{(p_1 - p_3)^2 - m^2} \epsilon_\mu(2) \overline{u}(4)(\gamma^\mu)(\gamma^\nu) u(1) \epsilon_\nu^*(3)$$

$$e^2 \frac{(\not{p}_1 - \not{p}_3 + m)}{(p_1 - p_3)^2 - m^2} \epsilon_\mu(2) \overline{u}(4)(\gamma^\mu)(\gamma^\nu) u(1) \epsilon_\nu^*(3)$$

$$e^2 \frac{(\not{p}_1 - \not{p}_3 + m)}{(p_1 - p_3)^2 - m^2} \not{\epsilon}(2) \overline{u}(4) u(1) \not{\epsilon}^*(3)$$

Now the second diagram

$$i(2\pi)^4\delta^4(p_1+p_2-q_1)(2\pi)^4\delta^4(q_1-p_3-p_4)$$



$$\int i(2\pi)^4\delta^4(p_1+p_2-q_1)(2\pi)^4\delta^4(q_1-p_3-p_4)\epsilon_\mu(2)\overline{u}(4)(ie\gamma^\mu)\frac{i(\gamma^\lambda q_\lambda+m)}{q^2-m^2}(ie\gamma^\nu)u(1)\epsilon^*_\nu(3)\frac{d^4q_1}{(2\pi)^4}$$

Can proceed in same way, and total matrix element is sum of the two

Potentially want to calculate processes with given initial and final state spins and polarizations, but more often than not we don’t specify either: average over initial state configurations and sum over final state configurations

Back to electron-muon scattering

$$\mathcal{M} = -\frac{e^2}{(p_1 - p_3)^2}[\overline{u}^{(s3)}(p3)\gamma^\mu u^{(s1)}(p1)][\overline{u}^{(s4)}(p4)\gamma_\mu u^{(s2)}(p2)]$$

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1 - p_3)^4}[\overline{u}^{(s3)}(p3)\gamma^\mu u^{(s1)}(p1)][\overline{u}^{(s4)}(p4)(\gamma_\mu u^{(s2)}(p2)][\overline{u}^{(s3)}(p3)\gamma^\nu u^{(s1)}(p1)]^*[\overline{u}^{(s4)}(p4)(\gamma_\nu u^{(s2)}(p2)]^*$$

So how to calculate (for example): ???

$$[\overline{u}^{(s3)}(p3)\gamma^\mu u^{(s1)}(p1)][\overline{u}^{(s3)}(p3)\gamma^\nu u^{(s1)}(p1)]^*$$

Or more generally

$$G = [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^*$$

Evaluating G

$$G = [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^*$$

Reminder that $\overline{a} = a^\dagger \gamma^0$

$$G = [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^*$$

$$G = [u(a)^\dagger \gamma^0 \Gamma_1 u(b)][u(a)^\dagger \gamma^0 \Gamma_2 u(b)]^\dagger$$

$$G = [u(a)^\dagger \gamma^0 \Gamma_1 u(b)][u(b)^\dagger \Gamma_2^\dagger \gamma^{0\dagger} u(a)]$$

$$G = [u(a)^\dagger \gamma^0 \Gamma_1 u(b)][u(b)^\dagger \Gamma_2^\dagger \gamma^0 u(a)]$$

For 1dx1d matrix, dagger and star are the same

$\gamma^{0\dagger} = \gamma^0$

Continuing with G

$$G = [u(a)^\dagger\gamma^0\Gamma_1 u(b)][u(b)^\dagger\Gamma_2^\dagger\gamma^0 u(a)]$$

$$G = [u(a)^\dagger\gamma^0\Gamma_1 u(b)][u(b)^\dagger\gamma^0\gamma^0\Gamma_2^\dagger\gamma^0 u(a)]$$

$$G = [u(a)^\dagger\gamma^0\Gamma_1 u(b)][\overline{u}(b)\overline{\Gamma}_2 u(a)]$$

$$G = [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(b)\overline{\Gamma}_2 u(a)]$$

WHY?!

$$\overline{\Gamma a} = (\Gamma a)^\dagger\gamma^0 = a^\dagger\Gamma^\dagger\gamma^0$$

$$\overline{\Gamma a} = a^\dagger\gamma^0\gamma^0\Gamma^\dagger\gamma^0$$

$$\overline{\Gamma a} = (a^\dagger\gamma^0)(\gamma^0\Gamma^\dagger\gamma^0) = \overline{a}\overline{\Gamma}$$

$\overline{\Gamma}_2 = \gamma^0\Gamma_2^\dagger\gamma^0$ For a matrix

$\overline{a} = a^\dagger\gamma^0$ For a spinor

CAREFUL - New definitions!

We were looking at: $G = [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^*$

$$[\overline{u}^{(s3)}(p3)\gamma^\mu u^{(s1)}(p1)][\overline{u}^{(s3)}(p3)\gamma^\nu u^{(s1)}(p1)]^*$$

So now we have G

We were looking at:

$$[\overline{u}^{(s3)}(p3)\gamma^{\mu}u^{(s1)}(p1)][\overline{u}^{(s3)}(p3)\gamma^{\nu}u^{(s1)}(p1)]^*$$

And defined $G = [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^*$

And found that $G = [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(b)\overline{\Gamma}_2 u(a)]$

We are interested in summing over spins s1 and s3 since we are not isolating any particular spin

$$G = \overline{u}(a)\Gamma_1 \sum_{b-spin} [u(b)\overline{u}(b)]\,\overline{\Gamma}_2 u(a)$$

Now we have a new problem (it is progress, I promise). Need to find $\sum_{b-spin} [u(b)\overline{u}(b)]$

Aside: The completeness relation

$$\sum_{b-spin} [u(b)\overline{u}(b)]$$

Reminder of our two solutions for u (not v), appropriately normalized

$$u(1) = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \frac{p_z}{\sqrt{E+m}} \\ \frac{p_x+ip_y}{\sqrt{E+m}} \end{pmatrix} \qquad u(2) = \begin{pmatrix} 0 \\ \sqrt{E+m} \\ \frac{p_x-ip_y}{\sqrt{E+m}} \\ \frac{-p_z}{\sqrt{E+m}} \end{pmatrix}$$

$$\sum_{b-spin} [u(b)\overline{u}(b)] = u(1)\overline{u}(1) + u(2)\overline{u}(2)$$

The completeness relation

$$\sum_{b-spin} [u(b)\overline{u}(b)] = u(1)\overline{u}(1) + u(2)\overline{u}(2)$$

$$\overline{u} = u^\dagger \gamma^0 = u^\dagger \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Minus signs

But also extra -1 on every i

Minus signs

$$\begin{pmatrix} \sqrt{E+m} \\ 0 \\ \frac{p_z}{\sqrt{E+m}} \\ \frac{p_x+ip_y}{\sqrt{E+m}} \end{pmatrix} \left(\sqrt{E+m} \;\; 0 \;\; \frac{-p_z}{\sqrt{E+m}} \;\; \frac{-p_x+ip_y}{\sqrt{E+m}} \right) + \begin{pmatrix} 0 \\ \sqrt{E+m} \\ \frac{p_x-ip_y}{\sqrt{E+m}} \\ \frac{-p_z}{\sqrt{E+m}} \end{pmatrix} \left(0 \;\; \sqrt{E+m} \;\; \frac{-p_x-ip_y}{\sqrt{E+m}} \;\; \frac{p_z}{\sqrt{E+m}} \right)$$

$$\begin{pmatrix} E+m & 0 & -p_z & -p_x+ip_y \\ 0 & 0 & 0 & 0 \\ p_z & 0 & \frac{-p_z^2}{E+m} & \frac{-p_xp_z+ip_yp_z}{E+m} \\ p_x+ip_y & 0 & \frac{-p_xp_z-ip_yp_z}{E+m} & \frac{-p_x^2-p_y^2}{E+m} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & E+m & -p_x-ip_y & p_z \\ 0 & p_x-ip_y & \frac{-p_x^2-p_y^2}{E+m} & \frac{p_xp_z-ip_yp_z}{E+m} \\ 0 & -p_z & \frac{p_xp_z+ip_yp_z}{E+m} & \frac{-p_z^2}{E+m} \end{pmatrix}$$

