A brief detour to Lagrangians

You've all seen this before, hopefully more than a few times

$$\mathcal{L} = T - U$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$$

Now we are dealing with fields! Which are themselves functions of x,y,z .. and also t. Define a "Lagrangian density" that is a function of these fields and their derivatives:

Does analogy make sense?

$$\partial_\mu \phi_i = \frac{\partial \phi_i}{\partial x^\mu}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_i}$$

Relating Lagrangian density to more familiar Lagrangian

$$\mathcal{L} = T - U$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$$

$$L = \int \mathcal{L}\, d^3\mathbf{x}$$

Lagrangian density: generalized coordinates q_i are replaced by the field themselves, and time derivates dq_i/dt are replaced by derivatives of the fields with respect to each space-time coordinate

Simplest Lagrangian (Klein-Gordon for scalar)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_i}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

$$\mathcal{L} = \frac{1}{2}[\partial_0 \phi \partial_0 \phi - \partial_1 \phi \partial_1 \phi - \partial_2 \phi \partial_2 \phi - \partial_3 \phi \partial_3 \phi - m^2 \phi^2]$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi = \partial^0 \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_1 \phi)} = -\partial_1 \phi = \partial^1 \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

Simplest Lagrangian (Klein-Gordon for scalar)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_i}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \qquad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\boxed{\partial_\mu \partial^\mu \phi = -m^2 \phi}$$

Klein-Gordon equation describing the field of a spin-0 particle with mass m

Dirac Lagrangian for spinor field

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m^2\overline{\psi}\psi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} = 0 \qquad \frac{\partial\mathcal{L}}{\partial\overline{\psi}} = i\gamma^\mu\partial_\mu\psi - m^2\psi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\overline{\psi}\gamma^\mu \qquad \frac{\partial\mathcal{L}}{\partial\psi} = -m^2\overline{\psi}$$

Implies the Dirac equation!

$$i\gamma^\mu\partial_\mu\psi - m^2\psi = 0$$

$$i\gamma^\mu\partial_\mu\overline{\psi} + m^2\overline{\psi} = 0$$

Proca Lagrangian for vector field

$$\mathcal{L} = \frac{-1}{16\pi}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{m^2}{8\pi} A^\nu A_\nu$$

Define $$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

And can work through algebra (we won't!) to get:

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$

Think back to classical mechanics and an object of mass m orbiting in a gravitational field produced by a second mass M

$$L = T - V = \frac{1}{2}mv^2 + \frac{GMm}{r^2}$$

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{GMm}{r^2}$$

No Φ dependence (Lagrangian is **invariant** under the transformation Φ⟶ Φ+δΦ), so

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \quad \longrightarrow \quad \frac{\partial L}{\partial \dot{\phi}} = J = mr^2\dot{\phi}$$ **constant**

We like transformation that don't change our Lagrangian (we like symmetries!) If we can identify a symmetry, then we can say that something is conserved. But we need to expand what sorts of things we examine…

Local Gauge invariance

$$\mathcal{L} = i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - m^2\overline{\psi}\psi$$

Global gauge invariance (Lagrangian doesn't change due to overall phase)

$$\psi \rightarrow e^{i\theta}\psi$$
$$\overline{\psi} \rightarrow e^{-i\theta}\overline{\psi}$$

Look at the difference carefully here!

What about local phase transformations?

$$\psi(x) \rightarrow e^{i\theta(x)}\psi(x)$$
$$\overline{\psi}(x) \rightarrow e^{-i\theta(x)}\overline{\psi}(x)$$

Local Gauge invariance

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m^2\overline{\psi}\psi$$

$$\psi(x) \rightarrow e^{i\theta(x)}\psi(x)$$

$$\overline{\psi}(x) \rightarrow e^{-i\theta(x)}\overline{\psi}(x)$$

$$\mathcal{L}' = ie^{-i\phi(x)}\overline{\psi}\gamma^\mu\partial_\mu(e^{i\theta(x)}\psi) - m^2\overline{\psi}\psi$$

$$\partial_\mu(e^{i\theta(x)}\psi) = i[\partial_\mu\theta(x)]e^{i\theta(x)}\psi + e^{i\theta(x)}\partial_\mu\psi$$

$$\mathcal{L}' = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m^2\overline{\psi}\psi - \overline{\psi}\gamma^\mu[\partial_\mu\theta]\psi$$

So Lagrangian not generally invariant under this transformation! But we can demand it

Let's try and fix that!

