Recall that most of the particles we play with are unstable. Want to calculate their decay rate (Γ), the probability per unit time of decay

$$dN = -\Gamma N dt$$

$$N(t) = N(0)e^{-\Gamma t}$$

Number of particles that make it to N(t) and decay at N(t+dt)

$$N(t + dt) = N_0 e^{-\Gamma(t+dt)} = N_0 e^{-\Gamma t} e^{-\Gamma dt}$$

$$e^x \sim 1 + x \rightarrow e^{-\Gamma dt} \sim 1 - \Gamma dt$$

$$N(t + dt) = N_0 e^{-\Gamma t} \left(1 - \Gamma dt\right)$$

$$N(t) - N(t + dt) \sim N_0 e^{-\Gamma t} - N_0 e^{-\Gamma t} \left(1 - \Gamma dt\right) = \boxed{N_0 e^{-\Gamma t} \Gamma dt}$$

Average lifetime

$$\tau = \int_0^\infty t \left(e^{-\Gamma t} \Gamma dt \right)$$

$$\tau = \Gamma \int_0^\infty t e^{-\Gamma t} dt$$

$$u = t, dv = e^{-\Gamma t} dt, du = dt, v = \frac{-1}{\Gamma} e^{-\Gamma t}$$

$$\tau = \Gamma \left[\frac{-t}{\Gamma} e^{-\Gamma t} \right]_0^\infty - \Gamma \int_0^\infty \frac{-1}{\Gamma} e^{-\Gamma t} dt$$

$$\tau = -\Gamma \left[\frac{1}{\Gamma^2} e^{-\Gamma t} \right]_0^\infty = \frac{1}{\Gamma}$$

Number of particles that make it to N(t) and decay at N(t+dt):

$$N_0 e^{-\Gamma t} \Gamma dt$$

So, fraction that decay at N(t+dt) is then

$$e^{-\Gamma t} \Gamma dt$$

So average lifetime = 1/Γ

Typically define partial widths Γ_i which are defined as the rates for specific decays. The total decay rate is the sum of the partial widths, and the lifetime is given by τ:

$$\Gamma = \sum_{i=1}^{n} \Gamma_i \qquad BR(i) = \frac{\Gamma_i}{\Gamma}$$

$$\tau = \frac{1}{\Gamma}$$

The branching ratio (BR) is the fraction of all decays to that go to a specific final state

We're also very often interested in the collision between two objects. Collisions can be:

A+B→A+B (elastic, no energy lost)
A+B→Other (inelastic, energy "lost" in the form of conversion to other particles)

Typically we refer to the cross section (σ) for a collision process. A natural way to think of a collision that relates to classical scattering theory. Units of area

Sometimes in particle physics we think of differential cross sections (dσ/dX), which refer to how often a process occurs per unit of X

X can be energy (ex: cross section for collision to produce a particle with a certain energy)
X can be number of objects (ex: how often does a collision produce a process with a certain number of jets)
X can be angle (ex: cross section where collision decay products travel in a certain direction)

In particle physics, we typically use "barns" (b).
1 barn = 10^{-28} m^2 (typically ~area of Uranium nucleus)
If you believe Wikipedia and its references ...

Etymology [edit]
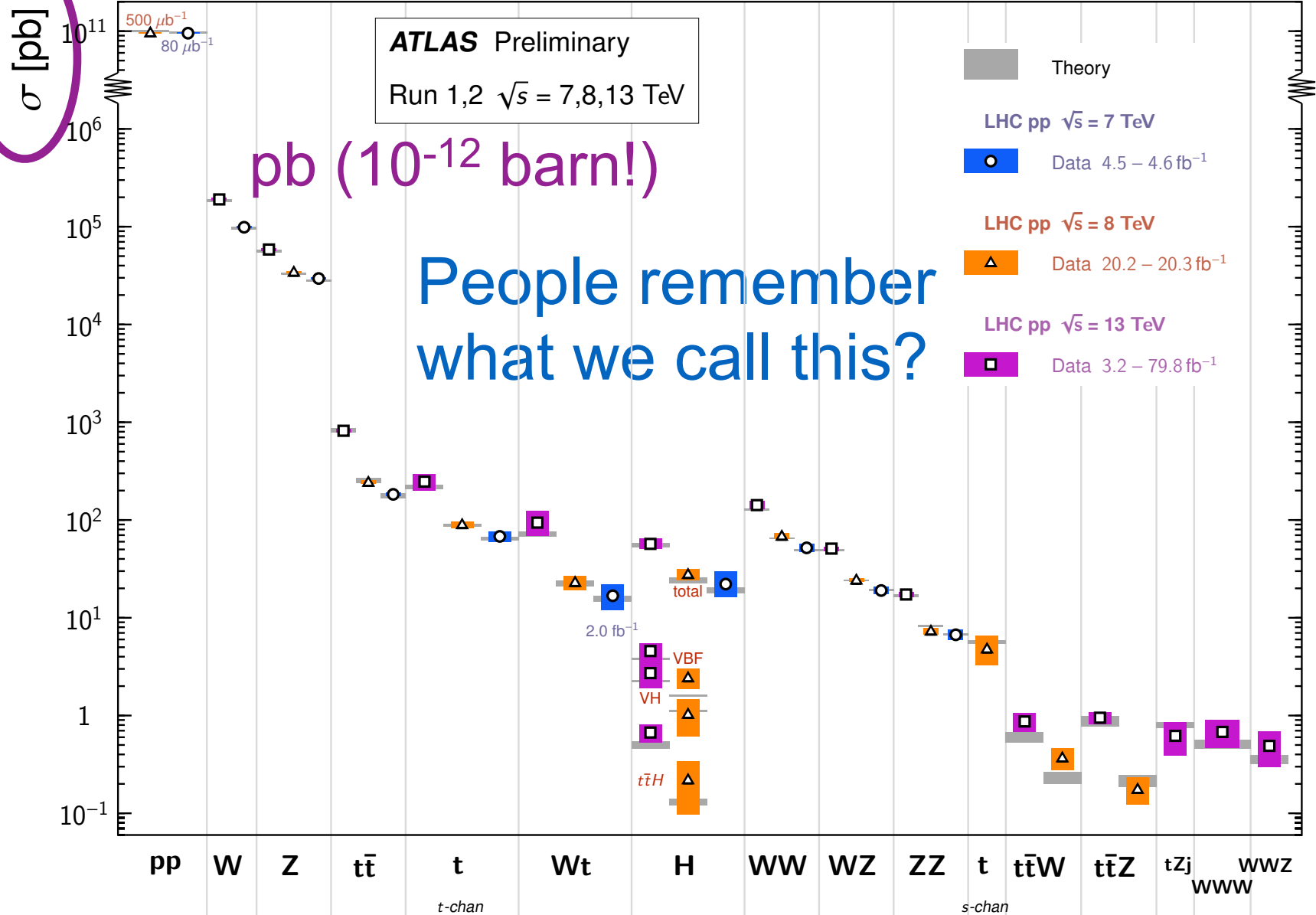
The etymology of the unit barn is whimsical: during wartime research on the atomic bomb, American physicists at Purdue University needed a secretive unit to describe the approximate cross sectional area presented by the typical nucleus (10^{-28} m^2) and decided on "barn." This was particularly applicable because they considered this a large target for particle accelerators that needed to have direct strikes on nuclei and the American idiom "couldn't hit the broad side of a barn"[2] refers to someone whose aim is terrible. Initially they hoped the name would obscure any reference to the study of nuclear structure; eventually, the word became a standard unit in nuclear and particle physics.[3][4]

1 barn is a huge number in particle physics!

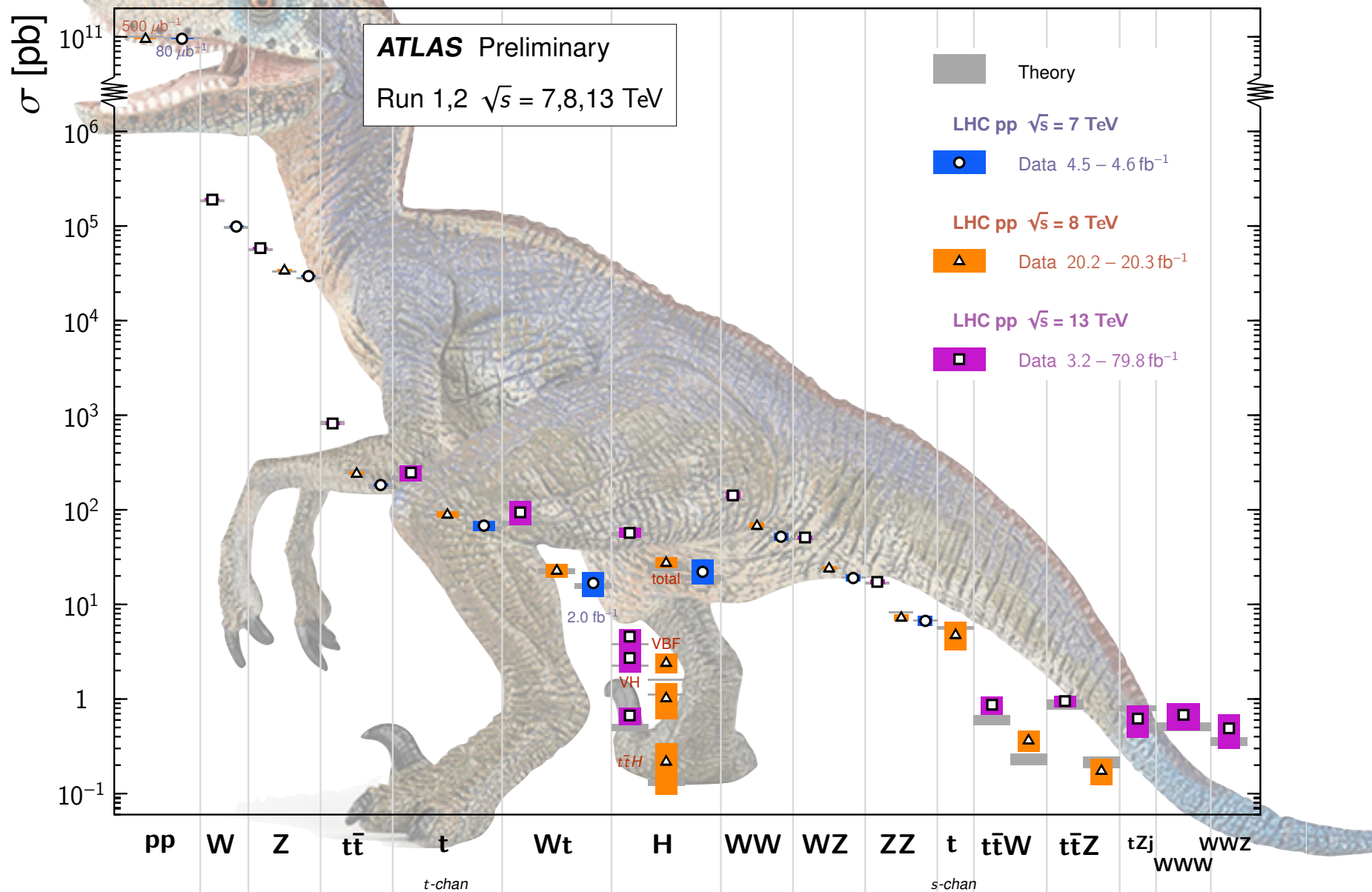**Standard Model Total Production Cross Section Measurements** *Status: March 2019*

pb (10⁻¹² barn!)

People remember what we call this?

Standard Model Total Production Cross Section Measurements *Status: March 2019*
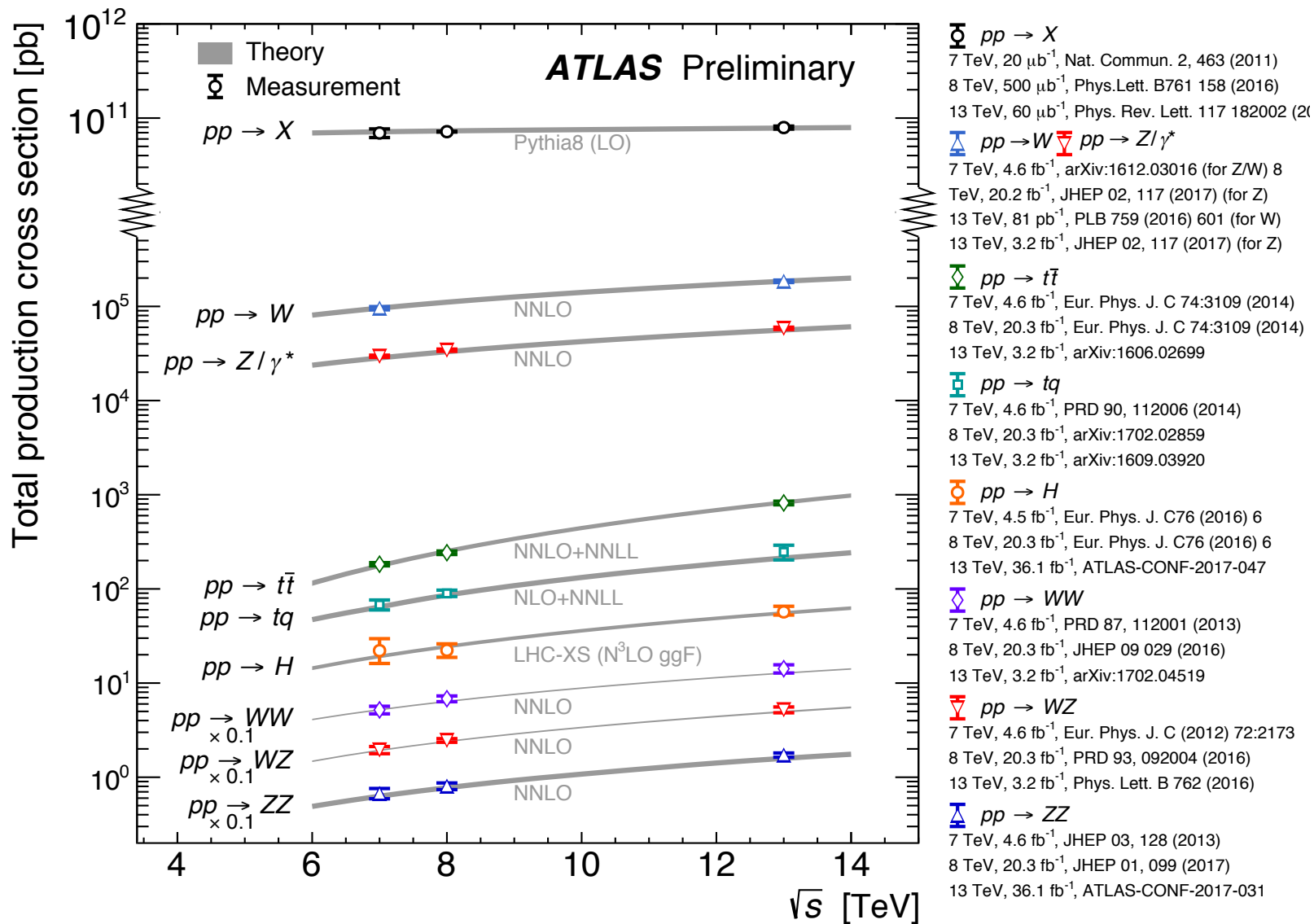
https://atlas.web.cern.ch/Atlas/GROUPS/PHYSICS/CombinedSummaryPlots/SM/

Standard Model Total Production Cross Section Measurements *Status: March 2019*

Cross sections at the LHC

Small but non-zero dependence

Limits on processes instead

ATLAS Exotics Searches* - 95% CL Upper Exclusion Limits

Status: May 2019

ATLAS Preliminary

$\int \mathcal{L} \, dt = (3.2 - 139) \, \text{fb}^{-1}$ $\sqrt{s} = 8, 13 \, \text{TeV}$