The completeness relation

$$\sum_{b-spin}[u(b)\overline{u}(b)] = u(1)\overline{u}(1) + u(2)\overline{u}(2)$$

$$\begin{pmatrix} E+m & 0 & -p_z & -p_x+ip_y \\ 0 & 0 & 0 & 0 \\ p_z & 0 & \frac{-p_z^2}{E+m} & \frac{-p_xp_z+ip_yp_z}{E+m} \\ p_x+ip_y & 0 & \frac{-p_xp_z-ip_yp_z}{E+m} & \frac{-p_x^2-p_y^2}{E+m} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & E+m & -p_x-ip_y & p_z \\ 0 & p_x-ip_y & \frac{-p_x^2-p_y^2}{E+m} & \frac{p_xp_z-ip_yp_z}{E+m} \\ 0 & -p_z & \frac{p_xp_z+ip_yp_z}{E+m} & \frac{-p_z^2}{E+m} \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} E+m & 0 & -p_z & -p_x+ip_y \\ 0 & E+m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & \frac{-p_x^2-p_y^2-p_z^2}{E+m} & 0 \\ p_x+ip_y & -p_z & 0 & \frac{-p_x^2-p_y^2-p_z^2}{E+m} \end{pmatrix}$$

The completeness relation

$$\sum_{b-spin}[u(b)\overline{u}(b)] = u(1)\overline{u}(1) + u(2)\overline{u}(2)$$

$$\begin{pmatrix} E+m & 0 & -p_z & -p_x+ip_y \\ 0 & E+m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & \frac{-p_x^2-p_y^2-p_z^2}{E+m} & 0 \\ p_x+ip_y & -p_z & 0 & \frac{-p_x^2-p_y^2-p_z^2}{E+m} \end{pmatrix}$$

$$\begin{pmatrix} E+m & 0 & -p_z & -p_x+ip_y \\ 0 & E+m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & \frac{-\mathbf{p}^2}{E+m} & 0 \\ p_x+ip_y & -p_z & 0 & \frac{-\mathbf{p}^2}{E+m} \end{pmatrix}$$

The completeness relation

$$\sum_{b-spin} [u(b)\overline{u}(b)] = u(1)\overline{u}(1) + u(2)\overline{u}(2)$$

$$\begin{pmatrix} E+m & 0 & -p_z & -p_x + ip_y \\ 0 & E+m & -p_x - ip_y & p_z \\ p_z & p_x - ip_y & \frac{-\mathbf{p}^2}{E+m} & 0 \\ p_x + ip_y & -p_z & 0 & \frac{-\mathbf{p}^2}{E+m} \end{pmatrix}$$

$$\frac{-\mathbf{p}^2}{E+m} = \frac{m^2 - E^2}{E+m} = \frac{(m-E)(m+E)}{E+m} = m - E$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{p} \cdot \sigma = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The completeness relation

$$\sum_{b-spin} [u(b)\overline{u}(b)] = u(1)\overline{u}(1) + u(2)\overline{u}(2)$$

$$\begin{pmatrix} E+m & 0 & -p_z & -p_x + ip_y \\ 0 & E+m & -p_x - ip_y & p_z \\ p_z & p_x - ip_y & \frac{-\mathbf{p}^2}{E+m} & 0 \\ p_x + ip_y & -p_z & 0 & \frac{-\mathbf{p}^2}{E+m} \end{pmatrix} =$$

$$\begin{pmatrix} \begin{matrix} m+E & 0 \\ 0 & m+E \end{matrix} & -\mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & \begin{matrix} m-E & 0 \\ 0 & m-E \end{matrix} \end{pmatrix} =$$

$$m + \gamma^0 E - \mathbf{p}\cdot \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} = \gamma^\mu p_\mu + m$$

The completeness relation (phew)

$$\sum_{b-spin} [u(b)\overline{u}(b)] = u(1)\overline{u}(1) + u(2)\overline{u}(2) = \gamma^\mu p_\mu + m$$

Phew! Will not prove but (you can...)

$$\sum_{b-spin} [v(b)\overline{v}(b)] = v(1)\overline{v}(1) + v(2)\overline{v}(2) = \gamma^\mu p_\mu - m$$

$$G = \overline{u}(a)\Gamma_1 \sum_{b-spin} [u(b)\overline{u}(b)]\,\overline{\Gamma}_2 u(a) = \overline{u}(a)\Gamma_1[\gamma^\mu p^b_\mu + m_b]\overline{\Gamma}_2 u(a)$$

$$G = \overline{u}(a)\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2 u(a)$$

Continuing on with G

$$G = \overline{u}(a)\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2 u(a)$$

Now let's sum over a spins

$$\sum_{a,b\ spins} G = \sum_{sa=1,2} \overline{u}(sa)\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2 u(sa)$$

Scalar quantity we're calculating (sum of 4 numbers)

1x4 matrix

4x4 matrix

4x1 matrix (spinor)

Continuing on with G

$$\sum_{a,b\ spins} G = \sum_{sa=1,2} \overline{u}(sa)\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2 u(sa)$$

$$\sum_{a,b\ spins} G = \sum_{sa=1,2}\sum_{i,j=1}^{4} \overline{u}(sa)_i \left[\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2\right]_{ij} u(sa)_j$$

$$\sum_{a,b\ spins} G = \sum_{i,j=1}^{4} \left[\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2\right]_{ij} \sum_{sa=1,2} u(sa)_j \overline{u}(sa)_i$$

$$\sum_{a,b\ spins} G = \sum_{i,j=1}^{4} \left[\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2\right]_{ij} (\not{p}_a + m_a)_{ji} = \text{Trace}[\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2(\not{p}_a + m_a)]$$

In summary

$$\sum_{spins} [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^* = \text{Trace}[\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2(\not{p}_a + m_a)]$$

If u is replaced by v, the mass gets a minus sign (as we saw before)

Did this really gain us anything except for a headache?!?!