Add a vector field A to the Lagrangian:

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m^2\overline{\psi}\psi - (q\overline{\psi}\gamma^\mu\psi)A_\mu$$

$$A_\mu \rightarrow A_\mu + \partial_\mu\lambda$$

$$\mathcal{L}' = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m^2\overline{\psi}\psi - \overline{\psi}\gamma^\mu[\partial_\mu\theta]\psi - q\overline{\psi}\gamma^\mu\psi A_\mu - q\overline{\psi}\gamma^\mu\psi\partial_\mu\lambda$$

We want these to cancel

So ..

$$\lambda = \frac{-\theta}{q}$$

But we need to add the free term for A

$$\mathcal{L} = i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - m^2\overline{\psi}\psi - (q\overline{\psi}\gamma^{\mu}\psi)A_{\mu} \qquad \lambda = \frac{-\theta}{q}$$

$$A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\lambda$$

$$\mathcal{L}^{\mathcal{A}} = \frac{-1}{16\pi}(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu})(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}) + \frac{m^2}{8\pi}A^{\nu}A_{\nu}$$

How do these new terms transform under our gauge transformation?

$$\mathcal{L}^{\mathcal{A}'} = \frac{-1}{16\pi}[\partial^{\mu}(A^{\nu} + \partial^{\nu}\lambda) - \partial^{\nu}(A^{\mu} + \partial^{\mu}\lambda)(\partial_{\mu}(A_{\nu} + \partial_{\nu}\lambda) - \partial_{\nu}(A_{\mu} + \partial_{\mu}\lambda)] + \frac{m^2}{8\pi}(A^{\nu} + \partial^{\nu}\lambda)(A_{\nu} + \partial_{\nu}\lambda)$$

How are the new pieces of our Lagrangian transforming?

$$\mathcal{L}^{\mathcal{A}} = \frac{-1}{16\pi}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{m^2}{8\pi} A^\nu A_\nu$$

$$\mathcal{L}^{\mathcal{A}'} = \frac{-1}{16\pi}[\partial^\mu (A^\nu + \partial^\nu \lambda) - \partial^\nu (A^\mu + \partial^\mu \lambda)(\partial_\mu (A_\nu + \partial_\nu \lambda) - \partial_\nu (A_\mu + \partial_\mu \lambda)] + \frac{m^2}{8\pi}(A^\nu + \partial^\nu \lambda)(A_\nu + \partial_\nu \lambda)$$

$$\mathcal{L}^{\mathcal{A}'} = \frac{-1}{16\pi}[\partial^\mu A^\nu + \partial^\mu \partial^\nu \lambda - \partial^\nu A^\mu - \partial^\nu \partial^\mu \lambda)(\partial_\mu A_\nu + \partial_\mu \partial_\nu \lambda - \partial_\nu A_\mu - \partial_\nu \partial_\mu \lambda)] +$$
$$\frac{m^2}{8\pi}(A^\nu A_\nu + \partial^\nu \lambda A_\nu + A^\nu \partial_\nu \lambda + \partial^\nu \lambda \partial_\nu \lambda)$$

$$\partial^\mu \partial^\nu \lambda = \partial^\nu \partial^\mu \lambda$$

$$\Delta(\mathcal{L}^{\mathcal{A}'}) = \frac{m^2}{8\pi}(\partial^\nu \lambda A_\nu + A^\nu \partial_\nu \lambda + \partial^\nu \lambda \partial_\nu \lambda)$$

So this is only invariant if m = 0!

Can we make this more general?

Physicists like theories with symmetries (see chapter 4!) and local gauge invariance is a nicely general one! How to build on it?

$$\partial_\mu(e^{i\theta(x)}\psi) = i[\partial_\mu\theta(x)]e^{i\theta(x)}\psi + e^{i\theta(x)}\partial_\mu\psi$$

$$\partial_\mu(e^{-i\lambda q}\psi) = -i[\partial_\mu(\lambda q)]e^{-i\lambda q}\psi + e^{-i\lambda q}\partial_\mu\psi$$

$$\partial_\mu(e^{-i\lambda q}\psi) = e^{-i\lambda q}[\partial_\mu - iq(\partial_\mu\lambda)]\psi$$

Can we make this more general?

$$\partial_\mu(e^{-i\lambda q}\psi) = e^{-i\lambda q}[\partial_\mu - iq(\partial_\mu \lambda)]\psi$$