	Model	ℓ, γ	Jets†	E_T^{miss}	$\int \mathcal{L} \, dt[\text{fb}^{-1}]$	Limit		Reference
Extra dimensions	ADD $G_{KK} + g/q$	$0 \, e, \mu$	$1 - 4 \, j$	Yes	36.1	M_D 7.7 TeV	$n = 2$	1711.03301
	ADD non-resonant $\gamma\gamma$	$2 \, \gamma$	–	–	36.7	M_S 8.6 TeV	$n = 3$ HLZ NLO	1707.04147
	ADD QBH	–	$2 \, j$	–	37.0	M_{th} 8.9 TeV	$n = 6$	1703.09127
	ADD BH high $\sum p_T$	$\geq 1 \, e, \mu$	$\geq 2 \, j$	–	3.2	M_{th} 8.2 TeV	$n = 6$, $M_D = 3$ TeV, rot BH	1606.02265
	ADD BH multijet	–	$\geq 3 \, j$	–	3.6	M_{th} 9.55 TeV	$n = 6$, $M_D = 3$ TeV, rot BH	1512.02586
	RS1 $G_{KK} \to \gamma\gamma$	$2 \, \gamma$	–	–	36.7	G_{KK} mass 4.1 TeV	$k/\overline{M}_{Pl} = 0.1$	1707.04147
	Bulk RS $G_{KK} \to WW/ZZ$	multi-channel			36.1	G_{KK} mass 2.3 TeV	$k/\overline{M}_{Pl} = 1.0$	1808.02380
	Bulk RS $G_{KK} \to WW \to qqqq$	$0 \, e, \mu$	$2 \, J$	–	139	G_{KK} mass 1.6 TeV	$k/\overline{M}_{Pl} = 1.0$	ATLAS-CONF-2019-003
	Bulk RS $g_{KK} \to tt$	$1 \, e, \mu$	$\geq 1 \, b, \geq 1J/2j$	Yes	36.1	g_{KK} mass 3.8 TeV	$\Gamma/m = 15\%$	1804.10823
	2UED / RPP	$1 \, e, \mu$	$\geq 2 \, b, \geq 3 \, j$	Yes	36.1	KK mass 1.8 TeV	Tier (1,1), $\mathcal{B}(A^{(1,1)} \to tt) = 1$	1803.09678
Gauge bosons	SSM $Z' \to \ell\ell$	$2 \, e, \mu$	–	–	139	Z' mass 5.1 TeV		1903.06248
	SSM $Z' \to \tau\tau$	$2 \, \tau$	–	–	36.1	Z' mass 2.42 TeV		1709.07242
	Leptophobic $Z' \to bb$	–	$2 \, b$	–	36.1	Z' mass 2.1 TeV		1805.09299
	Leptophobic $Z' \to tt$	$1 \, e, \mu$	$\geq 1 \, b, \geq 1J/2j$	Yes	36.1	Z' mass 3.0 TeV	$\Gamma/m = 1\%$	1804.10823
	SSM $W' \to \ell\nu$	$1 \, e, \mu$	–	Yes	139	W' mass 6.0 TeV		CERN-EP-2019-100
	SSM $W' \to \tau\nu$	$1 \, \tau$	–	Yes	36.1	W' mass 3.7 TeV		1801.06992
	HVT $V' \to WZ \to qqqq$ model B	$0 \, e, \mu$	$2 \, J$	–	139	V' mass 3.6 TeV	$g_V = 3$	ATLAS-CONF-2019-003
	HVT $V' \to WH/ZH$ model B	multi-channel			36.1	V' mass 2.93 TeV	$g_V = 3$	1712.06518
	LRSM $W_R \to tb$	multi-channel			36.1	W_R mass 3.25 TeV		1807.10473
	LRSM $W_R \to \mu N_R$	$2 \, \mu$	$1 \, J$	–	80	W_R mass 5.0 TeV	$m(N_R) = 0.5$ TeV, $g_L = g_R$	1904.12679
CI	CI $qqqq$	–	$2 \, j$	–	37.0	Λ 21.8 TeV η_{LL}^-		1703.09127
	CI $\ell\ell qq$	$2 \, e, \mu$	–	–	36.1	Λ 40.0 TeV η_{LL}^-		1707.02424
	CI $tttt$	$\geq 1 \, e, \mu$	$\geq 1 \, b, \geq 1 \, j$	Yes	36.1	Λ 2.57 TeV	$|C_{4t}| = 4\pi$	1811.02305
DM	Axial-vector mediator (Dirac DM)	$0 \, e, \mu$	$1 - 4 \, j$	Yes	36.1	m_{med} 1.55 TeV	$g_q = 0.25$, $g_\chi = 1.0$, $m(\chi) = 1$ GeV	1711.03301
	Colored scalar mediator (Dirac DM)	$0 \, e, \mu$	$1 - 4 \, j$	Yes	36.1	m_{med} 1.67 TeV	$g = 1.0$, $m(\chi) = 1$ GeV	1711.03301
	$VV\chi\chi$ EFT (Dirac DM)	$0 \, e, \mu$	$1 \, J, \leq 1 \, j$	Yes	3.2	M_* 700 GeV	$m(\chi) < 150$ GeV	1608.02372
	Scalar reson. $\phi \to t\chi$ (Dirac DM)	$0-1 \, e, \mu$	$1 \, b, 0-1 \, J$	Yes	36.1	m_ϕ 3.4 TeV	$y = 0.4$, $\lambda = 0.2$, $m(\chi) = 10$ GeV	1812.09743
LQ	Scalar LQ 1st gen	$1,2 \, e$	$\geq 2 \, j$	Yes	36.1	LQ mass 1.4 TeV	$\beta = 1$	1902.00377
	Scalar LQ 2nd gen	$1,2 \, \mu$	$\geq 2 \, j$	Yes	36.1	LQ mass 1.56 TeV	$\beta = 1$	1902.00377
	Scalar LQ 3rd gen	$2 \, \tau$	$2 \, b$	–	36.1	LQ_3^u mass 1.03 TeV	$\mathcal{B}(LQ_3^u \to b\tau) = 1$	1902.08103
	Scalar LQ 3rd gen	$0-1 \, e, \mu$	$2 \, b$	Yes	36.1	LQ_3^d mass 970 GeV	$\mathcal{B}(LQ_3^d \to t\tau) = 0$	1902.08103
Heavy quarks	VLQ $TT \to Ht/Zt/Wb + X$	multi-channel			36.1	T mass 1.37 TeV	SU(2) doublet	1808.02343
	VLQ $BB \to Wt/Zb + X$	multi-channel			36.1	B mass 1.34 TeV	SU(2) doublet	1808.02343
	VLQ $T_{5/3} T_{5/3} \, | T_{5/3} \to Wt + X$	2(SS)/$\geq$3 e, μ	$\geq 1 \, b, \geq 1 \, j$	Yes	36.1	$T_{5/3}$ mass 1.64 TeV	$\mathcal{B}(T_{5/3} \to Wt) = 1$, $c(T_{5/3} Wt) = 1$	1807.11883
	VLQ $Y \to Wb + X$	$1 \, e, \mu$	$\geq 1 \, b, \geq 1j$	Yes	36.1	Y mass 1.85 TeV	$\mathcal{B}(Y \to Wb) = 1$, $c_R(Wb) = 1$	1812.07343
	VLQ $B \to Hb + X$	$0 \, e, \mu, 2 \, \gamma$	$\geq 1 \, b, \geq 1j$	Yes	79.8	B mass 1.21 TeV	$\kappa_B = 0.5$	ATLAS-CONF-2018-024
	VLQ $QQ \to WqWq$	$1 \, e, \mu$	$\geq 4 \, j$	Yes	20.3	Q mass 690 GeV		1509.04261
Excited fermions	Excited quark $q^* \to qg$	–	$2 \, j$	–	139	q^* mass 6.7 TeV	only u^* and d^*, $\Lambda = m(q^*)$	ATLAS-CONF-2019-007
	Excited quark $q^* \to q\gamma$	$1 \, \gamma$	$1 \, j$	–	36.7	q^* mass 5.3 TeV	only u^* and d^*, $\Lambda = m(q^*)$	1709.10440
	Excited quark $b^* \to bg$	–	$1 \, b, 1 \, j$	–	36.1	b^* mass 2.6 TeV		1805.09299
	Excited lepton ℓ^*	$3 \, e, \mu$	–	–	20.3	ℓ^* mass 3.0 TeV	$\Lambda = 3.0$ TeV	1411.2921
	Excited lepton ν^*	$3 \, e, \mu, \tau$	–	–	20.3	ν^* mass 1.6 TeV	$\Lambda = 1.6$ TeV	1411.2921
Other	Type III Seesaw	$1 \, e, \mu$	$\geq 2 \, j$	Yes	79.8	N^0 mass 560 GeV		ATLAS-CONF-2018-020
	LRSM Majorana ν	$2 \, \mu$	$2 \, j$	–	36.1	N_R mass 3.2 TeV	$m(W_R) = 4.1$ TeV, $g_L = g_R$	1809.11105
	Higgs triplet $H^{\pm\pm} \to \ell\ell$	$2,3,4 \, e, \mu$ (SS)	–	–	36.1	$H^{\pm\pm}$ mass 870 GeV	DY production	1710.09748
	Higgs triplet $H^{\pm\pm} \to \ell\tau$	$3 \, e, \mu, \tau$	–	–	20.3	$H^{\pm\pm}$ mass 400 GeV	DY production, $\mathcal{B}(H_L^{\pm\pm} \to \ell\tau) = 1$	1411.2921
	Multi-charged particles	–	–	–	36.1	multi-charged particle mass 1.22 TeV	DY production, $|q| = 5e$	1812.03673
	Magnetic monopoles	–	–	–	34.4	monopole mass 2.37 TeV	DY production, $|g| = 1g_D$, spin 1/2	1905.10130

$\sqrt{s} = 8$ TeV $\sqrt{s} = 13$ TeV **partial data** $\sqrt{s} = 13$ TeV **full data**

Mass scale [TeV]

10^{-1} 1 10
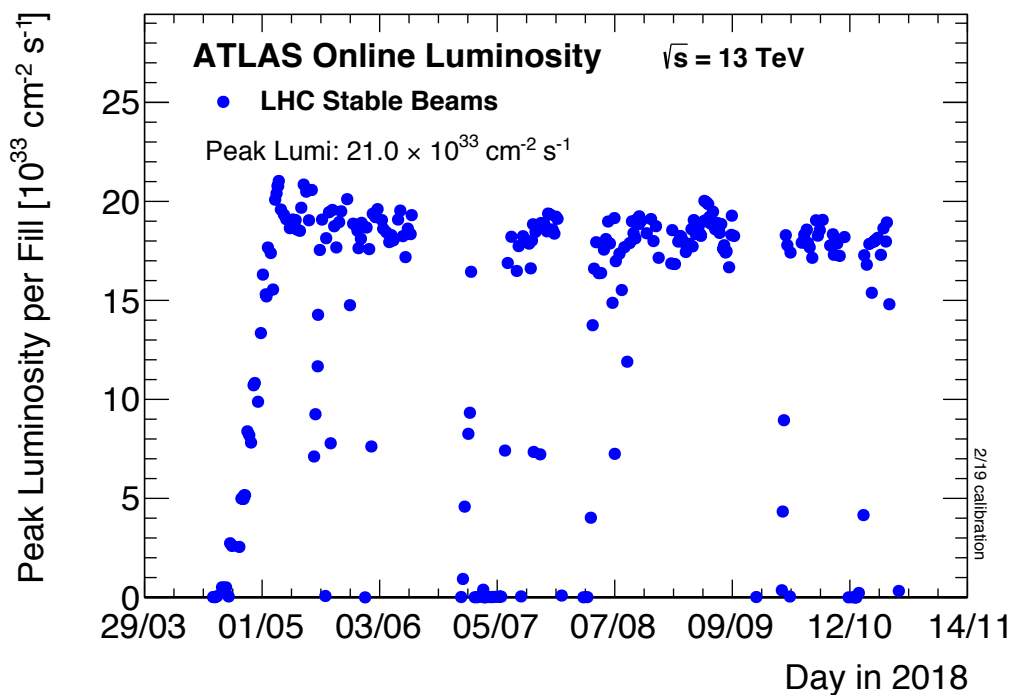
*Only a selection of the available mass limits on new states or phenomena is shown.

†Small-radius (large-radius) jets are denoted by the letter j (J).

Luminosity defines how many particles we have to collide. More specifically, the number of particles per unit time per unit area. We often think of the instantaneous luminosity, which is the luminosity at any one given time

We also think of the integrated luminosity over time, which when multiplied by a cross section, tells us how many events of a certain process we expected to produce.

Example: ATLAS collected ~140 fb^{-1} of data at energy of 13 TeV. The cross section for Higgs bosons at 13 TeV is ~50 pb = 50,000 fb, so ~7.5 million Higgs bosons were produced in that data set at ATLAS

The branching ratio for Higgs bosons to pairs of photons is 0.0023, so ~17,000 Higgs bosons were produced in the diphoton final state

Have you seen this before in Quantum Mechanics? We'll need the relativistic version of it. If not, suggest you look it up

Fermi was a smart man (hard to think of someone with more things named after him). He told us that the rate for a process to occur is equal to the square of the quantum mechanical amplitude (aka the matrix element), multiplied by the density of states

Dirac was also a smart man (maybe fewer things named after him than Fermi, but not by that much)

$$\delta(x) = \begin{array}{ll} 0, & x \neq 0 \\ \infty, & x = 0 \end{array} \qquad \delta(x - a) = \begin{array}{ll} 0, & x \neq a \\ \infty, & x = a \end{array}$$

$$f(x)\delta(x) = f(0)\delta(x)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(0)\delta(x)dx = f(0)\int_{-\infty}^{\infty} \delta(x)dx = f(0)$$

$$f(x)\delta(x - a) = f(a)\delta(x - a)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x - a)dx = \int_{-\infty}^{\infty} f(a)\delta(x - a)dx = f(a)\int_{-\infty}^{\infty} \delta(x - a)dx = f(a)$$

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = ?$$

$$z = kx, dx = dz/k$$

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \frac{1}{k}\int_{z=-\infty}^{z=\infty} f(z/k)\delta(z)dz =$$

$$\frac{1}{k}\int_{z=-\infty}^{z=\infty} f(z/k)\delta(z)dz = \frac{f(0)}{k}\int_{-\infty}^{\infty} \delta(z)z = \frac{f(0)}{k}$$
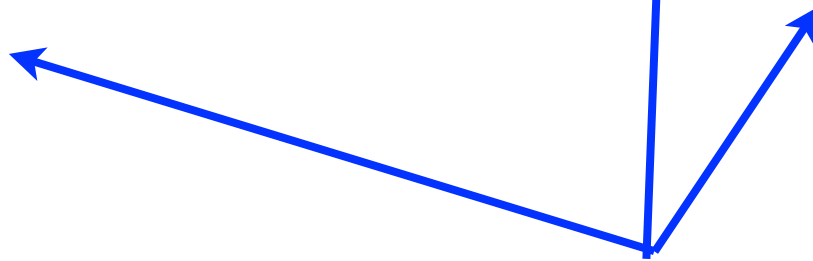
Note that here, limits of integration go from -infinity to +infinity only if k is positive

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = ?$$

$$z = kx, dx = dz/k \text{ (k negative)}$$

$$\int_{x=-\infty}^{x=\infty} f(x)\delta(kx)dx = \frac{1}{k}\int_{z=+\infty}^{z=-\infty} f(z/k)\delta(z)dz = -\frac{1}{k}\int_{z=-\infty}^{z=+\infty} f(z/k)\delta(z)dz$$

$$-\frac{1}{k}\int_{-\infty}^{\infty} f(z/k)\delta(z)dz = \frac{f(0)}{|k|}\int_{-\infty}^{\infty} \delta(z)z = \frac{f(0)}{|k|}$$

$$\boxed{\delta(kx) = \frac{1}{|k|}\delta(x)}$$

Any arbitrary function with potentially any number of zeros.

$$\delta(g(x)), g(x_i) = 0, i = 1, 2, 3...$$

$$g(x) = g(x_i) + (x - x_i)g'(x_i) + \frac{1}{2}(x - x_i)^2 g''(x_i) + ...$$

if x_i something other than zero, this just shifts the delta function
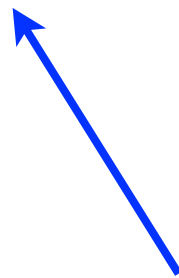
g'(x_i) is "k" in the previous slide

For one zero:

$$\delta(g(x)) = \frac{1}{|g'(x_i)|}\delta(x - x_i)$$

In total:

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x - x_i)$$

$$\theta(x) = 0 \, (x < 0)$$
$$\theta(x) = 1 \, (x > 0)$$
$$\delta(x) = \frac{d\theta}{dx}$$

What does he look like to you?

Wikipedia image of Oliver Heaviside

Maybe it's just me?

Let's do

Griffiths problems A1 and A3 together

$$A.1a) \int_0^3 (2x^2 + 7x + 3)\delta(x - 1)dx$$

$$A.1b) \int_0^3 \ln(1 + x)\delta(\pi - x)dx$$

$$A.3) \text{Simplify } \delta(\sin x)$$

Fermi's Golden Rule: nothing to do with how you should treat others (that's a different Golden Rule). It tells us that the rate for a process (a given collision or decay) is a product of the square of the matrix element (dynamics specific to the theory of the forces at play) and the phase space (recall that things like to happen the more phase space there is for it to happen)

For example, let's begin by considering 1→2+3+4+....n

In other words, object 1 decaying to objects 2, 3, 4... (n-1 total particles)

Up to some overall normalization, consider the phase space of the jth object as $d^4p_j=d(p^0_j)d^3(\mathbf{p_j})$

Hopefully that makes some intuitive sense as a definition of phase space?