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1 - p_3)^4}[\overline{u}^{(s3)}(p3)\gamma^\mu u^{(s1)}(p1)][\overline{u}^{(s4)}(p4)\gamma_\mu u^{(s2)}(p2)][\overline{u}^{(s3)}(p3)\gamma^\nu u^{(s1)}(p1)]^*[\overline{u}^{(s4)}(p4)(\gamma_\nu u^{(s2)}(p2)]^*$$

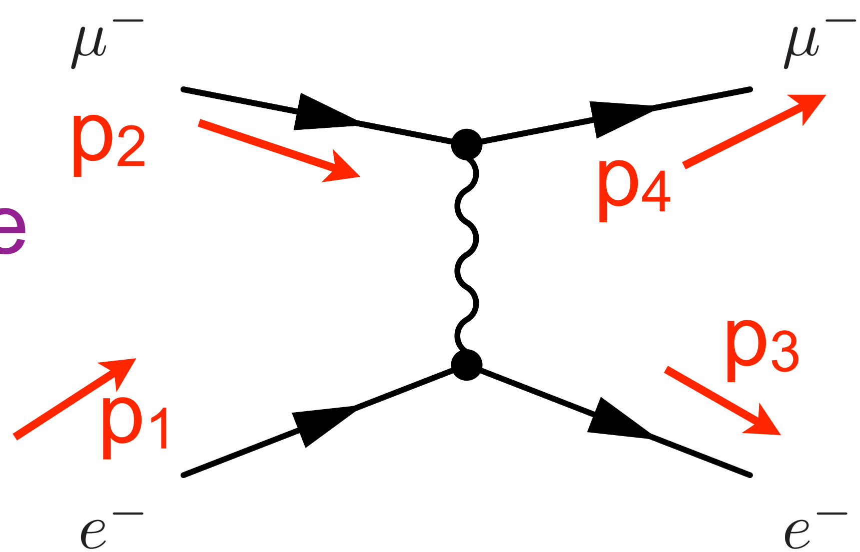
Electron-muon scattering

$$\sum_{spins} [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^* = \text{Trace}[\Gamma_1(\not{p}_b + m_b)\overline{\Gamma}_2(\not{p}_a + m_a)]$$

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1 - p_3)^4}[\overline{u}^{(s3)}(p3)\gamma^\mu u^{(s1)}(p1)][\overline{u}^{(s4)}(p4)\gamma_\mu u^{(s2)}(p2)][\overline{u}^{(s3)}(p3)\gamma^\nu u^{(s1)}(p1)]^*[\overline{u}^{(s4)}(p4)(\gamma_\nu u^{(s2)}(p2)]^*$$

$$|\mathcal{M}|^2 = \frac{e^4}{4(p_1 - p_3)^4}\text{Trace}[\gamma^\mu(\not{p}_1 + m_1)\overline{\gamma}^\nu(\not{p}_3 + m_3)]\text{Trace}[\gamma_\mu(\not{p}_2 + m_2)\overline{\gamma}_\nu(\not{p}_4 + m_4)]$$

1/4 to average over initial spins



$m_1 = m_3 = m_e$

$m_2 = m_4 = m_\mu$

One missing piece

$$\overline{\gamma}^{\mu} = \gamma^0 \gamma^{\mu\dagger} \gamma^0$$

$$\overline{\gamma}^{\mu} = \gamma^{\mu}$$

$$\overline{\gamma}^0 = \gamma^0 \gamma^{0\dagger} \gamma^0 = \gamma^0 \gamma^0 \gamma^0 = \gamma^0$$

$$\overline{\gamma}^i = \gamma^0 \gamma^{i\dagger} \gamma^0 = \gamma^0 \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}^{\dagger} \gamma^0$$

$$\overline{\gamma}^i = \gamma^0 \gamma^{i\dagger} \gamma^0 = \gamma^0 \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}^{\dagger} \gamma^0$$

$$\overline{\gamma}^i = \gamma^0 \begin{pmatrix} 0 & -\sigma^{i\dagger} \\ \sigma^{i\dagger} & 0 \end{pmatrix} \gamma^0$$

Obvious?

$$\overline{\gamma}^i = \gamma^0 \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \gamma^0 = -\gamma^0 \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \gamma^0 = -\gamma^0 \gamma^i \gamma^0$$

$$\overline{\gamma}^i = -(-\gamma^i \gamma^0) \gamma^0 = \gamma^i$$

Electron-muon scattering

$$|\mathcal{M}|^2 = \frac{e^4}{4(p_1 - p_3)^4} \text{Trace}[\gamma^\mu(\not{p}_1 + m_1)\overline{\gamma}^\nu(\not{p}_3 + m_3)]\text{Trace}[\gamma_\mu(\not{p}_2 + m_2)\overline{\gamma}_\nu(\not{p}_4 + m_4)]$$

$$|\mathcal{M}|^2 = \frac{e^4}{4(p_1 - p_3)^4} \text{Trace}[\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_3)]\text{Trace}[\gamma_\mu(\not{p}_2 + m_2)\gamma_\nu(\not{p}_4 + m_4)]$$

Griffiths has a number of “Trace theorems” in Chapter 7.7. Please look at them (don’t want to spend time deriving them all, but we’ll go over a couple)

Trace theorems from Griffiths

Phew

$$Tr(A+B) = Tr(A) + Tr(B)$$

$$Tr(\alpha A) = \alpha Tr(A)$$

$$Tr(AB) = Tr(BA)$$

$$g_{\mu\nu} g^{\mu\nu} = 4$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$\not{a}\not{b} + \not{b}\not{a} = 2a \cdot b$$

$$\gamma_\mu \gamma^\mu = 4$$

$$\gamma_\mu \gamma^\mu \gamma^\nu = -2\gamma^\nu$$

$$\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$$

$$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4g^{\nu\lambda}$$

$$\gamma_\mu \not{a}\not{b} \gamma^\mu = 4(a \cdot b)$$

$$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma \lambda \gamma \nu$$

$$\gamma_\mu \not{a}\not{b}\not{c} = -2\not{c}\not{b}\not{a}$$

$$Tr(\gamma^\mu \gamma^\nu \gamma^\lambda) = 0$$

$$Tr(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$$

$$Tr(\not{a}\not{b}) = 4(a \cdot b)$$

$$Tr(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4\left(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda}\right)$$

$$Tr(\not{a}\not{b}\not{c}\not{d}) = 4(a \cdot b c \cdot d - a \cdot c b \cdot d + a \cdot d b \cdot c)$$

Some trace theorem proofs

$$Tr(A) = \sum_i A_{ii}$$

$$Tr(A+B) = \sum (A+B)_{ii} = \sum A_{ii} + \sum B_{ii} = TrA + TrB$$

$$Tr(\alpha A) = \sum (\alpha A)_{ii} = \sum (\alpha A_{ii}) = \alpha Tr(A)$$

$$Tr(AB) = \sum (AB)_{ii} = \sum_i \sum_j (A_{ij} B_{ji}) = \sum_i \sum_j B_{ji} A_{ij} = \sum_j (BA)_{jj} = TR(BA)$$

$$Tr(ABC) = \sum (ABC)_{ii} = \sum_i \sum_j \sum_k (A_{ij} B_{jk} C_{ki}) = \sum_i \sum_j \sum_k C_{ki} A_{ij} B_{jk} = \sum_j \sum_k (CA)_{kj} B_{jk} = TR(CAB)$$

Tr(ABC) = Tr(CAB) = Tr(BCA) but this is not the same as Tr(BAC)

Some trace theorem proofs

$$
\begin{aligned}
g_{\mu\nu}g^{\mu\nu} = {} & g_{00}g^{00} + \cancel{g_{01}g^{01}} + \cancel{g_{02}g^{02}} + \cancel{g_{03}g^{03}} + \\
& \cancel{g_{10}g^{10}} + g_{11}g^{11} + \cancel{g_{12}g^{12}} + \cancel{g_{13}g^{13}} + \\
& \cancel{g_{20}g^{20}} + \cancel{g_{21}g^{21}} + g_{22}g^{22} + \cancel{g_{23}g^{23}} + \\
& \cancel{g_{30}g^{30}} + \cancel{g_{31}g^{31}} + \cancel{g_{32}g^{32}} + g_{33}g^{33}
\end{aligned}
$$

$$
g_{\mu\nu}g^{\mu\nu} = g_{00}g^{00} + g_{11}g^{11} + g_{22}g^{22} + g_{33}g^{33} = (1)(1) + (-1)(-1) + (-1)(-1) + (-1)(-1) = 4
$$