OK if we replace:

$$\partial_\mu \to \mathcal{D}_\mu = \partial_\mu + iqA_\mu$$

U(1) gauge invarariance

$$\psi \to U\psi, U^\dagger U = 1, U = e^{i\theta}$$

Covariant derivative

If you take field theory, they will become your friend

The price we pay is that we must introduce a new massless vector field

Two spin-1/2 non-interacting particles

$$\mathcal{L} = i\overline{\psi}_1\gamma^\mu\partial_\mu\psi_1 - m^2\overline{\psi_1}\psi_1 + i\overline{\psi}_2\gamma^\mu\partial_\mu\psi_2 - m^2\overline{\psi_2}\psi_2$$

Write this more cleanly as:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\overline{\psi} = (\overline{\psi}_1 \; \overline{\psi}_2)$$

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - M^2\overline{\psi}\psi$$

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

Let's discuss particle masses in Lagrangians

Classical $\mathcal{L} = T - U$

More field theory inspired

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2$$

$$\mathcal{L} = \frac{1}{2}[\partial_0\phi\partial_0\phi - \partial_1\phi\partial_1\phi - \partial_2\phi\partial_2\phi - \partial_3\phi\partial_3\phi - m^2\phi^2]$$

Can think of the mass term as the "field squared" term in the Lagrangian. The first piece (with derivatives) is the kinetic energy term (we know that kinetic energy involves derivatives)

How do we know it is that term?

Could the mass be the k term?

[Field*m]^2

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - k^4\phi^4$$

[Field*GeV]^2

[Field*k]^4

	Not our choice	In our choice of natural units
Energy	GeV	GeV
Momentum	GeV/c	GeV
Mass	GeV/c^2	GeV
Time	hbar/GeV	GeV^{-1}
Length	c*hbar/GeV	GeV^{-1}
Area	(c*hbar/GeV)^2	GeV^{-2}

All 3 terms must have the same units!

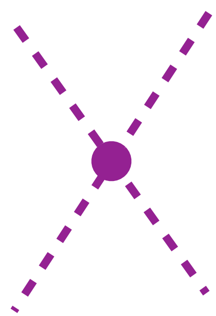
m has units of GeV that we want!

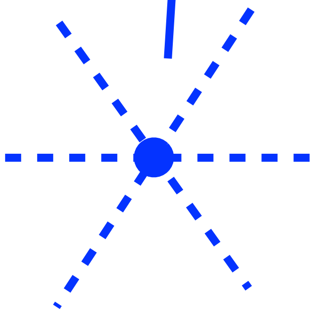
What does a Lagrangian with higher order terms represent?

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - k^4\phi^4 - g^6\phi^6$$

Kinetic terms of the fields

Mass terms





Additional terms represent new couplings, of more objects to a single vertex

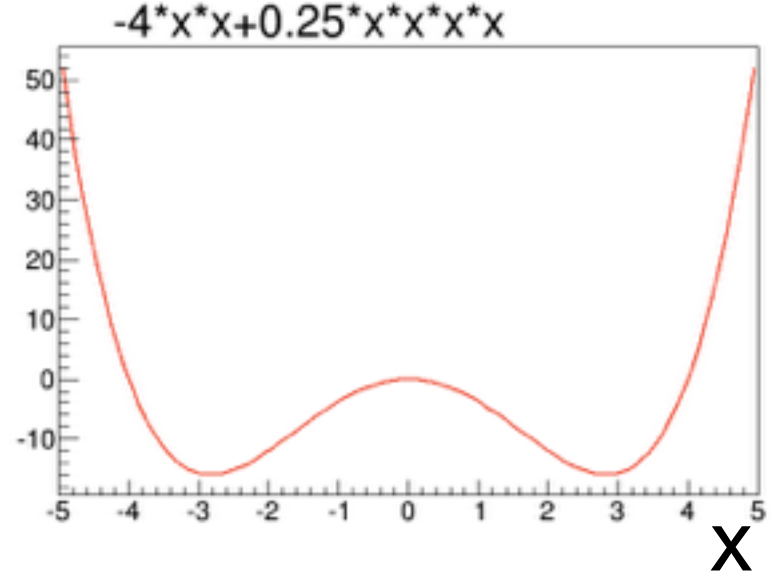
What about this Lagrangian?