But of course, the jth object can't just have any arbitrary value of energy and momentum

$1 \rightarrow 2+3+4+....n$

The decay products have a definite mass. In other words, $p_j^2 = m_j^2$. Can enforce this in an integral with a delta function, $\delta(p_j^2 - m_j^2)$

Don't allow negative energy states of decay productions, so $p_j^0 > 0$. Can enforce this with Heaviside function, $\theta(p_j^0)$

Conserve energy and momentum. Can enforce this with $\delta(p_1 - p_2 - p_3 - ... p_n)$

1→2+3+4+....n

Normalization

Matrix element squared (to be worked on later)

Momentum/energy conservation

Decay rate

$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3... - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0)\frac{d^4p_j}{(2\pi)^4}$$

On-shell final products

E>0

Phase space

Recall from basic QM: What is the transition rate from state |i> to state |f>, given some interaction Hamiltonian ($\hat{H}'$)? Given by Γ_{fi}:

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_i)$$

Matrix element ("transition")

$$|T_{fi}| = <f|\hat{H}'|i> + \sum_{j \neq i} \frac{<f|\hat{H}'|j><j|\hat{H}'|i>}{E_i - E_j} + ...$$

$$\rho(E_i) = \left| \frac{dn}{dE} \right|_{E_i}$$

Density of states (phase space)

We can rewrite phase space as:

$$\rho(E_i) = \left| \frac{dn}{dE} \right|_{E_i}$$

$$\left| \frac{dn}{dE} \right|_{E_i} = \int \frac{dn}{dE} \delta(E_i - E) dE$$

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E) dn$$

a

Recall in a one dimensional box that boundary conditions force quantization of momentum

$$\psi(x + a) = \psi(x) \rightarrow p_x = n_x \frac{2\pi}{a}$$

Similarly, in 3 dimensions (if in a 3D cube of length a on each side, total volume V)

$$(p_x, p_y, p_z) = (n_x, n_y, n_z)\frac{2\pi}{a}$$

What is the volume of a state in momentum space?

$$d^3\mathbf{p} = dp_x dp_y dp_z = \left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$

Common to normalize to a particle per unit volume, so that the number of states for ith particle is

$$dn_i = \frac{d^3 \mathbf{p}_i}{(2\pi)^3}$$

And then the total number of states is

$$dn = \prod_{i=1}^{N-1} dn_i = \prod_{i=1}^{N-1} \frac{d^3 \mathbf{p}_i}{(2\pi)^3}$$

Note that we "lost" the last dn_i because it is not independent (fixed, due to momentum conservation)

We can add in the last missing $d^3\mathbf{p}_N$ by including a delta function, which forces momentum conservation (particle a is the one decaying), and accounting for the extra $(2\pi)^3$

$$dn = (2\pi)^3 \prod_{i=1}^{N} \frac{d^3\mathbf{p}_i}{(2\pi)^3} \delta^3 \left(\mathbf{p}_a - \sum_{i=1}^{N} \mathbf{p}_i \right)$$

For particle a decaying
to particles 1 and 2...

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_a - E_1 - E_2) dn$$

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_a - E_1 - E_2) \frac{d^3\mathbf{p_1}}{(2\pi)^3} \frac{d^3\mathbf{p_2}}{(2\pi)^3} \delta^3(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2)$$

Recall that we normalized our transition matrix element to one particle per unit volume.

What happens in another reference frame? Perpendicular to direction of motion, nothing.

Parallel to direction of motion, we get a Lorentz contraction of $1/\gamma = m/E$, therefore to be Lorentz invariant our normalization must be proportional to $1/E$ (we choose $1/2E$ by Griffiths' convention)

Our new Lorentz-invariant phase space

$$\prod_{i=1}^{N} \frac{d^3 \mathbf{p_i}}{2E_i (2\pi)^3}$$

$$\int \delta(E_i^2 - \mathbf{p}_i^2 - m_i^2) dE_i = \frac{1}{2E_i}$$

Energy-momentum relation delta function. Is this clear?

$$\prod_{i=1}^{N} \frac{d^3\mathbf{p_i}}{2E_i(2\pi)^3} = \prod_{i=1}^{N} \frac{d^3\mathbf{p_i}}{(2\pi)^3} \int \delta(E_i^2 - \mathbf{p}_i^2 - m_i^2) dE_i$$

$$\prod_{i=1}^{N} \frac{d^3\mathbf{p_i}}{2E_i(2\pi)^3} = \int \prod_{i=1}^{N} \frac{d^3\mathbf{p}_i dE_i}{(2\pi)^3} \delta(E_i^2 - \mathbf{p}_i^2 - m_i^2)$$

Now let's use 4-vector notation

$$\text{phase space} = \int \prod_{i=1}^{N} \frac{d^4 p_i}{(2\pi)^3} \delta(p_i^2 - m_i^2)$$

Let's take another look at our Lorentz-invariant phase space and check that it really is Lorentz-invariant

$$\frac{d^3\mathbf{p}}{(2\pi)^3 2E}$$

Let's look at a transformation along the z axis. What is the phase space?

$$d^3\mathbf{p}' = dp'_x dp'_y dp'_z = dp_x dp_y \frac{dp'_z}{dp_z} dp_z = d^3\mathbf{p} \frac{dp'_z}{dp_z}$$

$$d^3\mathbf{p}' = dp'_x dp'_y dp'_z = dp_x dp_y \frac{dp'_z}{dp_z} dp_z = d^3\mathbf{p} \frac{dp'_z}{dp_z}$$

$$p'_z = \gamma(p_z - \beta E), E' = \gamma(E - \beta p_z), E^2 = p_x^2 + p_y^2 + p_z^2 + m^2$$

$$\frac{dp'_z}{dp_z} = \gamma(1 - \beta \frac{\partial E}{\partial p_z})$$

$$\frac{\partial E}{\partial p_z} = \frac{1}{2} \cdot 2p_z \left(\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2} \right)^{-1} = \frac{p_z}{E}$$

$$\frac{dp'_z}{dp_z} = \gamma(1 - \beta \frac{p_z}{E}) = \frac{\gamma}{E}(E - \beta p_z) = \frac{E'}{E}$$

So ...
$$d^3\mathbf{p}' = d^3\mathbf{p} \frac{E'}{E} \rightarrow \frac{d^3\mathbf{p}'}{E'} = \frac{d^3\mathbf{p}}{E}$$

1→2+3+4+....n

$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3... - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0)\frac{d^4 p_j}{(2\pi)^4}$$

Start with this delta function. We know that
$p_j{}^2 = (p^0)^2\text{-}\mathbf{p}^2$

$$\delta(p_j^2 - m_j^2) = \delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m^2\right)$$

$$\rightarrow (p^0)_j = \pm\sqrt{\mathbf{p_j}^2 + m^2}$$

$$\delta(p_j^2 - m_j^2) = \delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m^2\right)$$

$$\rightarrow (p^0)_j = \pm\sqrt{\mathbf{p_j}^2 + m^2}$$

$$\frac{d}{dp_j^0}\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = 2p_j^0$$

$$\rightarrow \delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = \frac{1}{2p_j^0}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m_j^2}\right) + \delta\left(p_j^0 + \sqrt{\mathbf{p_j}^2 + m_j^2}\right)\right]$$

$$\delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = \frac{1}{2p_j^0}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m_j^2}\right) + \delta\left(p_j^0 + \sqrt{\mathbf{p_j}^2 + m_j^2}\right)\right]$$

$$\theta(p_j^0)\delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = \frac{\theta(p_j^0)}{2p_j^0}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m_j^2}\right) + \delta\left(p_j^0 + \sqrt{\mathbf{p_j}^2 + m_j^2}\right)\right]$$

Heaviside forces p_j^0 to always be greater than 0

$$\theta(p_j^0)\delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = \frac{1}{2p_j^0}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m_j^2}\right)\right]$$

1→2+3+4+....n

$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3... - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi \delta(p_j^2 - m_j^2) \theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4}$$

Delta picks out specific value for p_j^0, no need to integrate

So...

$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3... - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi \frac{\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m^2}\right)}{2p_j^0} \frac{d^4 p_j}{(2\pi)^4}$$

$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 ... - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi \frac{\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m^2}\right)}{2p_j^0} \frac{d^4 p_j}{(2\pi)^4}$$

Make delta function
substitution

$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 ... - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^4}$$

$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 ... - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^4}$$

And rearrange a bit

$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 ... - p_n) \times$$

$$\prod_{j=2}^{n} \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^3}$$

1→2+3

$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3... - p_n) \times$$

$$\frac{1}{2\sqrt{\mathbf{p_2}^2 + m_2^2}} \frac{d^3\mathbf{p_2}}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{p_3}^2 + m_3^2}} \frac{d^3\mathbf{p_3}}{(2\pi)^3}$$

Rearrange...

$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \delta^4(p_1 - p_2 - p_3... - p_n) \times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}} \frac{1}{\sqrt{\mathbf{p_3}^2 + m_3^2}} d^3\mathbf{p_2} d^3\mathbf{p_3}$$

$$\delta^4(p_1 - p_2 - p_3) = \delta(p_1^0 - p_2^0 - p_3^0)\delta^3(\mathbf{p_1} - \mathbf{p_2} - \mathbf{p_3})$$

Let's choose reference frame where p₁ is at rest, so **p₁=0** and p₁ = (m₁,**0**)
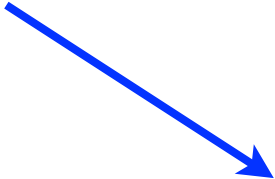
$$\delta^4(p_1 - p_2 - p_3) = \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_3}^2 + m_3^2})\delta^3(\mathbf{p_2} + \mathbf{p_3})$$

$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_3}^2 + m_3^2})\delta^3(\mathbf{p_2} + \mathbf{p_3}) \times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}} \frac{1}{\sqrt{\mathbf{p_3}^2 + m_3^2}} d^3\mathbf{p}_2 d^3\mathbf{p}_3$$

p₃=-p₂ (had to be, due to conservation of momentum if **p₁**=0)

$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_3}^2 + m_3^2})\delta^3(\mathbf{p_2} + \mathbf{p_3}) \times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}} \frac{1}{\sqrt{\mathbf{p_3}^2 + m_3^2}} d^3\mathbf{p}_2 d^3\mathbf{p}_3$$

$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2}) \times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}} \frac{1}{\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p}_2$$

$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2}) \times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}} \frac{1}{\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p_2}$$

Let's rearrange again

$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta(m_1 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2})}{\sqrt{\mathbf{p_2}^2 + m_2^2}\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p_2}$$

Stepping back, this is impressive but not surprising. Only have integral over momentum of one particle left for phase space (why not surprising?)!

$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta(m_1 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2})}{\sqrt{\mathbf{p_2}^2 + m_2^2}\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p_2}$$

Let's go to spherical coordinates, $\mathbf{p_2}$ = (r,θ,φ) and d$^3\mathbf{p_2}$=r^{2}sinθ dr dθ dφ

Matrix element squared cannot be a function of anything but |$\mathbf{p_2}$| anymore since object 1 was at rest and $\mathbf{p_3}$ is just -$\mathbf{p_2}$ so angular integrals can be easily done

$$\int \sin\theta \, d\theta \, d\phi = 4\pi$$

$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta(m_1 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2})}{\sqrt{\mathbf{p_2}^2 + m_2^2}\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p}_2$$

Do substitution and angular integrals

$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \frac{\delta(m_1 - \sqrt{r^2 + m_2^2} - \sqrt{r^2 + m_3^2})}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}} r^2 dr$$

Let's make another substitution...

$$u = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

$$u = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \frac{\delta(m_1 - \sqrt{r^2 + m_2^2} - \sqrt{r^2 + m_3^2})}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}} r^2 dr$$

$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \frac{\delta(m_1 - u)}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}} r^2 dr$$

$$\frac{du}{dr} = \frac{r}{\sqrt{r^2 + m_2^2}} + \frac{r}{\sqrt{r^2 + m_3^2}}$$

$$\frac{du}{dr} = \frac{r\left(\sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}\right)}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}}$$

$$\frac{du}{dr} = \frac{ru}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}}$$

$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \frac{\delta(m_1 - u)}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}} r^2 dr$$

$$\frac{du}{dr} = \frac{ru}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}}$$

$$dr = du \frac{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}}{ru}$$

$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \delta(m_1 - u) \frac{r}{u} du$$

u=m$_1$

$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \delta(m_1 - u)\frac{r}{u}du$$

u=m₁

What does
SR tell us?

$$u = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

$$m_1 = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

$$m_1^2 = r^2 + m_2^2 + r^2 + m_3^2 + 2\sqrt{(r^2 + m_3^2)(r^2 + m_2^2)}$$

$$m_1^2 - 2r^2 - m_2^2 - m_3^2 = 2\sqrt{(r^2 + m_3^2)(r^2 + m_2^2)}$$

$$m_1^4 + 4r^4 + m_2^4 + m_3^4 - 4r^2m_1^2 + 4r^2m_2^2 + 4r^2m_3^2 - 2m_1^2m_2^2 - 2m_1^2m_3^2 + 2m_2^2m_3^2 = $$
$$4r^4 + 4m_2^2m_3^2 + 4r^2m_2^2 + 4r^2m_3^2$$

$$-4r^2m_1^2 + m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2 = 0$$

$$r = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2}$$

$$r = |\mathbf{p_2}| = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}$$

$$\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2$$

Note that matrix element factorizes (not always possible, but a pretty nice result!)

1+2→3+4

$$\Gamma = \frac{1}{4E_1 E_2} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} 2\pi \delta(p_j^2 - m_j^2)\theta(p_j^0)\frac{d^4 p_j}{(2\pi)^4}$$

This is the rate, though not quite what we're looking for. We are interested in the **cross section (σ)**
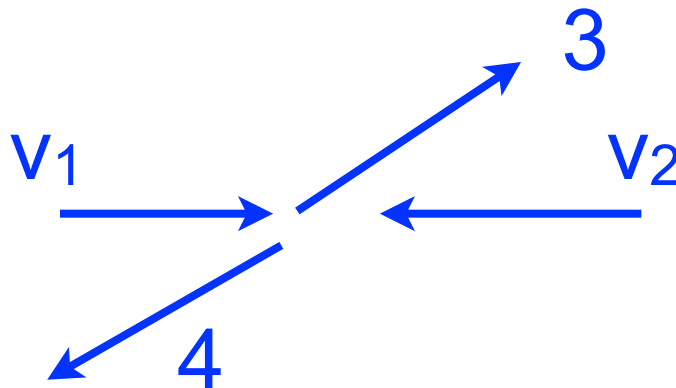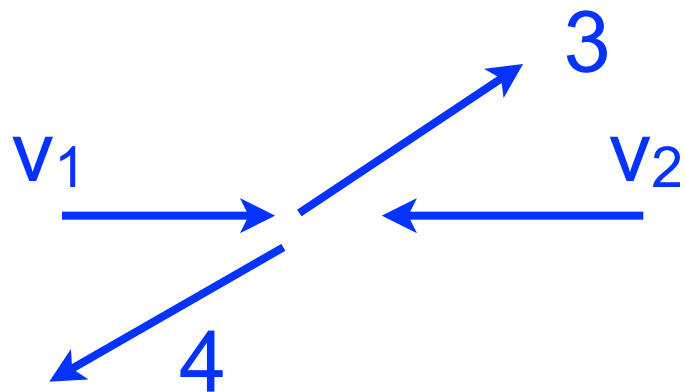
If we assume one particle per unit volume, then rate = $(v_1+v_2)\sigma$, ie the faster the set of objects 1 and 2 pass through each other, the larger the rate

Be careful (of relative v_1 and v_2 minus signs)

$$\Gamma = (v_1 + v_2)\sigma$$

$$\sigma = \frac{\Gamma}{v_1 + v_2}$$

So the cross section is ...

$$\sigma = \frac{1}{4E_1 E_2} \frac{1}{v_1 + v_2} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} 2\pi \delta(p_j^2 - m_j^2) \theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4}$$

Looks a bit odd to have velocity there!
Can that at all be Lorentz invariant?

$$F = 4E_1 E_2 (v_1 + v_2) = 4E_1 E_2 \left(\frac{|\mathbf{p_1}|}{E_1} + \frac{|\mathbf{p_2}|}{E_2} \right) = 4 \left(E_2 |\mathbf{p_1}| + E_1 |\mathbf{p_2}| \right)$$

$$F^2 = 16 \left(E_2^2 |\mathbf{p_1}|^2 + E_1^2 |\mathbf{p_2}|^2 + 2E_1 E_2 |\mathbf{p_1}| |\mathbf{p_2}| \right)$$

In case where particles 1 and 2 are collinear

$$(p_1 \cdot p_2) = E_1 E_2 + \mathbf{p_1 p_2}$$

Remember the extra minus sign here

$$(p_1 \cdot p_2)^2 = E_1^2 E_2^2 + \mathbf{p_1}^2 \mathbf{p_2}^2 + 2 E_1 E_2 \mathbf{p_1 p_2}$$

$$F^2 = 16 \left(E_2^2 |\mathbf{p_1}|^2 + E_1^2 |\mathbf{p_2}|^2 + 2 E_1 E_2 |\mathbf{p_1}||\mathbf{p_2}| \right)$$

$$F^2 = 16 \left(E_2^2 |\mathbf{p_1}|^2 + E_1^2 |\mathbf{p_2}|^2 + (p_1 \cdot p_2)^2 - E_1^2 E_2^2 - \mathbf{p_1}^2 \mathbf{p_2}^2 \right)$$

$$F^2 = 16 \left[(p_1 \cdot p_2)^2 - (E_1^2 - \mathbf{p_1}^2)(E_2^2 - \mathbf{p_2}^2) \right]$$

Lorentz invariant!

$$F^2 = 16 \left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2 \right]$$

$$\boxed{F = 4 \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}$$

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} 2\pi \delta(p_j^2 - m_j^2) \theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4}$$