Some trace theorem proofs

$$\not{a}\not{b} + \not{b}\not{a} = (a_\mu \gamma^\mu)(b_\nu \gamma^\nu) + (b_\nu \gamma^\nu)(a_\mu \gamma^\mu)$$

$$\not{a}\not{b} + \not{b}\not{a} = (a_\mu b_\nu)(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$$

$$\not{a}\not{b} + \not{b}\not{a} = (a_\mu b_\nu)(2g^{\mu\nu}) = 2a_\mu b^\mu = 2a \cdot b$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$g_{\mu\nu}\left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right) = g_{\mu\nu}(2g^{\mu\nu})$$

$$\gamma_\nu \gamma^\nu + \gamma_\nu \gamma^\nu = 2\gamma_\nu \gamma^\nu = 2g_{\mu\nu} g^{\mu\nu}$$

$$\gamma_\nu \gamma^\nu = g_{\mu\nu} g^{\mu\nu}$$

$$\gamma_\nu \gamma^\nu = 4$$

$$\gamma_\mu \gamma^\nu \gamma^\mu = \gamma_\mu \left(2g^{\mu\nu} - \gamma^\mu \gamma^\nu\right)$$

$$\gamma_\mu \gamma^\nu \gamma^\mu = 2\gamma_\mu g^{\mu\nu} - \gamma_\mu \gamma^\mu \gamma^\nu$$

$$\gamma_\mu \gamma^\nu \gamma^\mu = 2\gamma^\nu - 4\gamma^\nu$$

$$\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$$

Some trace theorem proofs

$$Tr(\gamma^\mu\gamma^\nu\gamma^\sigma) = Tr(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^5\gamma^5)$$

$$Tr(\gamma^\mu\gamma^\nu\gamma^\sigma) = (-1)^1 Tr(\gamma^\mu\gamma^\nu\gamma^5\gamma^\sigma\gamma^5)$$

$$Tr(\gamma^\mu\gamma^\nu\gamma^\sigma) = (-1)^2 Tr(\gamma^\mu\gamma^5\gamma^\nu\gamma^\sigma\gamma^5)$$

$$Tr(\gamma^\mu\gamma^\nu\gamma^\sigma) = (-1)^3 Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^5)$$

$$Tr(AB) = Tr(BA) \rightarrow Tr(\gamma^\mu\gamma^\nu\gamma^\sigma) = (-1)^3 Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^5) = (-1)^3 Tr(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^5\gamma^5)$$

$$Tr(\gamma^\mu\gamma^\nu\gamma^\sigma) = (-1)^3 Tr(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^5\gamma^5) \rightarrow Tr(\gamma^\mu\gamma^\nu\gamma^\sigma) = 0$$

True for any odd number of gamma matrices

Some trace theorem proofs

$$Tr(1) = Tr\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 + 1 + 1 + 1 = 4$$

$$Tr(\gamma^\mu \gamma^\nu) = Tr(\gamma^\nu \gamma^\mu) \rightarrow Tr(\gamma^\mu \gamma^\nu) = \frac{1}{2}\left[Tr(\gamma^\mu \gamma^\nu) + Tr(\gamma^\nu \gamma^\mu)\right] = \frac{1}{2} Tr(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$$

$$Tr(\gamma^\mu \gamma^\nu) = \frac{1}{2} Tr(2g^{\mu\nu}) = g^{\mu\nu} Tr(1) = 4g^{\mu\nu}$$

$$Tr(\gamma^5) = Tr\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 0$$

OK, back to Electron-muon scattering

Phew. Again. Are we ready to calculate some matrix elements again? (It's been awhile since we've calculated any, and for sure been a long detour)

Back to Electron-muon scattering

$$|\mathcal{M}|^2 = \frac{e^4}{4(p_1 - p_3)^4}\text{Trace}[\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_3)]\text{Trace}[\gamma_\mu(\not{p}_2 + m_2)\gamma_\nu(\not{p}_4 + m_4)]$$

$$Tr[\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_3)] =$$
$$Tr[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3] + Tr[\gamma^\mu m_1 \gamma^\nu \not{p}_3] + Tr[\gamma^\mu \not{p}_1 \gamma^\nu m_3] + Tr[\gamma^\mu m_1 \gamma^\nu m_3] =$$
$$Tr[\gamma^\mu \gamma^\sigma p_{1,\sigma} \gamma^\nu \gamma^\lambda p_{3,\lambda}] + Tr[\gamma^\mu m_1 \gamma^\nu \gamma^\lambda p_{3,\lambda}] + Tr[\gamma^\mu \gamma^\sigma p_{1,\sigma} \gamma^\nu m_3] + Tr[\gamma^\mu m_1 \gamma^\nu m_3]$$

$$Tr(\gamma^\mu \gamma^\nu \gamma^\lambda) = 0$$

Electron-muon scattering

$$|\mathcal{M}|^2 = \frac{e^4}{4(p_1 - p_3)^4} \mathrm{Trace}[\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_3)]\mathrm{Trace}[\gamma_\mu(\not{p}_2 + m_2)\gamma_\nu(\not{p}_4 + m_4)]$$

$$Tr[\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_3)] =$$

$$Tr[\gamma^\mu\gamma^\sigma p_{1,\sigma}\gamma^\nu\gamma^\lambda p_{3,\lambda}] + Tr[\gamma^\mu m_1\gamma^\nu m_3]$$

$$Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$Tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\right)$$

$$Tr[\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_3)] =$$

$$Tr[\gamma^\mu\gamma^\sigma p_{1,\sigma}\gamma^\nu\gamma^\lambda p_{3,\lambda}] + Tr[\gamma^\mu m_1\gamma^\nu m_3] =$$

$$4p_{1,\sigma}p_{3,\lambda}\left[g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\sigma\lambda} + g^{\mu\lambda}g^{\sigma\nu}\right] + 4m_1m_3g^{\mu\nu}$$

Electron-muon scattering

$$|\mathcal{M}|^2 = \frac{e^4}{4(p_1 - p_3)^4}\text{Trace}[\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_3)]\text{Trace}[\gamma_\mu(\not{p}_2 + m_2)\gamma_\nu(\not{p}_4 + m_4)]$$

Careful of
μ index vs muon object!

$m_1 = m_3 = m_e$
$m_2 = m_4 = m_\mu$

$$Tr[\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_3)] =$$

$$4p_{1,\sigma}p_{3,\lambda}\left[g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\sigma\lambda} + g^{\mu\lambda}g^{\sigma\nu}\right] + 4m_1 m_3 g^{\mu\nu} =$$

$$4[p_1^\mu p_3^\nu - g^{\mu\nu}p_1 \cdot p_3 + p_1^\nu p_3^\mu + m_1 m_3 g^{\mu\nu}] =$$

$$4[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu + g^{\mu\nu}(m_e^2 - p_1 \cdot p_3)]$$

Electron-muon scattering

$$|\mathcal{M}|^2 = \frac{e^4}{4(p_1 - p_3)^4}\text{Trace}[\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_3)]\text{Trace}[\gamma_\mu(\not{p}_2 + m_2)\gamma_\nu(\not{p}_4 + m_4)]$$

$$Tr[\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_3)] =$$

$$4[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu + g^{\mu\nu}(m_e^2 - p_1 \cdot p_3)]$$

$m_1 = m_3 = m_e$

$m_2 = m_4 = m_\mu$

$$Tr[\gamma_\mu(\not{p}_2 + m_2)\gamma_\nu(\not{p}_4 + m_4)] =$$

$$4[p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} + g_{\mu\nu}(m_\mu^2 - p_2 \cdot p_4)]$$

$$|\mathcal{M}|^2 = \frac{e^4}{4(p_1 - p_3)^4}Tr[\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_3 + m_3)]Tr[\gamma_\mu(\not{p}_2 + m_2)\gamma_\nu(\not{p}_4 + m_4)] =$$

$$\frac{e^4}{4(p_1 - p_3)^4}4[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu + g^{\mu\nu}(m_e^2 - p_1 \cdot p_3)]\times$$

$$4[p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} + g_{\mu\nu}(m_\mu^2 - p_2 \cdot p_4)]$$

Electron-muon scattering

$$|\mathcal{M}|^2 = \frac{4e^4}{(p_1 - p_3)^4}[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu + g^{\mu\nu}(m_e^2 - p_1 \cdot p_3)] \times [p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} + g_{\mu\nu}(m_\mu^2 - p_2 \cdot p_4)]$$

$$|\mathcal{M}|^2 = \frac{4e^4}{(p_1 - p_3)^4}[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) +$$
$$m_\mu^2(p_1 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4) +$$
$$m_\mu^2(p_1 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4) + m_e^2(p_2 \cdot p_4) - (p_2 \cdot p_4)(p_1 \cdot p_3) +$$
$$m_e^2(p_2 \cdot p_4) - (p_2 \cdot p_4)(p_1 \cdot p_3) + 4(m_e^2 - (p_1 \cdot p_3))(m_\mu^2 - (p_2 \cdot p_4))]$$

$$|\mathcal{M}|^2 = \frac{4e^4}{(p_1 - p_3)^4}(2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) - 2m_\mu^2(p_1 \cdot p_3) - 2m_e^2(p_2 \cdot p_4) + 4m_e^2 m_\mu^2)$$

Electron-muon scattering (getting there, I promise)

$$|\mathcal{M}|^2 = \frac{8e^4}{(p_1 - p_3)^4}((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - m_\mu^2(p_1 \cdot p_3) - m_e^2(p_2 \cdot p_4) + 2m_e^2 m_\mu^2)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|M|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|^2}{|\mathbf{p}_i|^2}$$

What if muon is stationary and we neglect its recoil? (Even better - protons are heavier than muons, have same |q|, and don’t decay!)

Electron-proton scattering

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|M|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|^2}{|\mathbf{p}_i|^2}$$

When electron scatters off of a proton (we're assuming we're not so energetic that the electron sees the quarks!), the proton doesn't budge. It's an elastic collision and $|\mathbf{p}_f| = |\mathbf{p}_i|$.
$E_1+E_2 = E_1 + m_p = m_p$

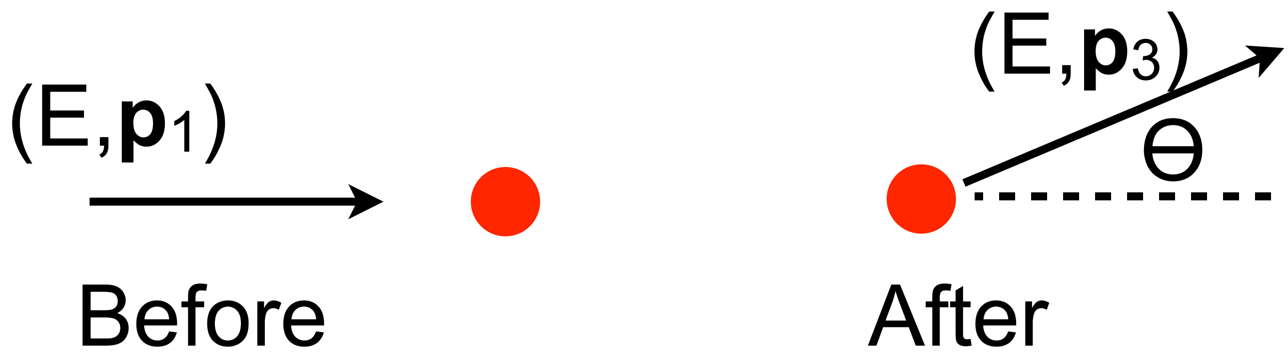
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|M|^2}{m_p^2}$$

Electron-proton scattering

$p_1 = (E, \mathbf{p}_1)$
$p_2 = (m_p, 0)$
$p_3 = (E, \mathbf{p}_3)$
$p_4 = (m_p, 0)$

$|\mathbf{p}_1| = |\mathbf{p}_3|$

Electron-proton scattering



$p_1 = (E,\mathbf{p}_1)$
$p_2 = (m_p,0)$
$p_3 = (E,\mathbf{p}_3)$
$p_4 = (m_p,0)$

$$\mathbf{p}_1 \cdot \mathbf{p}_3 = \mathbf{p}^2 \cos\theta$$

$$(p_1 - p_3)^2 = (E - E)^2 - (\mathbf{p}_1 - \mathbf{p}_3)^2 = -\mathbf{p}_1^2 - \mathbf{p}_3^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_3$$

$$(p_1 - p_3)^2 = -2\mathbf{p}^2 + 2\mathbf{p}^2 \cos\theta = -2\mathbf{p}^2(1 - \cos\theta)$$

$$|\mathcal{M}|^2 = \frac{8e^4}{(p_1 - p_3)^4}((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - m_p^2(p_1 \cdot p_3) - m_e^2(p_2 \cdot p_4) + 2m_e^2 m_p^2)$$

$$p_1 \cdot p_2 = p_3 \cdot p_4 = p_1 \cdot p_4 = p_2 \cdot p_3 = Em_p$$

$$p_1 \cdot p_3 = E^2 - \mathbf{p_1} \cdot \mathbf{p_3} = E^2 - \mathbf{p}^2 \cos\theta$$

$$p_2 \cdot p_4 = m_p^2$$

Electron-proton scattering

$$|\mathcal{M}|^2 = \frac{8e^4}{(p_1 - p_3)^4}((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - m_p^2(p_1 \cdot p_3) - m_e^2(p_2 \cdot p_4) + 2m_e^2 m_p^2)$$

$$p_1 \cdot p_2 = p_3 \cdot p_4 = p_1 \cdot p_4 = p_2 \cdot p_3 = Em_p$$

$$p_1 \cdot p_3 = E^2 - \mathbf{p_1} \cdot \mathbf{p_3} = E^2 - \mathbf{p}^2 \cos\theta$$

$$(p_1 - p_3)^2 = -2\mathbf{p}^2(1 - \cos\theta) \qquad p_2 \cdot p_4 = m_p^2$$

$$|\mathcal{M}|^2 = \frac{8e^4}{4\mathbf{p}^4(1 - \cos\theta)^2}(E^2 m_p^2 + E^2 m_p^2 - m_p^2(E^2 - \mathbf{p}^2\cos\theta) - m_e^2 m_p^2 + 2m_e^2 m_p^2)$$

$$|\mathcal{M}|^2 = \frac{2e^4 m_p^2}{\mathbf{p}^4(1 - \cos\theta)^2}(E^2 + \mathbf{p}^2\cos\theta + m_e^2)$$

$$E^2 = \mathbf{p}^2 + m_e^2$$

$$|\mathcal{M}|^2 = \frac{2e^4 m_p^2}{\mathbf{p}^4(1 - \cos\theta)^2}\left(\mathbf{p}^2(1 + \cos\theta) + 2m_e^2\right)$$

Electron-proton scattering

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{|M|^2}{m_p^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2\mathbf{p}^4(1-\cos\theta)^2}\left(\mathbf{p}^2(1+\cos\theta)+2m_e^2\right)$$

If electron is not relativistic, $m_e >> \mathbf{p}^2$ and $\mathbf{p}^4 = (m_e v)^4$