As before, 1d delta function is easy (and rearranges some 2π's). Heaviside enforces only one solution

$$p_j^0 = E_j = \sqrt{\mathbf{p}_j^2 + m_j^2}$$

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^3}$$

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^3}$$

$$p_j^0 = E_j = \sqrt{\mathbf{p}_j^2 + m_j^2}$$

Let's put back the earlier form for F

$$\sigma = \frac{1}{4E_1 E_2} \frac{1}{v_1 + v_2} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^3}$$

$$F = \frac{1}{4E_1 E_2} \frac{1}{v_1 + v_2} = \frac{1}{4E_1 E_2} \frac{1}{p_1/E_1 + p_2/E_2}$$

$$F = \frac{1}{4} \frac{1}{E_2 p_1 + E_1 p_2}$$

In Center of Mass frame, |**p**₂| = |**p**₁|

$$F = \frac{1}{4|p_1|} \frac{1}{E_1 + E_2}$$

$$\sigma = \frac{1}{4|p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^3}$$

$$\sigma = \frac{1}{64\pi^2 |p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{\sqrt{\mathbf{p}_j^2 + m_j^2}} d^3\mathbf{p}_j$$

$$p_j^0 = E_j = \sqrt{\mathbf{p}_j^2 + m_j^2}$$

Let's split up the 4d delta function

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$

$$p_j^0 = E_j = \sqrt{\mathbf{p}_j^2 + m_j^2}$$

In CoM frame, we know that **p₁+p₂** = 0

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_3 + \mathbf{p}_4)$$

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_3 + \mathbf{p}_4) \times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2}} \frac{1}{\sqrt{\mathbf{p}_4^2 + m_4^2}} d^3\mathbf{p}_3 d^3\mathbf{p}_4$$

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_3 + \mathbf{p}_4)\times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2}} \frac{1}{\sqrt{\mathbf{p}_4^2 + m_4^2}} d^3\mathbf{p}_3 d^3\mathbf{p}_4$$

The 3D delta function enforces **p**$_3$ = -**p**$_4$

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - E_3 - E_4)\times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2}} \frac{1}{\sqrt{\mathbf{p}_3^2 + m_4^2}} d^3\mathbf{p}_3$$

$$\sigma = \frac{1}{64\pi^2 |p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - E_3 - E_4) \times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2}} \frac{1}{\sqrt{\mathbf{p}_3^2 + m_4^2}} d^3\mathbf{p}_3$$

Recall that we had delta functions on E_3 and E_4 (the mass relations)

$$\sigma = \frac{1}{64\pi^2 |p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - \sqrt{\mathbf{p}_3^2 + m_3^2} - \sqrt{\mathbf{p}_4^2 + m_4^2}) \times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2}} \frac{1}{\sqrt{\mathbf{p}_3^2 + m_4^2}} d^3\mathbf{p}_3$$

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - \sqrt{\mathbf{p}_3^2 + m_3^2} - \sqrt{\mathbf{p}_4^2 + m_4^2}) \times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2}} \frac{1}{\sqrt{\mathbf{p}_3^2 + m_4^2}} d^3\mathbf{p}_3$$

The 3D delta function enforced $\mathbf{p}_3$ = -$\mathbf{p}_4$

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - \sqrt{\mathbf{p}_3^2 + m_3^2} - \sqrt{\mathbf{p}_3^2 + m_4^2}) \times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2}} \frac{1}{\sqrt{\mathbf{p}_3^2 + m_4^2}} d^3\mathbf{p}_3$$

$$\sigma = \frac{1}{64\pi^2 |p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - \sqrt{\mathbf{p}_3^2 + m_3^2} - \sqrt{\mathbf{p}_3^2 + m_4^2}) \times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2}} \frac{1}{\sqrt{\mathbf{p}_3^2 + m_4^2}} d^3\mathbf{p}_3$$

$$d^3\mathbf{p}_3 = r^2 dr d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 |p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - \sqrt{r^2 + m_3^2} - \sqrt{r^2 + m_4^2}) \times$$

$$\frac{1}{\sqrt{r^2 + m_3^2}} \frac{1}{\sqrt{r^2 + m_4^2}} r^2 dr$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 |p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - \sqrt{r^2 + m_3^2} - \sqrt{r^2 + m_4^2}) \times$$

$$\frac{1}{\sqrt{r^2 + m_3^2}} \frac{1}{\sqrt{r^2 + m_4^2}} r^2 dr$$

Same exact form as last ugly integral, so nothing new here

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|M|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

I think this is where Griffiths does a really nice job. We won't dive into QED, but will instead start with a simpler theory.

Feynman's calculus/his rules tell us how to calculate the matrix elements (why? we won't be diving into QFT, so for now please just accept them, as unappealing as that might be)
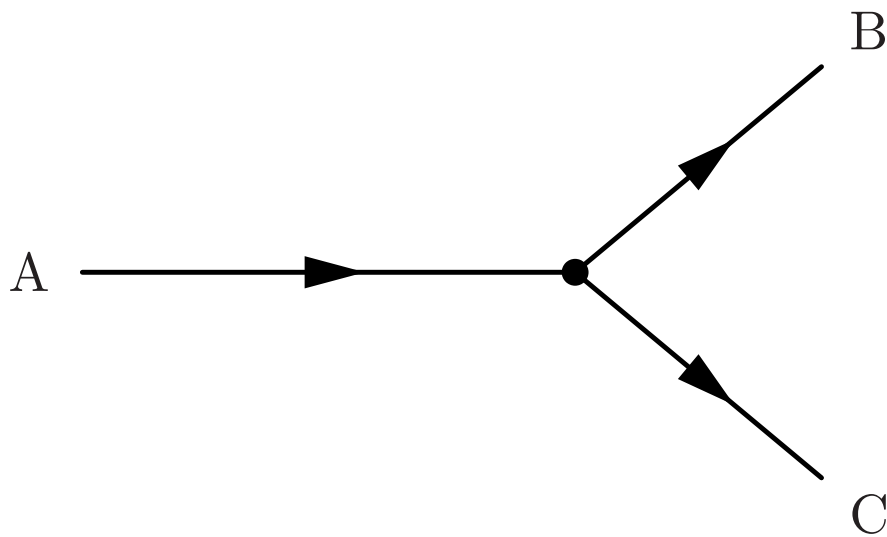
Our toy theory has 3 types of spin-0 particles, A, B and C. Let's assume that $m_A > m_B + m_C$. Here, A is incoming, and B and C are outgoing. This is a decay vertex
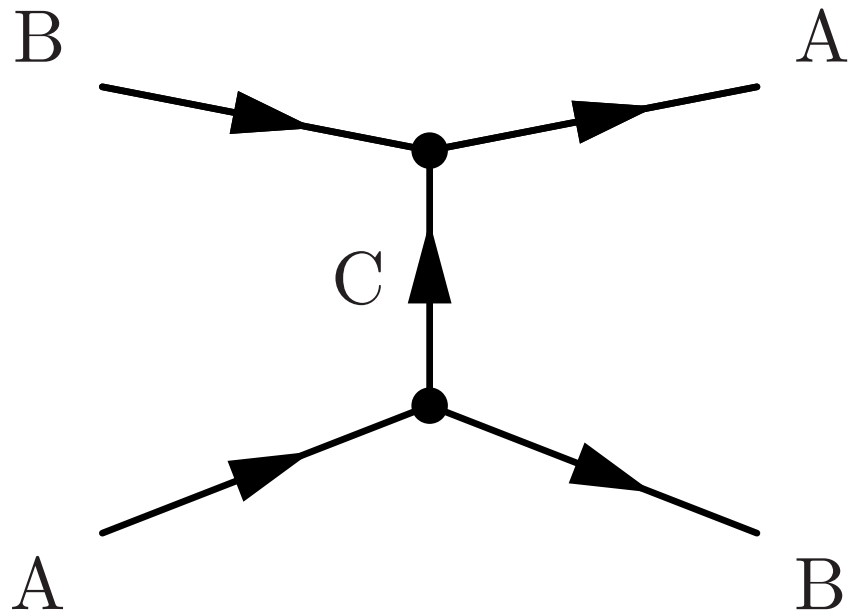
Here we have scattering A+B→A+B (one example diagram)

B p_1 p_3 A

C q_1

p_2

A p_4 B
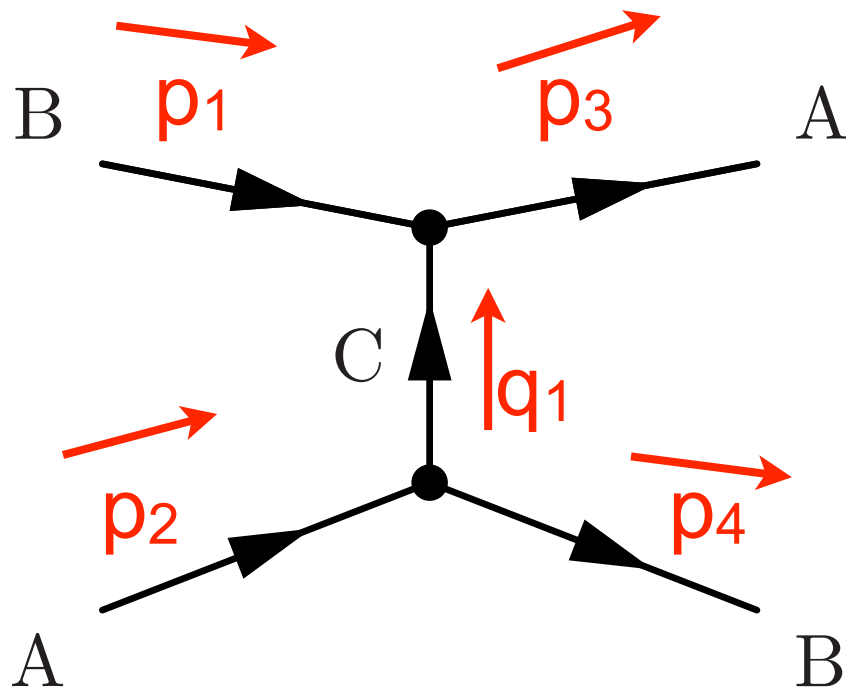
p vs q is pure convention!

Label all incoming and outgoing lines with p_1, p_2, ... p_n
Internal lines can go either way
Use arrows to keep track of what is going in and out (here this looks trivial, but can be more tricky with anti-particles)

Add factors of -ig for each vertex, specifying the coupling constants

For each internal line add a factor for the propagator (note that we don't have to be on-shell here!)
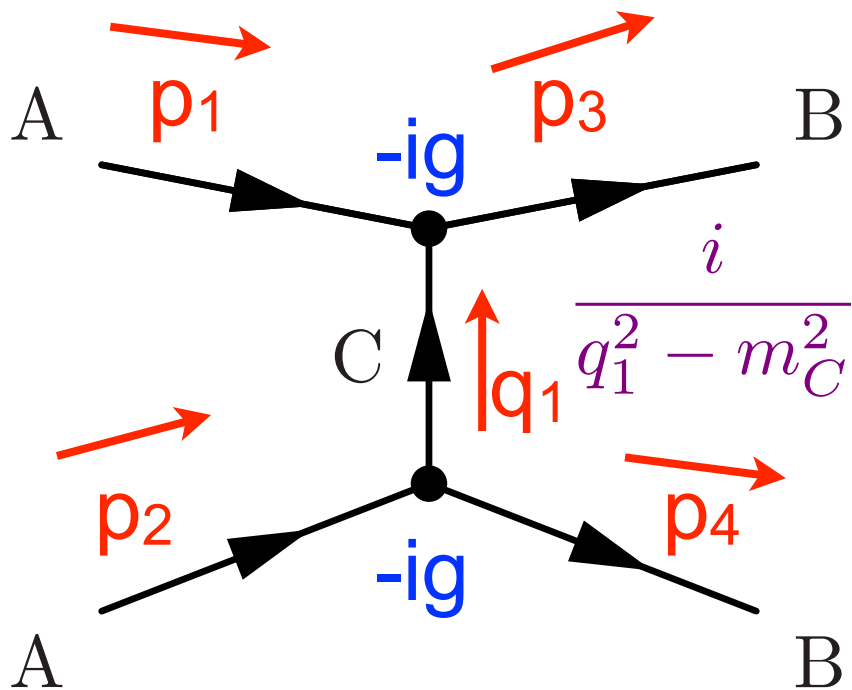
$$(2\pi)^4 \delta^4(p_1 + q_1 - p_3)$$

B p_1 -ig p_3 A

C $\dfrac{i}{q_1^2 - m_C^2}$

q_1

-ig

p_2

A p_4 B

$$(2\pi)^4 \delta^4(p_2 - q_1 - p_4)$$

Impose conservation of energy and momentum at each vertex with 4d Dirac Delta function (with appropriate 2pi normalization)

$$(2\pi)^4 \delta^4(p_1 + q_1 - p_3)$$

B p_1 p_3 A

-ig

C q_1

$$\frac{i}{q_1^2 - m_C^2} \frac{1}{(2\pi)^4} d^4 q_1$$

p_2 p_4

-ig

A B

$$(2\pi)^4 \delta^4(p_2 - q_1 - p_4)$$

Integrate over 4-momentum of internal lines with appropriate 2pi normalization factor

$$(2\pi)^4 \delta^4(p_1 + q_1 - p_3)$$

B p_1 p_3 A
         -ig

C q_1 $\dfrac{i}{q_1^2 - m_C^2} \dfrac{1}{(2\pi)^4} d^4 q_1$

p_2

A   -ig p_4 B

$$(2\pi)^4 \delta^4(p_2 - q_1 - p_4)$$

i
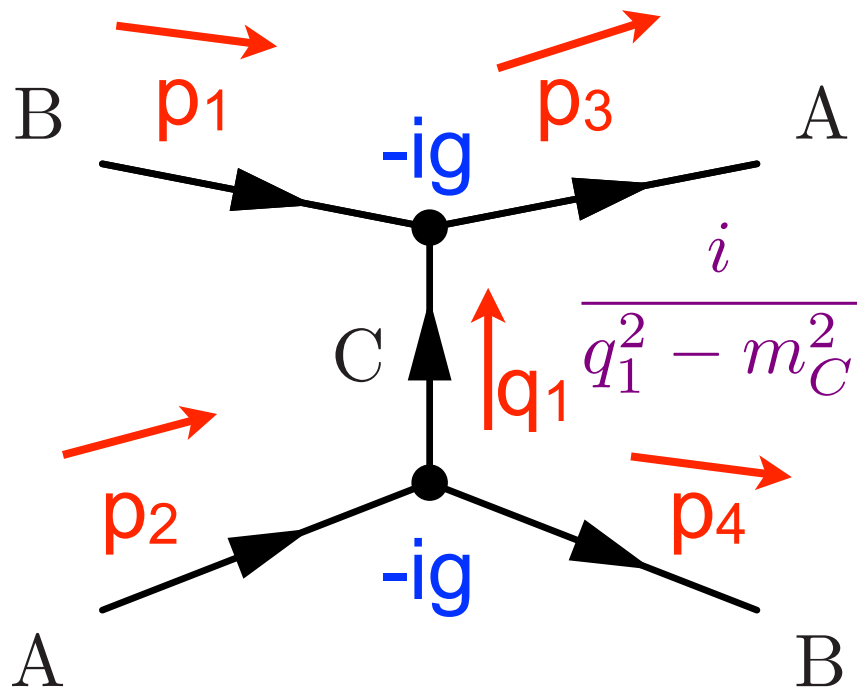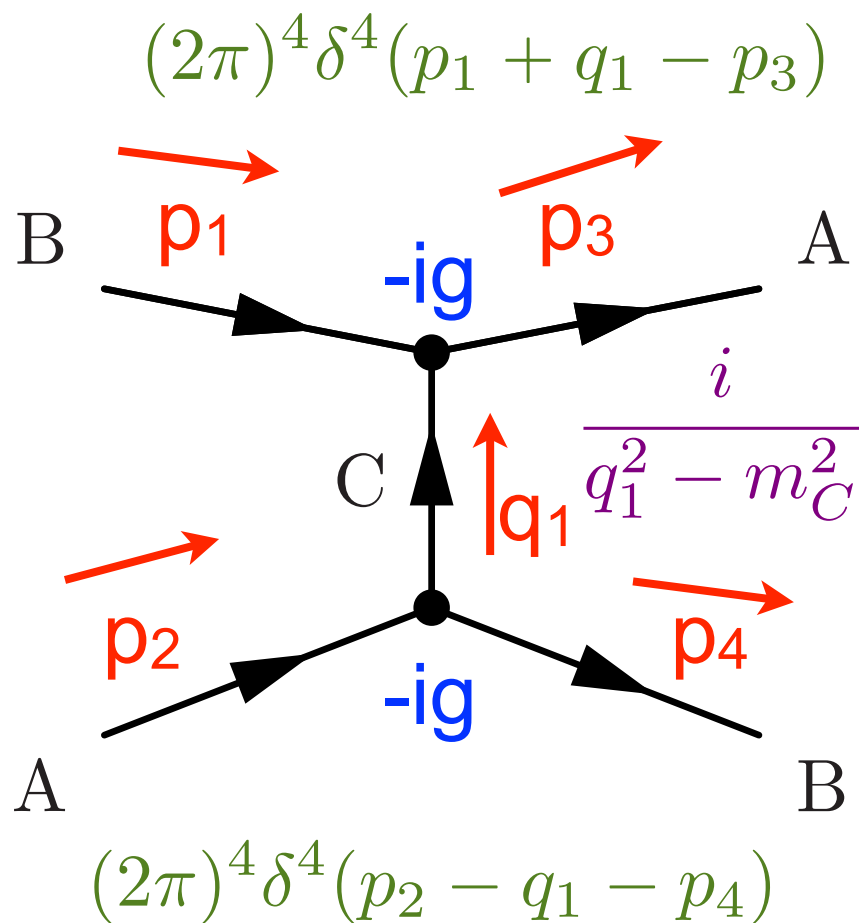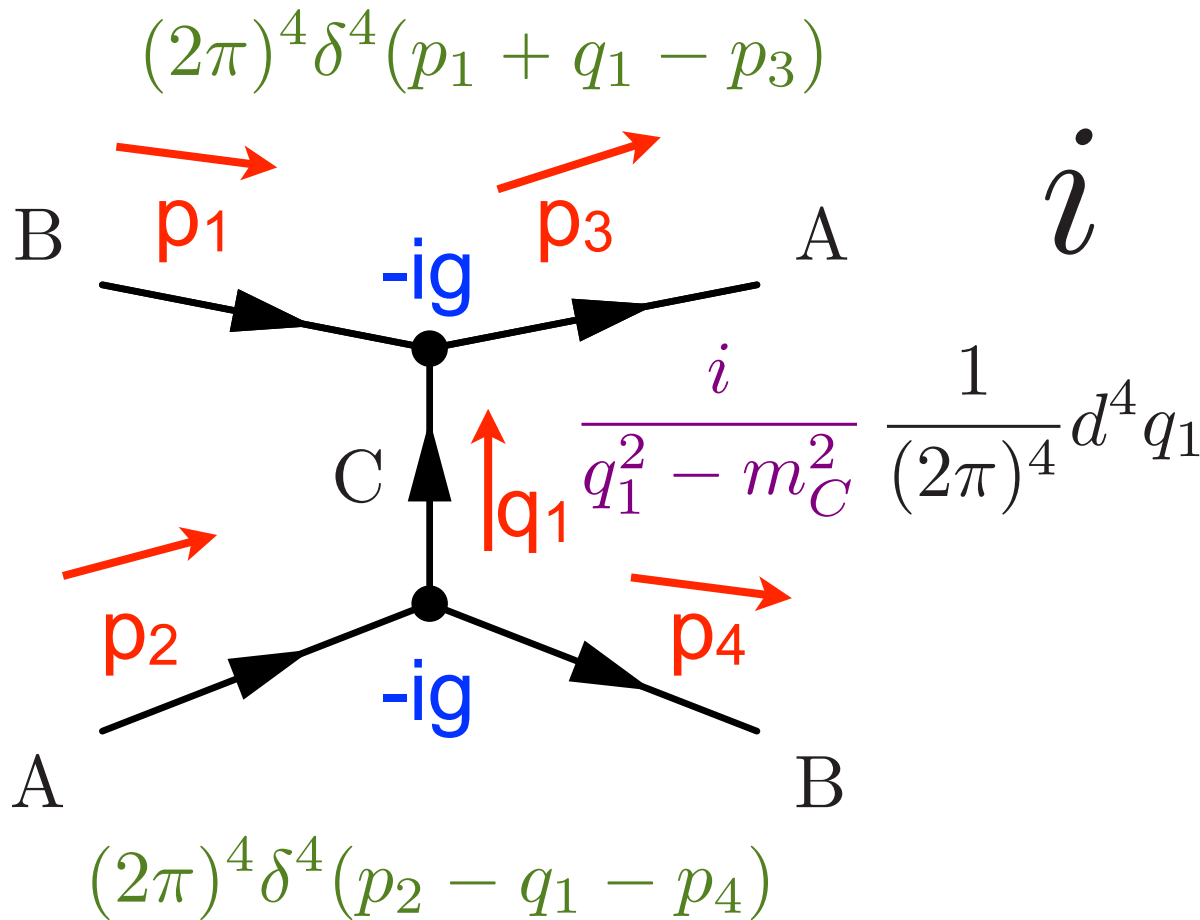
Cancel remaining delta function and add a factor of i, and you have the matrix element
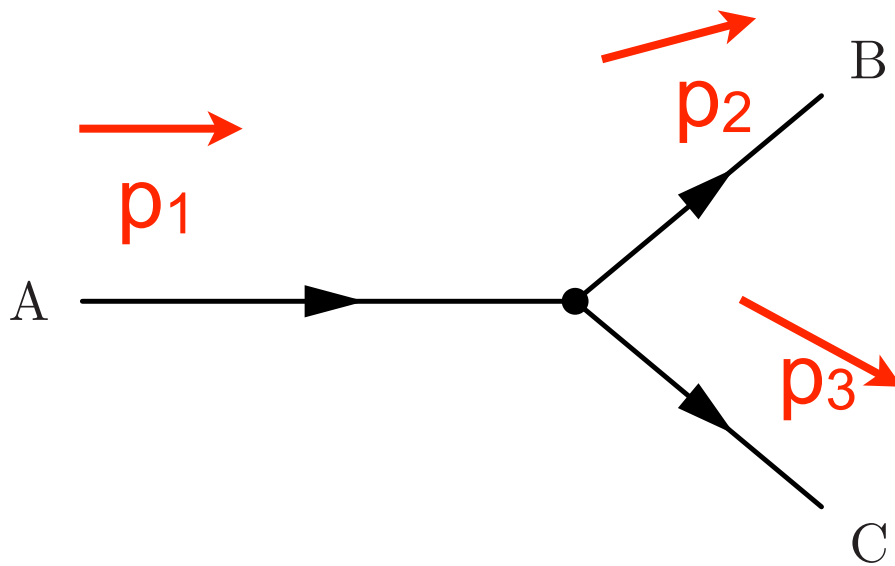
Label external (p) and internal lines (q) and draw arrows. Here we have no internal lines
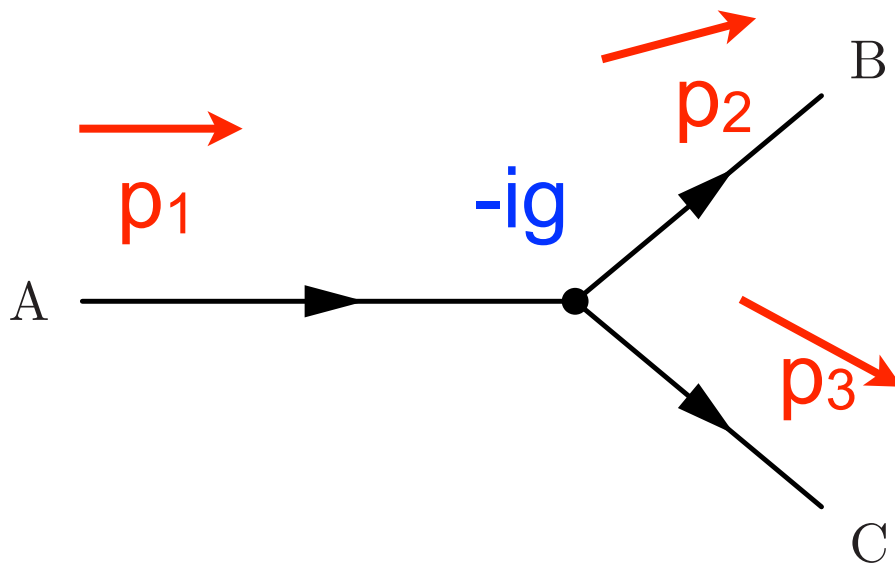
Single factor of -ig for our one vertex

p₁ → p_1

-ig → -ig

p₂ → p_2

p₃ → p_3

A

B

C
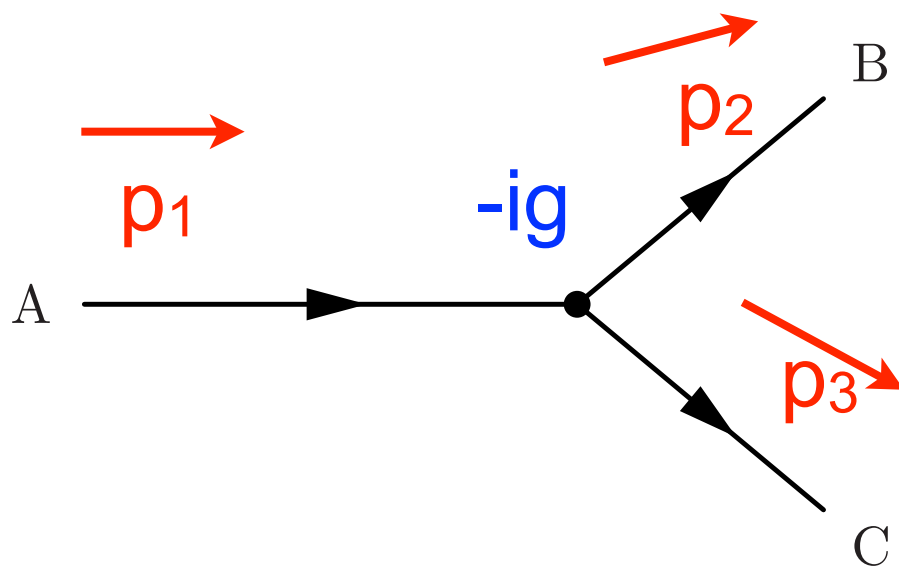
For each internal line add a factor for the propagator, but we don't have one here! (It's nice when things are simple)
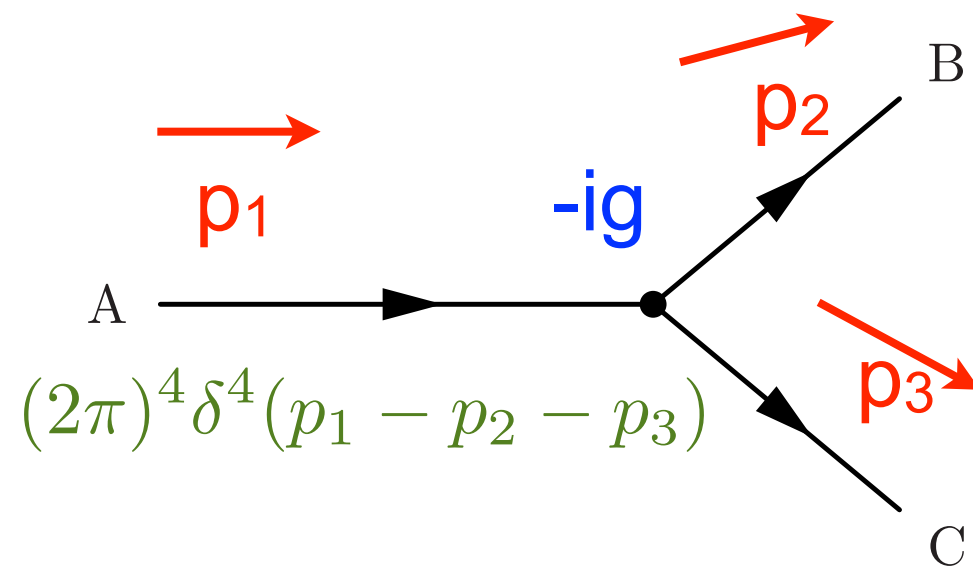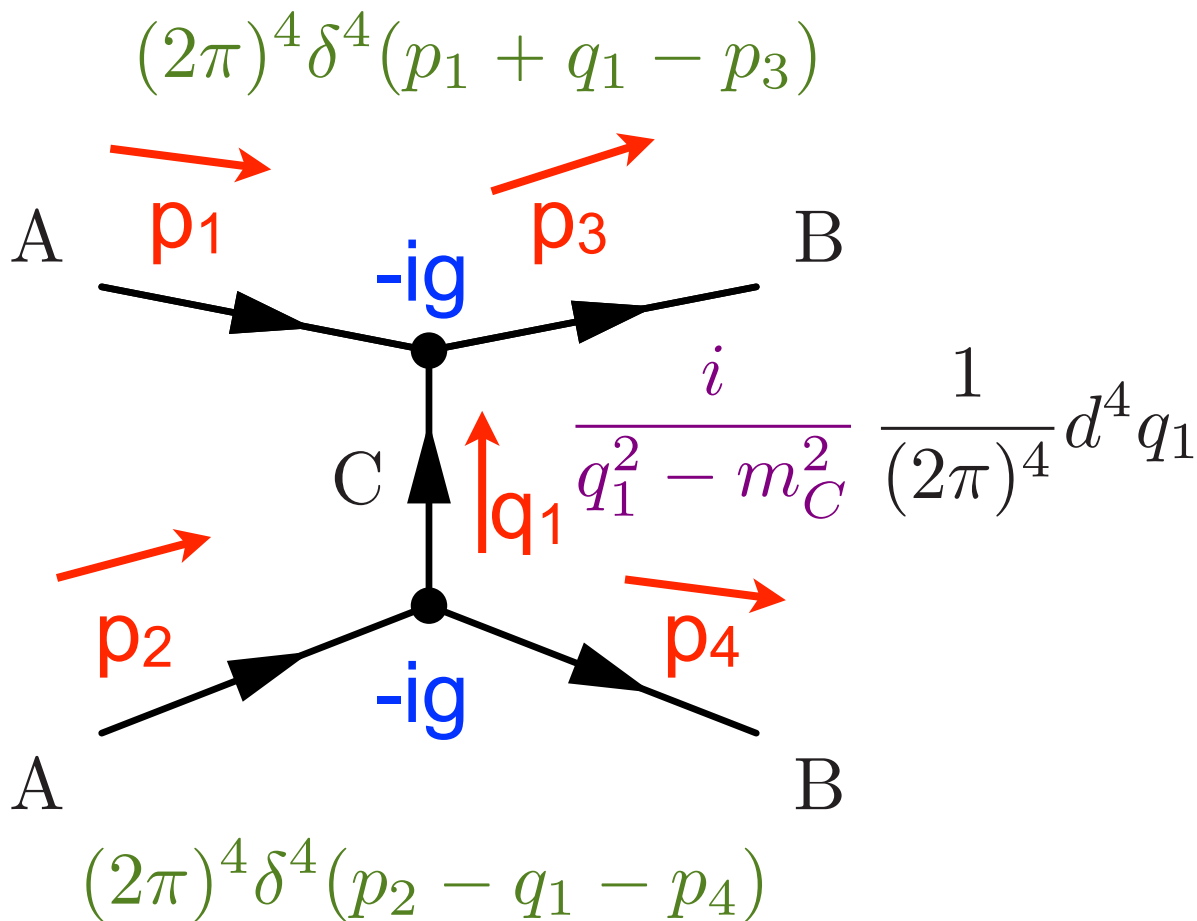
$$(2\pi)^4 \delta^4(p_1 - p_2 - p_3)$$

Impose conservation of energy and momentum at each vertex with 4d Dirac Delta function (with appropriate 2pi normalization)

p_1

p_2

p_3

-ig

$(2\pi)^4\delta^4(p_1 - p_2 - p_3)$

A

B

C
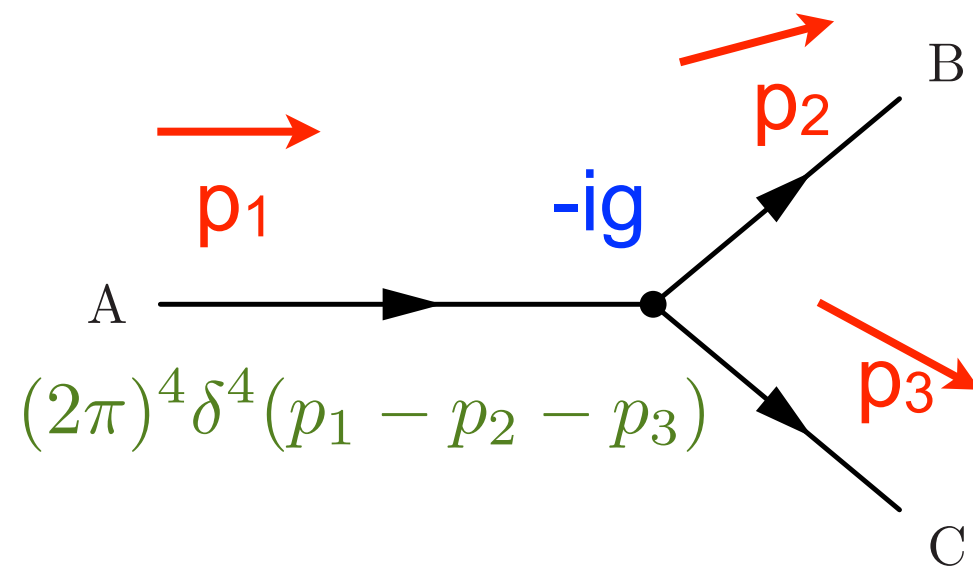
Integrate over 4-momentum of internal lines with appropriate 2pi normalization factor (here, none)