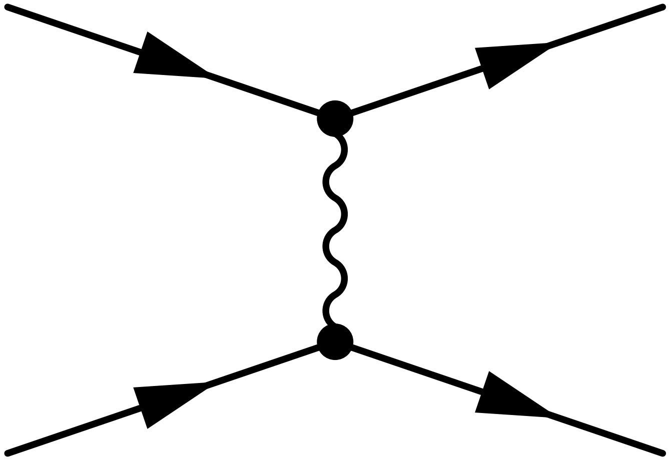
$$\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2 m_e^4 v^4(1-\cos\theta)^2}\left(2m_e^2\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{16\pi^2 m_e^2 v^4(1-\cos\theta)^2}$$

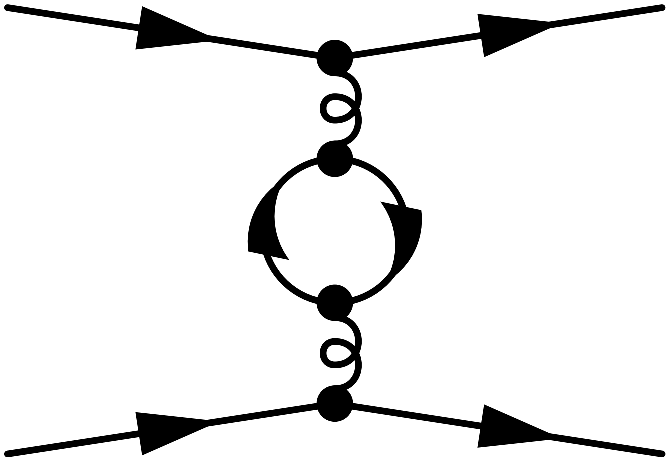
Skipping a topic

Will skip Example 7.8 (pair annihilation). But please read it!

Renormalization

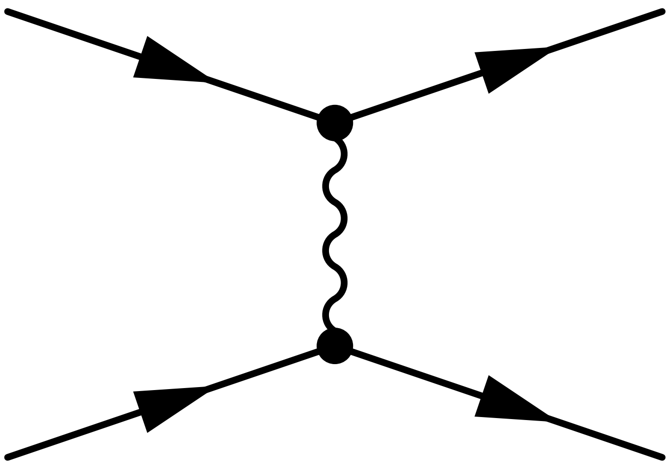


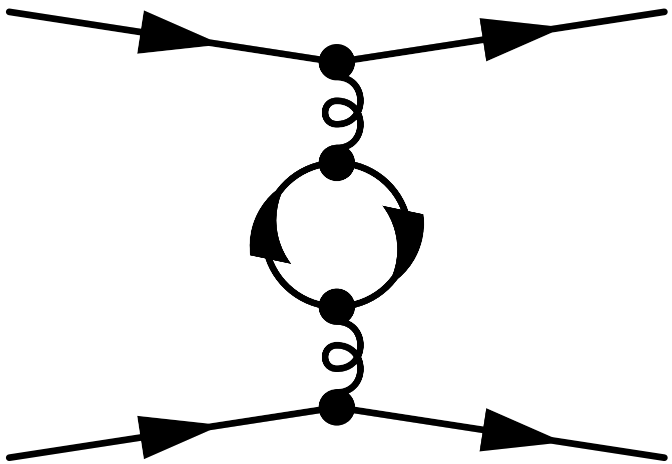
Leading order electron-muon scattering



Vacuum polarization diagram modifying effective charge of electron

Renormalization

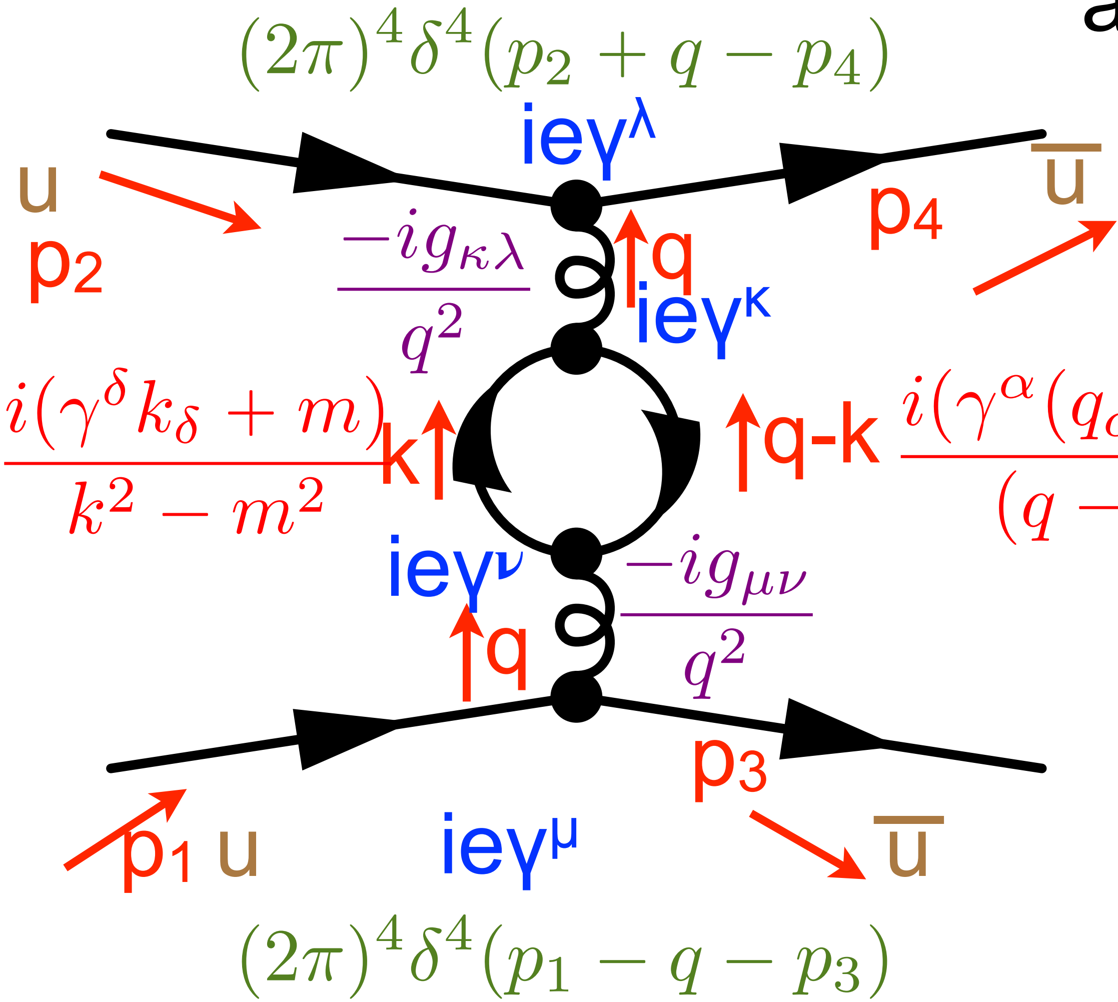




Difference is modifying the photon propagator term. Naively, this leads to infinities! But we have tricks to deal with them