$$i$$

$$p_1$$

$$-ig$$

$$p_2$$

$$p_3$$

A

B

C

$$(2\pi)^4 \delta^4(p_1 - p_2 - p_3)$$
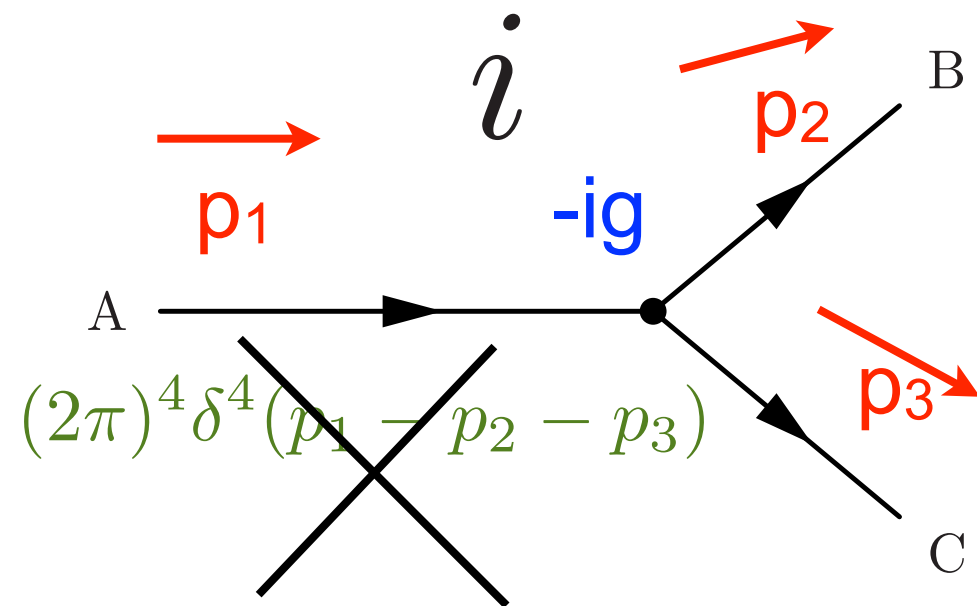
Add factor of i and cancel remaining delta function

i

-ig

p_1

p_2

p_3

A

B

C

$(2\pi)^4\delta^4(p_1 - p_2 - p_3)$
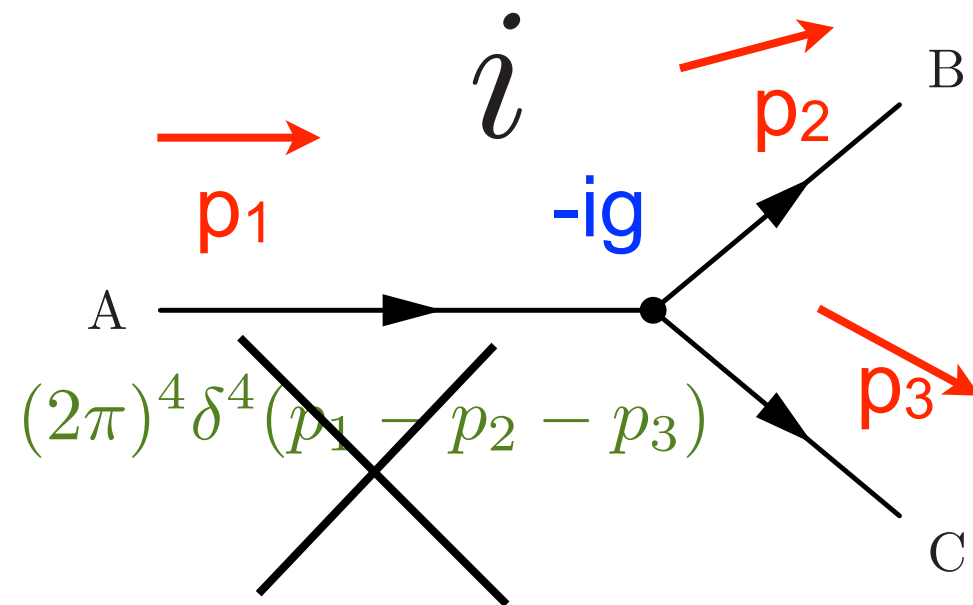
We're left only with M = i(-ig) = g

$$\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2 = \frac{g^2|\mathbf{p}|}{8\pi m_A^2}$$

$$|\mathbf{p}| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$$

$$\boxed{\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2 = \frac{g^2|\mathbf{p}|}{8\pi m_A^2}}$$

$$|\mathbf{p}| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$$

$$\tau = \frac{1}{\Gamma} = \frac{8\pi m_A^2}{g^2|\mathbf{p}|}$$

Let's calculate matrix element for AA→BB

Leading order diagram (there are others at higher order, as we'll see)

Label all incoming and outgoing lines with p_1, p_2, ... p_n
Internal lines can go either way
Use arrows to keep track of what is going in and out (here this looks trivial, but can be more tricky with anti-particles)

Add factors of -ig for each vertex, specifying the coupling constants

For each internal line add a factor for the propagator (note that we don't have to be on-shell here!)

$$(2\pi)^4 \delta^4(p_1 + q_1 - p_3)$$

$$\frac{i}{q_1^2 - m_C^2}$$

$$(2\pi)^4 \delta^4(p_2 - q_1 - p_4)$$

Impose conservation of energy and momentum at each vertex with 4d Dirac Delta function (with appropriate 2pi normalization)

$$(2\pi)^4 \delta^4(p_1 + q_1 - p_3)$$

A $\quad$ p_1 $\quad$ -ig $\quad$ p_3 $\quad$ B

$$\frac{i}{q_1^2 - m_C^2} \frac{1}{(2\pi)^4} d^4 q_1$$

C $\quad$ q_1

p_2

p_4

-ig

A $\quad$ B

$$(2\pi)^4 \delta^4(p_2 - q_1 - p_4)$$

Integrate over 4-momentum of internal lines with appropriate 2pi normalization factor

$$(2\pi)^4\delta^4(p_1 + q_1 - p_3)$$

A

p_1

$-ig$

p_3

B

C

$\dfrac{i}{q_1^2 - m_C^2} \dfrac{1}{(2\pi)^4} d^4 q_1$

q_1

p_2

$-ig$

p_4

A

B

$$(2\pi)^4\delta^4(p_2 - q_1 - p_4)$$

$$\int (-ig)(-ig) \frac{i}{q_1^2 - m_C^2}(2\pi)^4\delta^4(p_1 + q_1 - p_3)(2\pi)^4\delta^4(p_2 - q_1 - p_4)\frac{d^4 q_1}{(2\pi)^4}$$

$$\int (-ig)(-ig)\frac{i}{q_1^2 - m_C^2}(2\pi)^4\delta^4(p_1 + q_1 - p_3)(2\pi)^4\delta^4(p_2 - q_1 - p_4)\frac{d^4q_1}{(2\pi)^4}$$

$$(2\pi)^4(-g^2)\int \frac{i}{q_1^2 - m_C^2}\delta^4(p_1 + q_1 - p_3)\delta^4(p_2 - q_1 - p_4)d^4q_1$$

Integral is over 4-momentum of q_1 but this gets picked up by the Delta function. Let's use the first one, so $q_1 = p_3 - p_1$

$$(2\pi)^4(-g^2)\int \frac{i}{q_1^2 - m_C^2}\delta^4(p_1 + q_1 - p_3)\delta^4(p_2 - q_1 - p_4)d^4q_1$$

Integral is over 4-momentum of q_1 but this gets picked up by the Delta function. Let's use the first one, so q_1 = p_3-p_1 (note that this is true for each component of q_1, as we're using some notation shorthand here)

$$(2\pi)^4(-g^2)\frac{i}{(p_3 - p_1)^2 - m_C^2}\delta^4(p_2 + p_1 - p_3 - p_4)$$

Conservation of **p** and E for total system

$$(2\pi)^4(-g^2)\frac{i}{(p_3-p_1)^2-m_C^2}\delta^4(p_2+p_1-p_3-p_4)$$

Rule 6: cancel delta function (and 2pi)^4
and multiply by i to get Matrix Element

$$\mathcal{M} = i(2\pi)^4(-g^2)\frac{i}{(p_3-p_1)^2-m_C^2}$$

$$\boxed{\mathcal{M} = \frac{g^2}{(p_3-p_1)^2-m_C^2}}$$
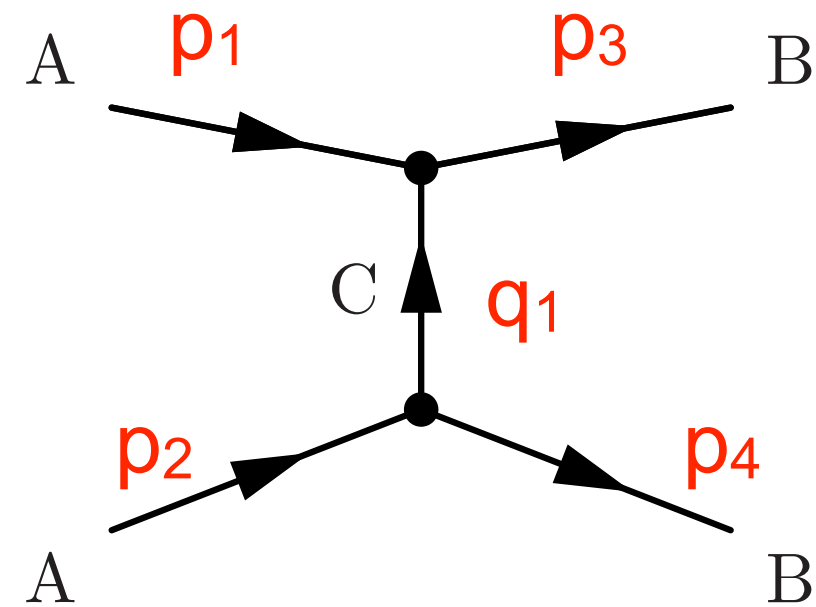
$$\mathcal{M} = \frac{g^2}{(p_3 - p_1)^2 - m_C^2}$$

This is not the only diagram. There is another similar one with the same initial state and the same final state. So they must be added together, as they interfere!

Let's work this out together on the board

A $\quad$ **p₁** $\qquad$ **p₃** $\quad$ B

C $\quad$ **q₁**

p₂ $\qquad$ **p₄**

A $\qquad\qquad$ B

A $\quad$ **p₁** $\qquad\qquad$ **p₃**B

q₁ C

p₂ $\qquad$ **p₄**

A $\qquad\qquad$ B
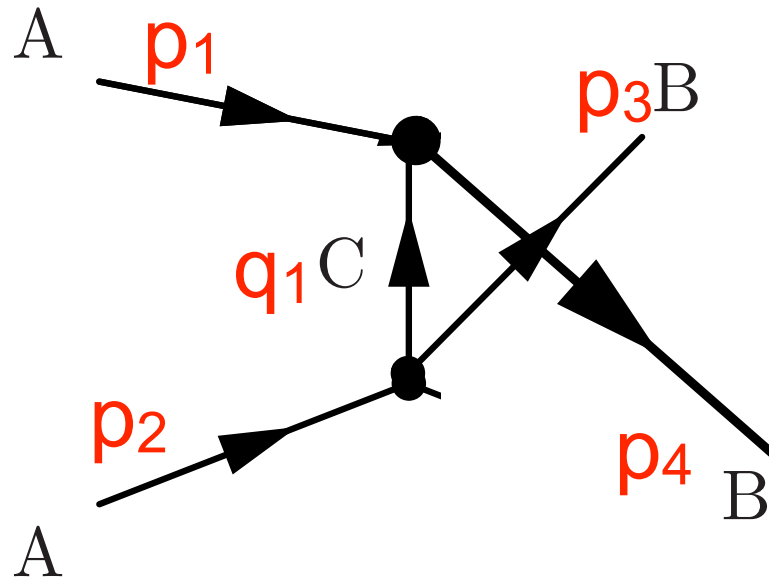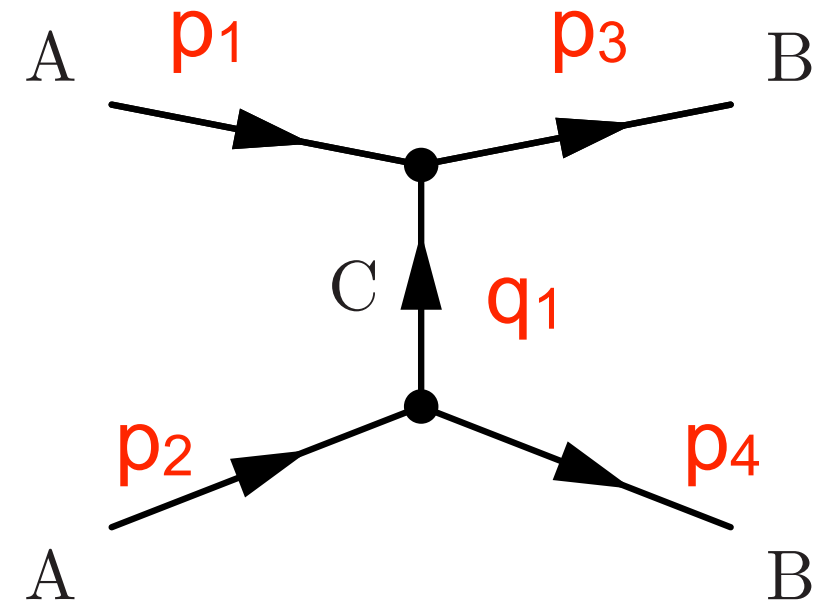
$$\mathcal{M} = \frac{g^2}{(p_3 - p_1)^2 - m_C^2}$$

Same diagram except that p₁ connects to p₄, not to p₃

A p_1 p_3 B

C q_1

p_2 p_4

A   B

A p_1 p_3B

q_1 C

p_2 p_4 B

A
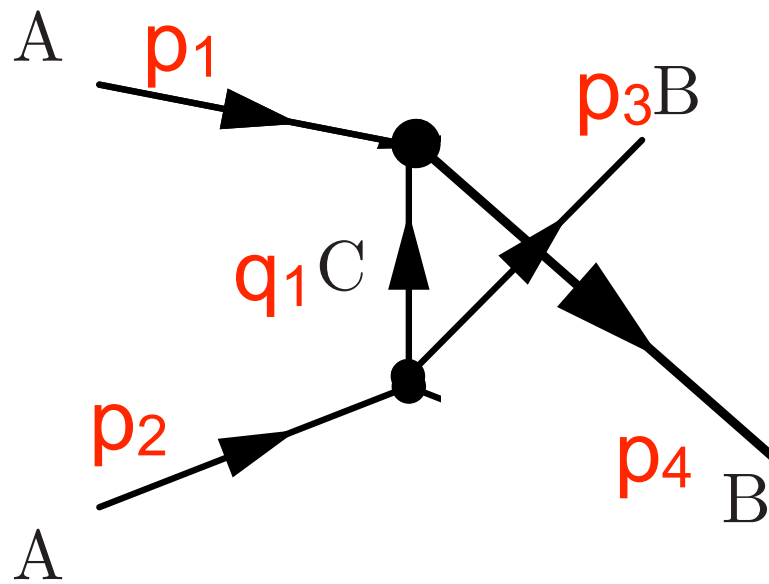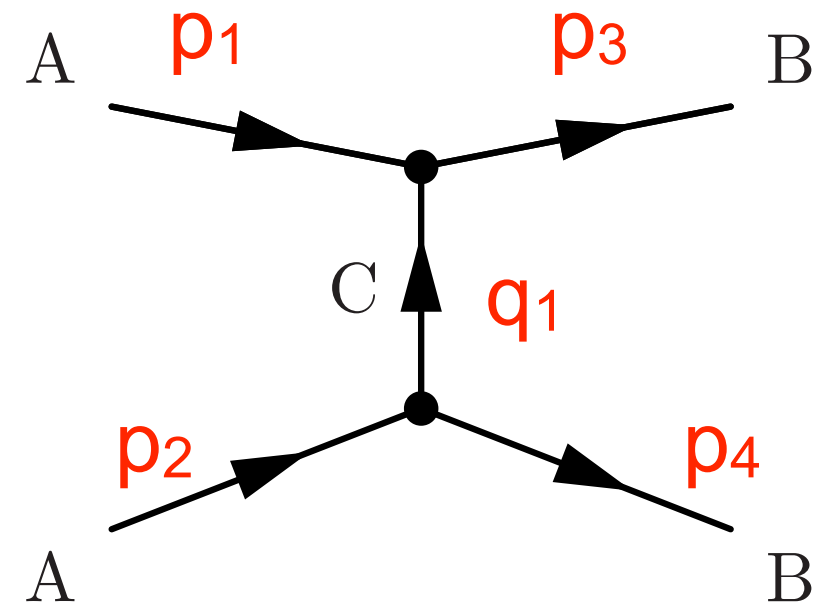
$$\mathcal{M} = \frac{g^2}{(p_3 - p_1)^2 - m_C^2}$$

$$\mathcal{M} = \frac{g^2}{(p_4 - p_1)^2 - m_C^2}$$

Does this look at all similar (notation-wise to your previous homework)?

$$\mathcal{M} = g^2 \left[\frac{1}{(p_3 - p_1)^2 - m_C^2} + \frac{1}{(p_4 - p_1)^2 - m_C^2} \right]$$

As in Griffiths, let's assume m_C is zero to simplify things, $m_A = m_B = m$

$$\mathcal{M} = g^2 \left[\frac{1}{(p_3 - p_1)^2} + \frac{1}{(p_4 - p_1)^2} \right]$$

$$\mathcal{M} = g^2 \left[\frac{1}{(p_3 - p_1)^2} + \frac{1}{(p_4 - p_1)^2} \right]$$

p_1

p_2

p_3

θ

p_4
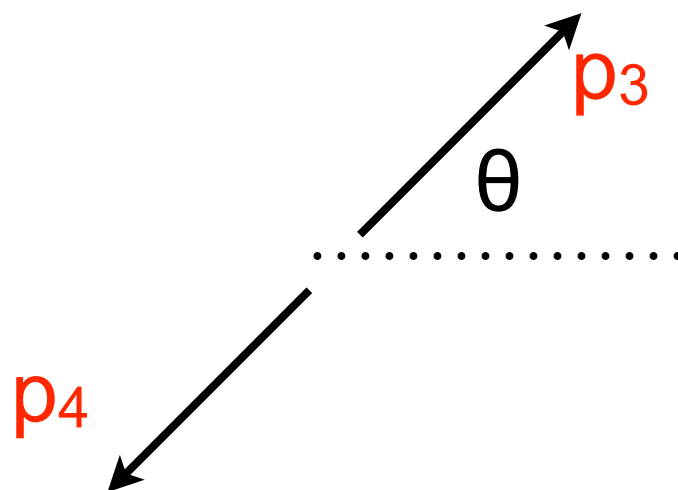
Use center of mass reference frame, where $|\mathbf{p}_1|=|\mathbf{p}_2|$, $|\mathbf{p}_3|=|\mathbf{p}_4|$
Since $m_A = m_B$ this means

That clear?

$|\mathbf{p}_1|=|\mathbf{p}_2|=|\mathbf{p}_3|=|\mathbf{p}_4|=p$ $E_1=E_2=E_3=E_4=E$
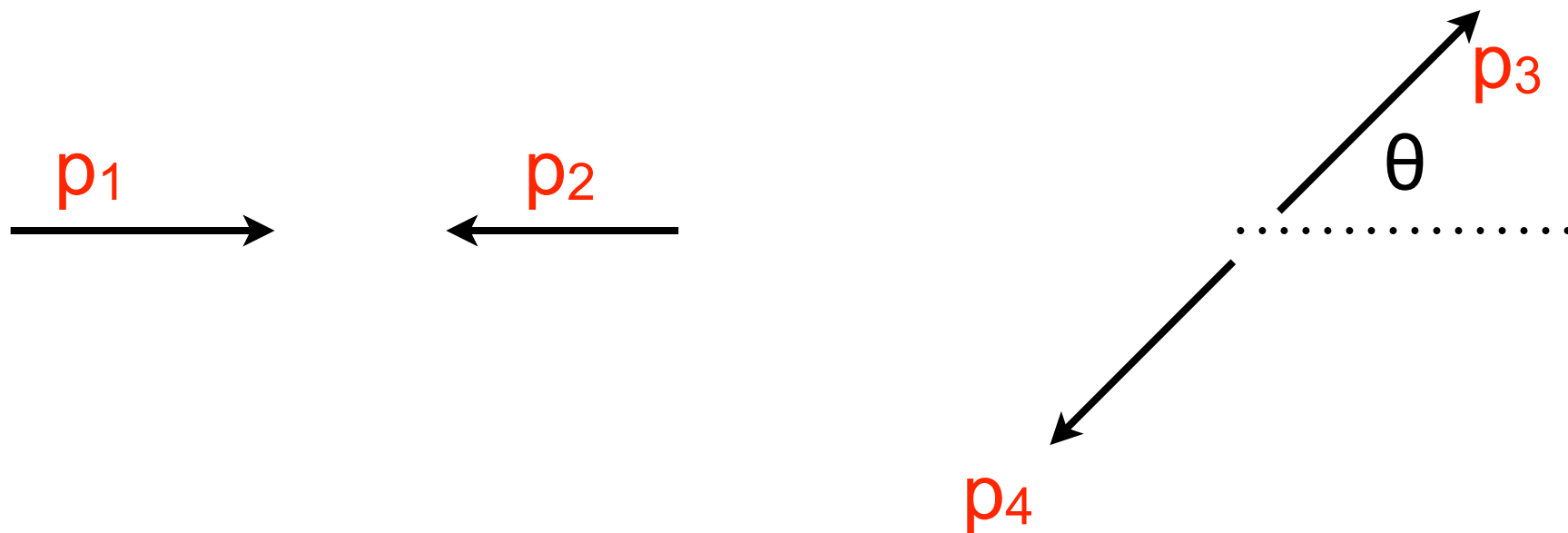
$$(p_4 - p_1)^2 = p_1^2 + p_4^2 - 2p_4 \cdot p_1 = 2m^2 - 2E_4 E_1 + 2\mathbf{p}_4 \cdot \mathbf{p}_1$$

$$(p_4 - p_1)^2 = 2m^2 - 2E^2 + 2\mathbf{p}_4 \cdot \mathbf{p}_1 = 2(E^2 - \mathbf{p}^2) - 2E^2 + 2\mathbf{p}_4 \cdot \mathbf{p}_1$$

$$(p_4 - p_1)^2 = 2(E^2 - \mathbf{p}^2) - 2E^2 + 2\mathbf{p}_4 \cdot \mathbf{p}_1 = -2\mathbf{p}^2 + 2\mathbf{p}_4 \cdot \mathbf{p}_1$$

$$(p_4 - p_1)^2 = -2\mathbf{p}^2 - 2\mathbf{p}^2 \cos\theta = -2\mathbf{p}^2(1 + \cos\theta)$$

$$\mathcal{M} = g^2 \left[\frac{1}{(p_3 - p_1)^2} + \frac{1}{(p_4 - p_1)^2} \right]$$

$(p_4 - p_1)^2 = -2\mathbf{p}^2(1 + \cos\theta)$

$(p_3 - p_1)^2 = -2\mathbf{p}^2(1 - \cos\theta)$

$$\mathcal{M} = g^2 \left[\frac{1}{-2\mathbf{p}^2(1 - \cos\theta)} + \frac{1}{-2\mathbf{p}^2(1 + \cos\theta)} \right]$$

$$\mathcal{M} = \frac{g^2}{-2\mathbf{p}^2} \left[\frac{1}{(1 - \cos\theta)} + \frac{1}{(1 + \cos\theta)} \right]$$

$$\mathcal{M} = \frac{g^2}{-2\mathbf{p}^2} \left[\frac{(1 + \cos\theta) + (1 - \cos\theta)}{(1 - \cos\theta)(1 + \cos\theta)} \right]$$

$$\mathcal{M} = \frac{g^2}{-2\mathbf{p}^2} \left[\frac{2}{1 - \cos^2\theta} \right]$$

$$\boxed{\mathcal{M} = \frac{-g^2}{\mathbf{p}^2 \sin^2\theta}}$$

Differential cross section here

$$\mathcal{M} = \frac{-g^2}{\mathbf{p}^2 \sin^2 \theta}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|M|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

$$|\mathbf{p}_f| = |\mathbf{p}_i| = |\mathbf{p}|$$

$$E_1 = E_2 = E$$

$$|\mathcal{M}|^2 = \frac{g^4}{\mathbf{p}^4 \sin^4 \theta}$$

Infinite cross section as θ→0

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{g^4}{(2E)^2 \mathbf{p}^4 \sin^4 \theta}$$

Here we have two instances of particle B in the final state. For every s identical particles, we add factor of 1/(s!) to account for this, or else we have over-counted the phase space
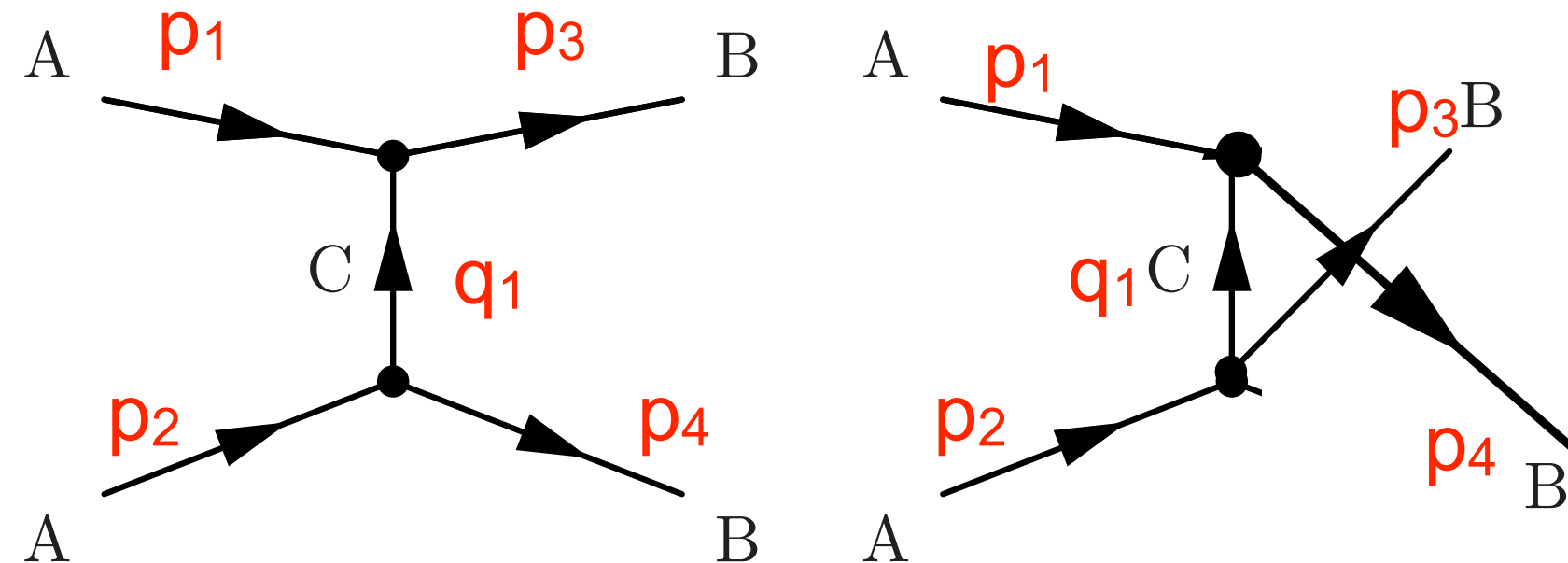
$$\frac{d\sigma}{d\Omega} = \frac{1}{2!}\frac{1}{64\pi^2}\frac{g^4}{(2E)^2 2 * \mathbf{p}^4 \sin^4\theta}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{1024\pi^2}\frac{g^4}{E^2\mathbf{p}^4\sin^4\theta}}$$

(From Griffiths)
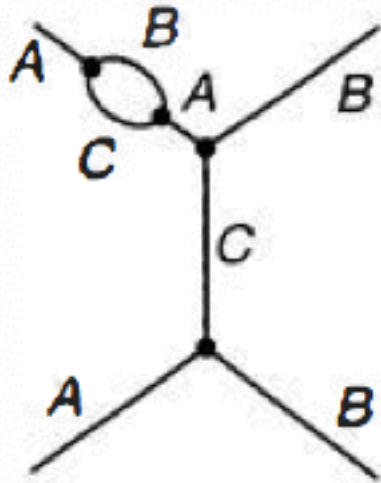
"Self-energy" diagrams

There are many of these. And even more at higher order

Box diagram
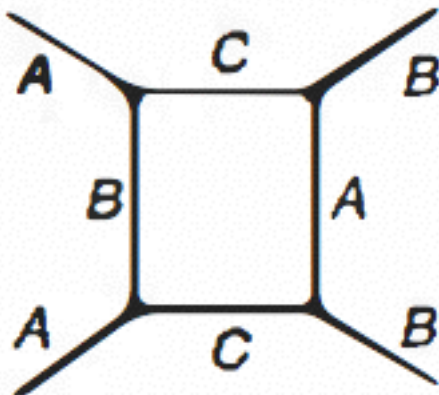
Vertex correction diagrams
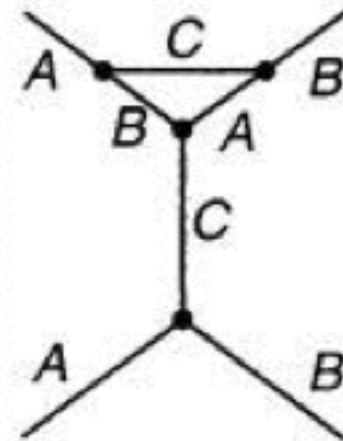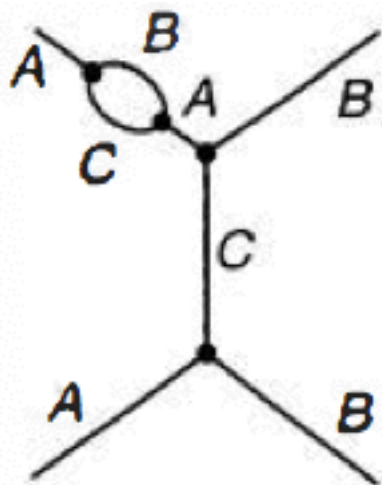
Each vertex carries a factor of "g" in the matrix element, so g^2 for physical quantities. So diagrams with extra vertices should be sub-dominant corrections, or so we hope
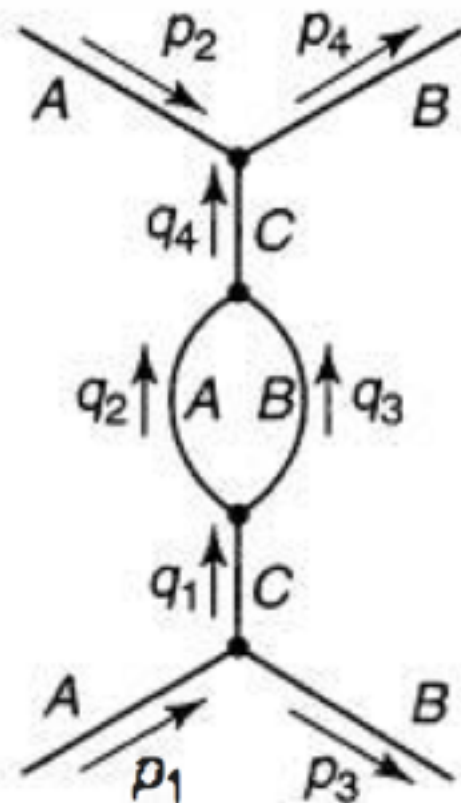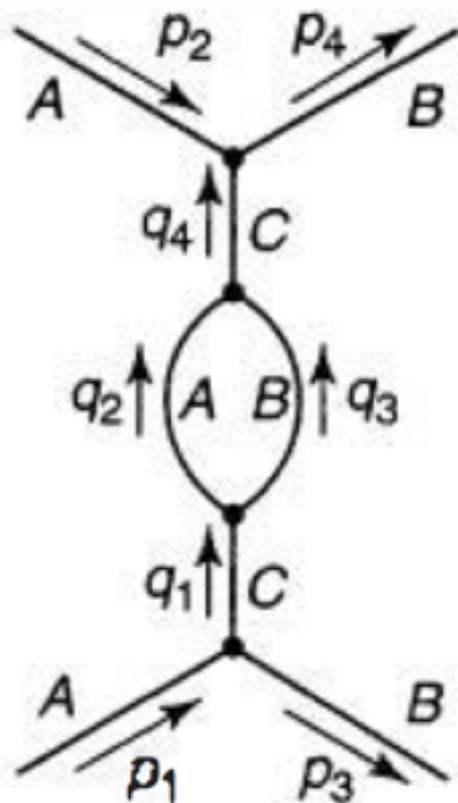
1. Label your p's and q's
2. Vertex factors
3. Propagators
4. Momentum and Energy conservation
5. Internal momentum integration
6. Cancel delta function and add extra 'i'

$(-ig)^4$
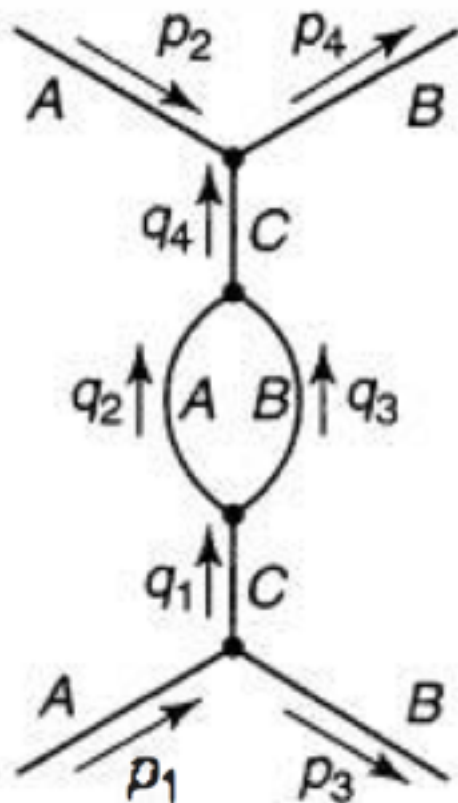
1. Label your p's and q's
2. Vertex factors
3. Propagators
4. Momentum and Energy conservation
5. Internal momentum integration
6. Cancel delta function and add extra 'i'

$$(-ig)^4 \quad \frac{i}{q_4^2 - m_C^2} \frac{i}{q_2^2 - m_A^2} \frac{i}{q_3^2 - m_B^2} \frac{i}{q_1^2 - m_C^2}$$
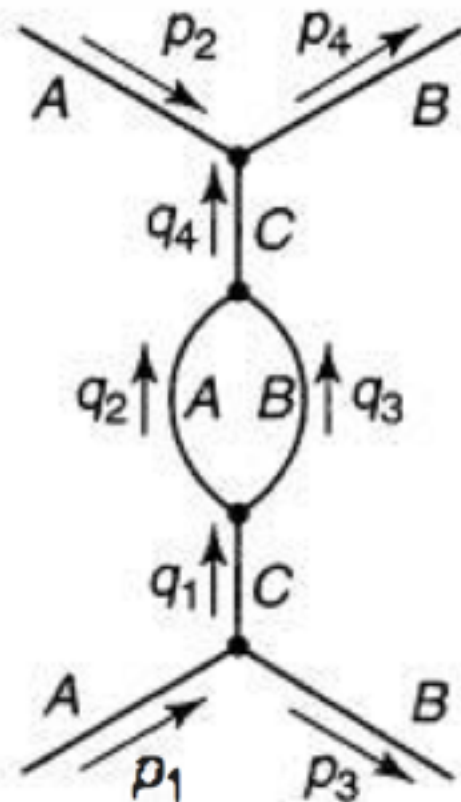
1. Label your p's and q's
2. Vertex factors
3. Propagators
4. Momentum and Energy conservation
5. Internal momentum integration
6. Cancel delta function and add extra 'i'

$$(-ig)^4 \; \frac{i}{q_4^2 - m_C^2} \frac{i}{q_2^2 - m_A^2} \frac{i}{q_3^2 - m_B^2} \frac{i}{q_1^2 - m_C^2}$$

$$(2\pi)^4 \delta^4(p_2 + q_4 - p_4)(2\pi)^4 \delta^4(q_2 + q_3 - q_4)(2\pi)^4 \delta^4(q_1 - q_2 - q_3)(2\pi)^4 \delta^4(p_1 - q_1 - p_3)$$