Let's see if we can calculate it

Cheated a bit here and already applied some delta functions!



$$i$$

$$\frac{1}{(2\pi)^4}d^4q\,\frac{1}{(2\pi)^4}d^4k$$

Let's see if we can calculate it

$$i \int [\overline{u}(3)(ie\gamma^\mu)u(1)] \frac{-ig_{\mu\nu}}{q^2}$$

$$ie\gamma^\nu \frac{i(\gamma^\delta k_\delta) + m)}{k^2 - m^2} \frac{i(\gamma^\alpha(q_\alpha - k_\alpha) + m)}{(q-k)^2 - m^2} ie\gamma^\kappa \frac{-ig_{\kappa\lambda}}{q^2}$$

$$[\overline{u}(4)(ie\gamma^\lambda)u(2)]$$

$$(2\pi)^4\delta^4(p_1 - q_1 - p_3)(2\pi)^4\delta^4(p_2 + q - p_4)\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}$$

Some cleanup

$$i \int [\overline{u}(3)(ie\gamma^{\mu})u(1)] \frac{-ig_{\mu\nu}}{q^2}$$

$$ie\gamma^{\nu} \frac{i(\gamma^{\delta}k_{\delta}) + m)}{k^2 - m^2} \frac{i(\gamma^{\alpha}(q_{\alpha} - k_{\alpha}) + m)}{(q-k)^2 - m^2} ie\gamma^{\kappa} \frac{-ig_{\kappa\lambda}}{q^2}$$

$$[\overline{u}(4)(ie\gamma^{\lambda})u(2)]$$

$$(2\pi)^4 \delta^4(p_1 - q_1 - p_3)(2\pi)^4 \delta^4(p_2 + q - p_4) \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}$$

$$ie^4 \int [\overline{u}(3)(\gamma^{\mu})u(1)] \frac{g_{\mu\nu}}{q^2}$$

$$\gamma^{\nu} \frac{(\gamma^{\delta}k_{\delta}) + m)}{k^2 - m^2} \frac{(\gamma^{\alpha}(q_{\alpha} - k_{\alpha}) + m)}{(q-k)^2 - m^2} \gamma^{\kappa} \frac{g_{\kappa\lambda}}{q^2}$$

$$[\overline{u}(4)(\gamma^{\lambda})u(2)]$$

$$(2\pi)^4 \delta^4(p_1 - q_1 - p_3)(2\pi)^4 \delta^4(p_2 + q - p_4) \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}$$

Some cleanup

$$ie^4 \int [\overline{u}(3)(\gamma^\mu)u(1)] \frac{g_{\mu\nu}}{q^2}$$

$$\gamma^\nu \frac{(\gamma^\delta k_\delta) + m)}{k^2 - m^2} \frac{(\gamma^\alpha (q_\alpha - k_\alpha) + m)}{(q-k)^2 - m^2} \gamma^\kappa \frac{g_{\kappa\lambda}}{q^2}$$

$$[\overline{u}(4)(\gamma^\lambda)u(2)]$$

$$(2\pi)^4 \delta^4 (p_1 - q_1 - p_3)(2\pi)^4 \delta^4 (p_2 + q - p_4) \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4}$$

$$ie^4 \int [\overline{u}(3)(\gamma_\nu)u(1)]$$

$$\frac{\gamma^\nu}{q^2} \frac{(\not{k} + m)}{k^2 - m^2} \frac{(\not{q} - \not{k} + m)}{(q-k)^2 - m^2} \frac{\gamma_\lambda}{q^2}$$

$$[\overline{u}(4)(\gamma^\lambda)u(2)]$$

$$(2\pi)^4 \delta^4 (p_1 - q - p_3)(2\pi)^4 \delta^4 (p_2 + q - p_4) \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4}$$

Time to use those delta functions

$$ie^4 \int [\overline{u}(3)(\gamma_\nu)u(1)]$$

$$\frac{\gamma^\nu}{q^2} \frac{(\not{k} + m)}{k^2 - m^2} \frac{(\not{q} - \not{k} + m)}{(q-k)^2 - m^2} \frac{\gamma_\lambda}{q^2}$$

$$[\overline{u}(4)(\gamma^\lambda)u(2)]$$

$$(2\pi)^4 \delta^4(p_1 - q - p_3)(2\pi)^4 \delta^4(p_2 + q - p_4) \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}$$

$$ie^4 \int [\overline{u}(3)(\gamma_\nu)u(1)]$$

$$\frac{\gamma^\nu}{(p_1 - p_3)^2} \frac{(\not{k} + m)}{k^2 - m^2} \frac{(\not{p}_1 - \not{p}_3 - \not{k} + m)}{(p_1 - p_3 - k)^2 - m^2} \frac{\gamma_\lambda}{(p_1 - p_3)^2} \frac{d^4k}{(2\pi)^4}$$

$$[\overline{u}(4)(\gamma^\lambda)u(2)]$$

$$(2\pi)^4 \delta^4(p_2 + p_1 - p_3 - p_4)$$

Almost there

$$ie^4 \int [\overline{u}(3)(\gamma_\nu)u(1)]$$

$$\frac{\gamma^\nu}{(p_1-p_3)^2}\frac{(\not k+m)}{k^2-m^2}\frac{(\not p_1-\not p_3-\not k+m)}{(p_1-p_3-k)^2-m^2}\frac{\gamma_\lambda}{(p_1-p_3)^2}\frac{d^4k}{(2\pi)^4}$$

$$[\overline{u}(4)(\gamma^\lambda)u(2)]$$

$$(2\pi)^4\delta^4(p_2+p_1-p_3-p_4)$$

$$ie^4 \int [\overline{u}(3)(\gamma_\nu)u(1)]$$

$$\frac{\gamma^\nu}{(p_1-p_3)^2}\frac{(\not k)+m}{k^2-m^2}\frac{(\not p_1-\not p_3-\not k)+m}{(p_1-p_3-k)^2-m^2}\frac{\gamma_\lambda}{(p_1-p_3)^2}\frac{d^4k}{(2\pi)^4}$$

$$[\overline{u}(4)(\gamma^\lambda)u(2)]$$

$$\cancel{(2\pi)^4\delta^4(p_2+p_1-p_3-p_4)}$$

Cancel

So ...

$$\mathcal{M} = ie^4[\overline{u}(3)(\gamma_\mu)u(1)]I^{\mu\nu}[\overline{u}(4)(\gamma_\nu)u(2)]$$

$$I^{\mu\nu} = \int \frac{\gamma^\mu}{(p_1 - p_3)^2} \frac{\not{k} + m}{k^2 - m^2} \frac{\not{p}_1 - \not{p}_3 - \not{k} + m}{(p_1 - p_3 - k)^2 - m^2} \frac{\gamma^\nu}{(p_1 - p_3)^2} \frac{d^4k}{(2\pi)^4}$$

If we compare this new matrix element to the leading order one, we see that it is a modification to the photon propagator! Note that it depends on (p_1-p_3), or q, as we might expect. Why?

What does integral look like?