1. Label your p's and q's
2. Vertex factors
3. Propagators
4. Momentum and Energy conservation
5. Internal momentum integration
6. Cancel delta function and add extra 'i'

$$\int (-ig)^4 \; \frac{i}{q_4^2 - m_C^2} \frac{i}{q_2^2 - m_A^2} \frac{i}{q_3^2 - m_B^2} \frac{i}{q_1^2 - m_C^2}$$

$$(2\pi)^4 \delta^4 (p_2 + q_4 - p_4)(2\pi)^4 \delta^4 (q_2 + q_3 - q_4)(2\pi)^4 \delta^4 (q_1 - q_2 - q_3)(2\pi)^4 \delta^4 (p_1 - q_1 - p_3)$$

$$\frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{d^4 q_3}{(2\pi)^4} \frac{d^4 q_4}{(2\pi)^4}$$

1. Label your p's and q's
2. Vertex factors
3. Propagators
4. Momentum and Energy conservation
5. Internal momentum integration
6. Cancel delta function and add extra 'i'

$$\int (-ig)^4 \ \frac{i}{q_4^2 - m_C^2} \frac{i}{q_2^2 - m_A^2} \frac{i}{q_3^2 - m_B^2} \frac{i}{q_1^2 - m_C^2}$$

$$(2\pi)^4 \delta^4(p_2 + q_4 - p_4)(2\pi)^4 \delta^4(q_2 + q_3 - q_4)(2\pi)^4 \delta^4(q_1 - q_2 - q_3)(2\pi)^4 \delta^4(p_1 - q_1 - p_3)$$

$$\frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{d^4 q_3}{(2\pi)^4} \frac{d^4 q_4}{(2\pi)^4}$$

$$g^4 \int \frac{1}{q_4^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{q_1^2 - m_C^2}$$

$$\delta^4(p_2 + q_4 - p_4)\delta^4(q_2 + q_3 - q_4)\delta^4(q_1 - q_2 - q_3)\delta^4(p_1 - q_1 - p_3)d^4 q_1 d^4 q_2 d^4 q_3 d^4 q_4$$

Now let's use those delta functions

$$g^4 \int \frac{1}{q_4^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{q_1^2 - m_C^2}$$

$$\delta^4(p_2 + q_4 - p_4)\delta^4(q_2 + q_3 - q_4)\delta^4(q_1 - q_2 - q_3)\delta^4(p_1 - q_1 - p_3)d^4q_1 d^4q_2 d^4q_3 d^4q_4$$

First eliminate q$_1$

$$g^4 \int \frac{1}{q_4^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2}$$

$$\delta^4(p_2 + q_4 - p_4)\delta^4(q_2 + q_3 - q_4)\delta^4(p_1 - p_3 - q_2 - q_3)d^4q_2 d^4q_3 d^4q_4$$

Now q$_4$

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2}$$

$$\delta^4(q_2 + q_3 + p_2 - p_4)\delta^4(p_1 - p_3 - q_2 - q_3)d^4q_2 d^4q_3$$

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2}$$

$$\delta^4(q_2 + q_3 + p_2 - p_4)\delta^4(p_1 - p_3 - q_2 - q_3)d^4q_2 d^4q_3$$

Now eliminate q$_2$

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2}$$

$$\delta^4(p_1 - p_3 - q_3 + q_3 + p_2 - p_4)d^4q_3$$

Rearrange

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2} \delta^4(p_1 + p_2 - p_3 - p_4)d^4q_3$$

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2} \delta^4(p_1 + p_2 - p_3 - p_4) d^4 q_3$$
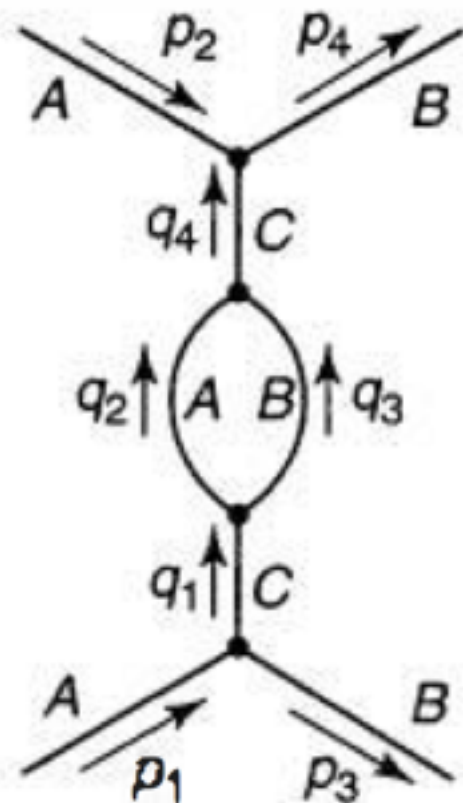
Delta function implies (p₄-p₂) = (p₁-p₃)

$$g^4 \int \frac{1}{(p_1 - p_3)^2 - m_C^2} \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2} \delta^4(p_1 + p_2 - p_3 - p_4) d^4 q_3$$

$$g^4 \int \left(\frac{1}{(p_1 - p_3)^2 - m_C^2} \right)^2 \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \delta^4(p_1 + p_2 - p_3 - p_4) d^4 q_3$$

$$g^4 \int \left(\frac{1}{(p_1 - p_3)^2 - m_C^2} \right)^2 \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \delta^4(p_1 + p_2 - p_3 - p_4) d^4 q_3$$
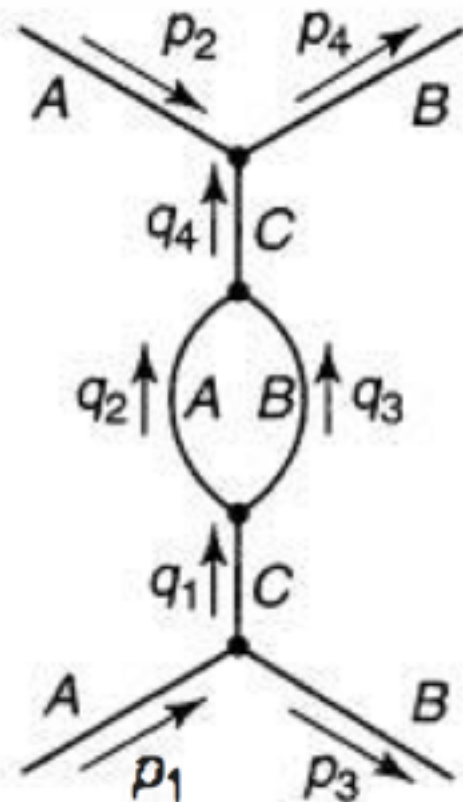

1. Label your p's and q's
2. Vertex factors
3. Propagators
4. Momentum and Energy conservation
5. Internal momentum integration
6. Cancel delta function and add extra 'i'

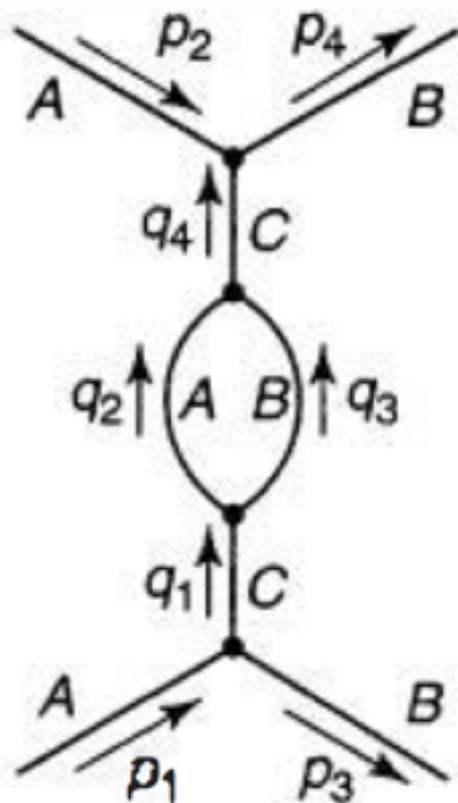$$i \left(\frac{g}{2\pi} \right)^4 \left(\frac{1}{(p_1 - p_3)^2 - m_C^2} \right)^2 \int \frac{1}{((p_1 - p_3 - q)^2 - m_A^2)(q^2 - m_B^2)} d^4 q$$

(2pi)$^{-4}$ from canceling delta

1. Label your p's and q's
2. Vertex factors
3. Propagators
4. Momentum and Energy conservation
5. Internal momentum integration
6. Cancel delta function and add extra 'i'

$$\mathcal{M} = i \left(\frac{g}{2\pi} \right)^4 \left(\frac{1}{(p_1 - p_3)^2 - m_C^2} \right)^2 \int \frac{1}{\left((p_1 - p_3 - q)^2 - m_A^2 \right) \left(q^2 - m_B^2 \right)} d^4 q$$

Can try and evaluate that ugly integral, but we find that it diverges at large momentum for internal q!

$$\mathcal{M} = i \left(\frac{g}{2\pi} \right)^4 \left(\frac{1}{(p_1 - p_3)^2 - m_C^2} \right)^2 \int \frac{1}{((p_1 - p_3 - q)^2 - m_A^2)(q^2 - m_B^2)} d^4q$$
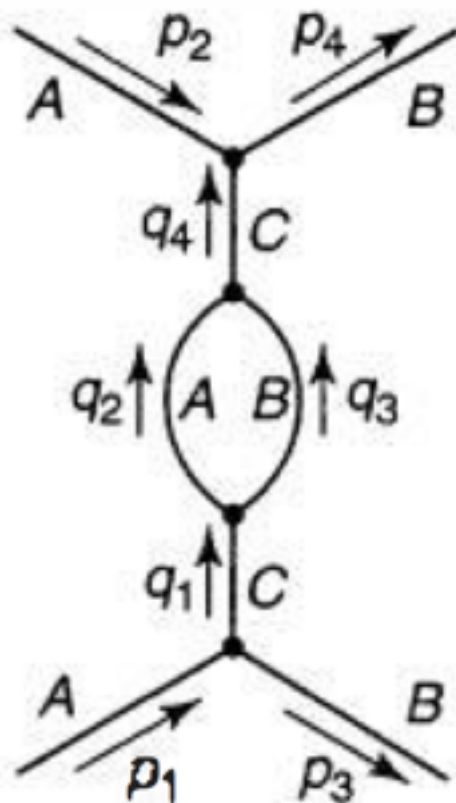
More thorough investigation finds that the infinities are really just affecting the masses of objects and the coupling constants - but these are measured quantities anyway, so we can "use" the measured values. Such theories are **renormalizable**

$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4\left(\frac{1}{(p_1 - p_3)^2 - m_C^2}\right)^2\int\frac{1}{\left((p_1 - p_3 - q)^2 - m_A^2\right)\left(q^2 - m_B^2\right)}d^4q$$
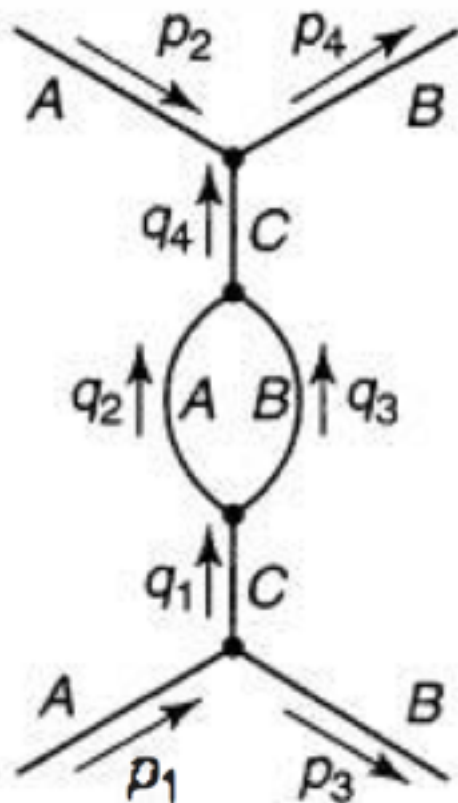
We can't assume that physics inside those loops can have infinite momentum - there must be a "cutoff" at which new physics appears (of course, this is just a toy theory, but the same thing happens in real theories!)
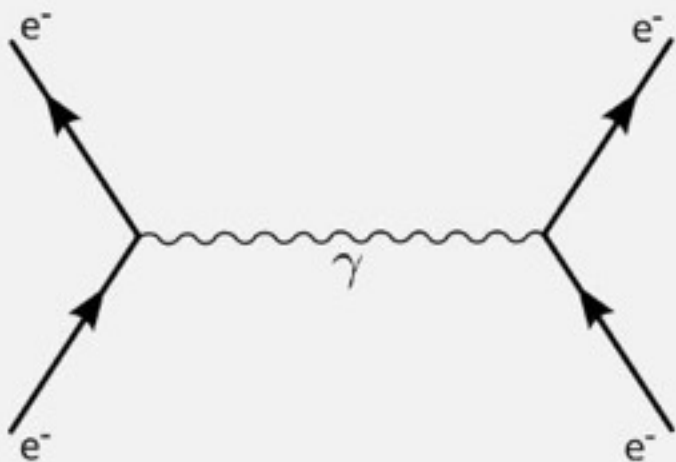
$$\mathcal{M} = i \left(\frac{g}{2\pi}\right)^4 \left(\frac{1}{(p_1 - p_3)^2 - m_C^2}\right)^2 \int \frac{1}{((p_1 - p_3 - q)^2 - m_A^2)(q^2 - m_B^2)} d^4q$$
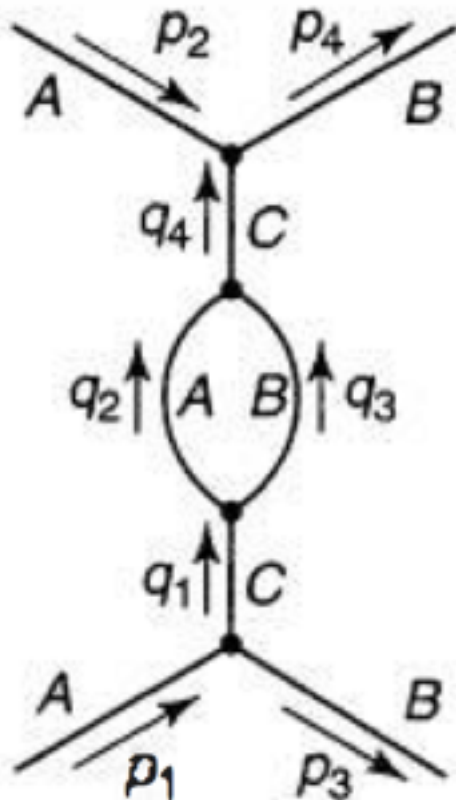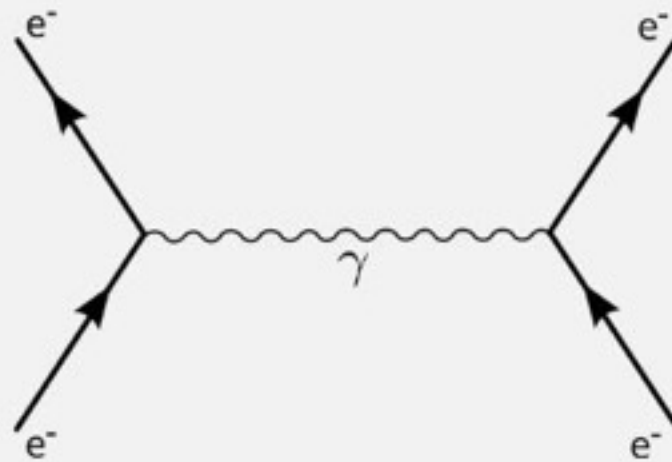

There **are** modifications to the matrix element that are not infinite that do provide corrections to m and g. Implies that there is energy dependence to masses and couplings! (See running of the couplings from earlier in the semester)

THERE ARE **TWO KINDS OF PEOPLE** IN THIS WORLD

"This diagram represents two electrons interacting via exchange of a virtual photon"

"This diagram represents a term in a series expansion"