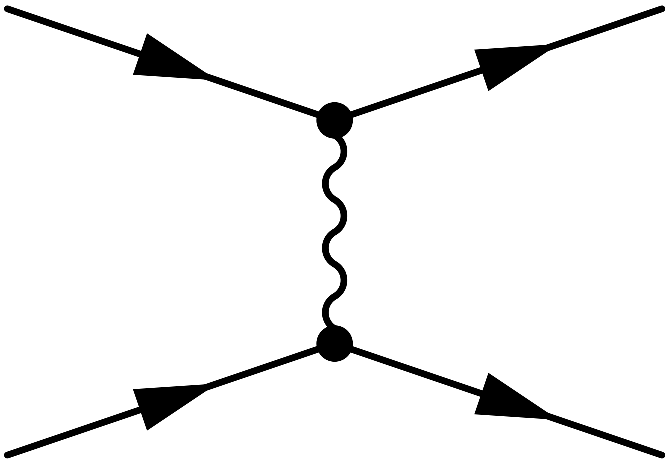
$$I^{\mu\nu} = \int \frac{\gamma^\mu}{(p_1 - p_3)^2} \frac{\not{k} + m}{k^2 - m^2} \frac{\not{p}_1 - \not{p}_3 - \not{k} + m}{(p_1 - p_3 - k)^2 - m^2} \frac{\gamma^\nu}{q^2} \frac{d^4k}{(2\pi)^4}$$

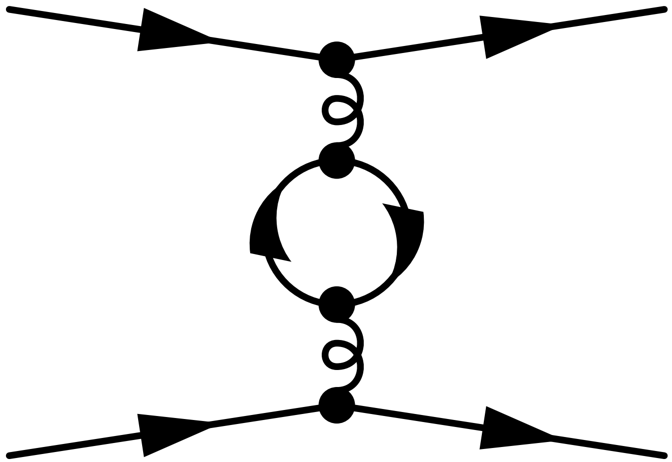
$$I \propto \int \frac{k^2}{k^4} d^4k$$

$$I \propto \int \frac{k^2}{k^4} k^3 dk$$

$$I \to \infty \text{ for } k \to \infty$$

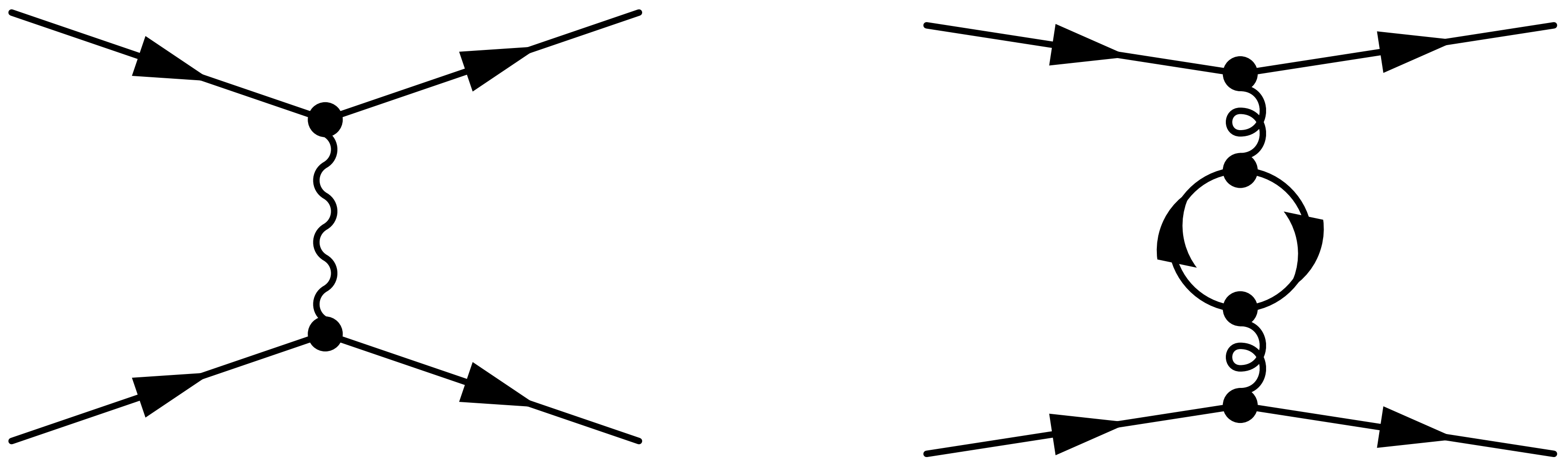
Renormalization





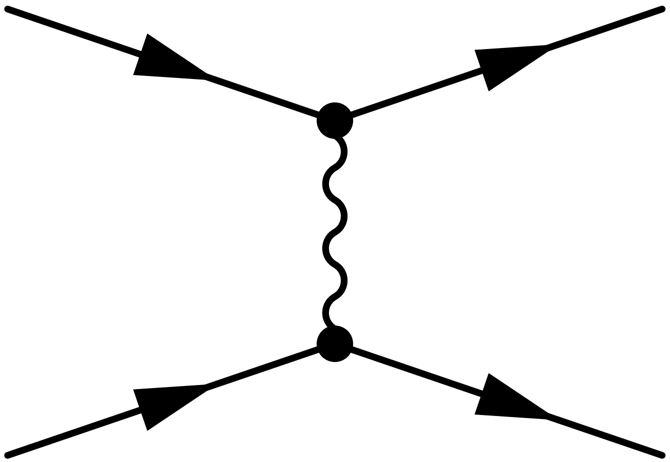
“Infinity” can be removed if we include it in the definition of electron charge, ie the renormalized electron charge. That seems odd, but is OK - the renormalized charge (these diagrams, plus others up to infinite order) are really what we measure in the lab, not the bare value of e!

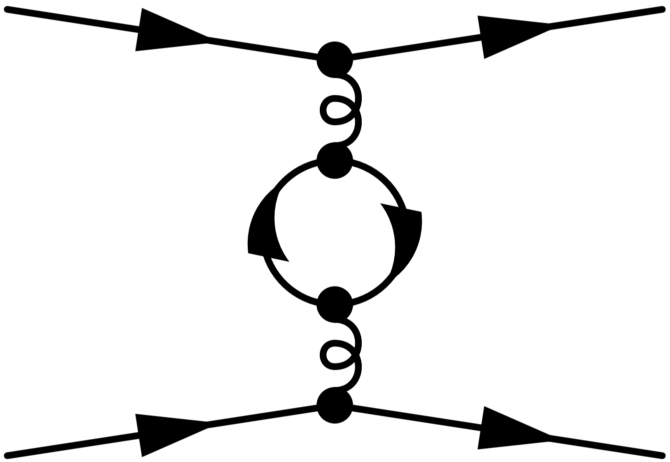
Renormalization



When we renormalize the electric charge there is a leftover term that is finite, but depends on q^2. This is as before the running of the coupling constant. One way to think of this - the larger q^2 the closer the particles can get, which changes the effective electromagnetic screening. A small but measurable effect!

Of course, those were two terms... there are an infinite number!





$$\alpha(q^2) = \frac{\alpha(0)}{1 - \frac{\alpha(0)}{3\pi}\ln(q^2/m^2)}$$

On renormalization

It's OK to cry sometimes