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2}m^2\phi^2 - \frac{1}{4}k^2\phi^4$$

$$U = -\frac{1}{2}m^2\phi^2 + \frac{1}{4}k^2\phi^4$$

Mass term has wrong sign! That is because our calculations are really a fancy version of perturbation theory. In most theories, the ground state of the "potential" has the field at zero (the ground state of the E&M field has no E&M waves or photons!)

Minimum not at Φ=0 but rather Φ=±m/k



Expanding about the real ground state

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2}m^2\phi^2 - \frac{1}{4}k^2\phi^4 \qquad \eta = \phi \pm \frac{m}{k} \qquad \phi = \eta \pm \frac{m}{k}$$

$$\phi^2 = \eta^2 + \frac{m^2}{k^2} \pm 2\frac{m}{k}\eta$$

$$\phi^4 = \eta^4 + \frac{m^4}{k^4} + 4\frac{m^2}{k^2}\eta^2 + 2\frac{m^2}{k^2}\eta^2 \pm 4\frac{m}{k}\eta^3 \pm 4\frac{m^3}{k^3}\eta$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}m^2(\eta^2 + \frac{m^2}{k^2} \pm 2\frac{m}{k}\eta) - \frac{1}{4}k^2(\eta^4 + \frac{m^4}{k^4} + 4\frac{m^2}{k^2}\eta^2 + 2\frac{m^2}{k^2}\eta^2 \pm 4\frac{m}{k}\eta^3 \pm 4\frac{m^3}{k^3}\eta)$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \eta(\pm\frac{m^3}{k} \mp \frac{m^3}{k}) + \eta^2(\frac{m^2}{2} - m^2 - \frac{m^2}{2}) \mp mk\eta^3 - \frac{k^2}{4}\eta^4 + C$$

Ignore constant term and cancel terms...

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - m^2\eta^2 \pm mk\eta^3 - \frac{k^2}{4}\eta^4$$

New Lagrangian

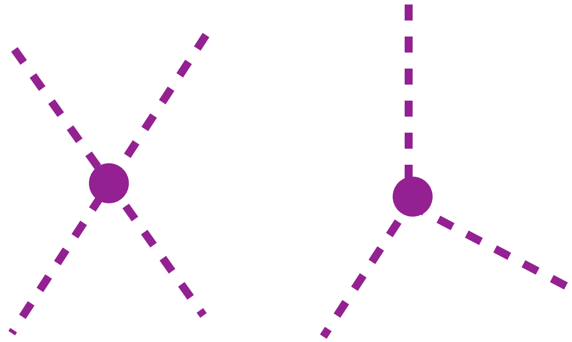
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - m^2\eta^2 \pm mk\eta^3 - \frac{k^2}{4}\eta^4$$

Compare with original one before adding new terms

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2$$

Mass of particle is sqrt(2)*m

And we have these two new interactions



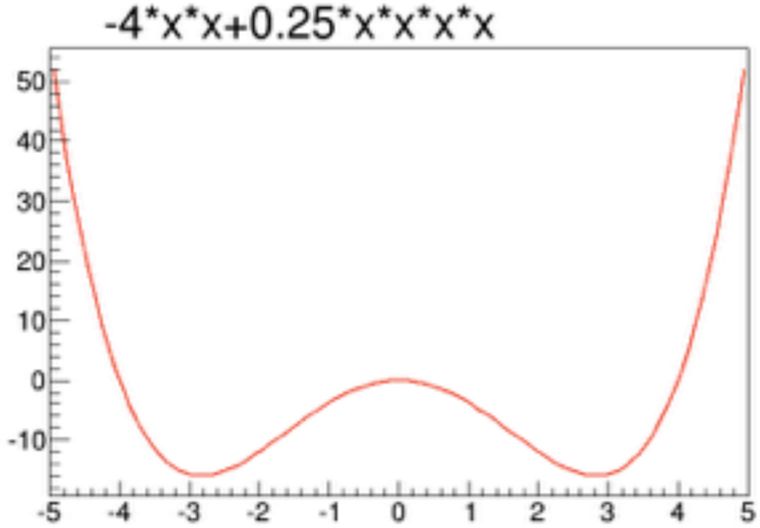
Reminder: description in terms of new variable **must be** the same. Choice of vacuum ground state breaks symmetries

Spontaneous symmetry breaking

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2}m^2\phi^2 - \frac{1}{4}k^2\phi^4$$

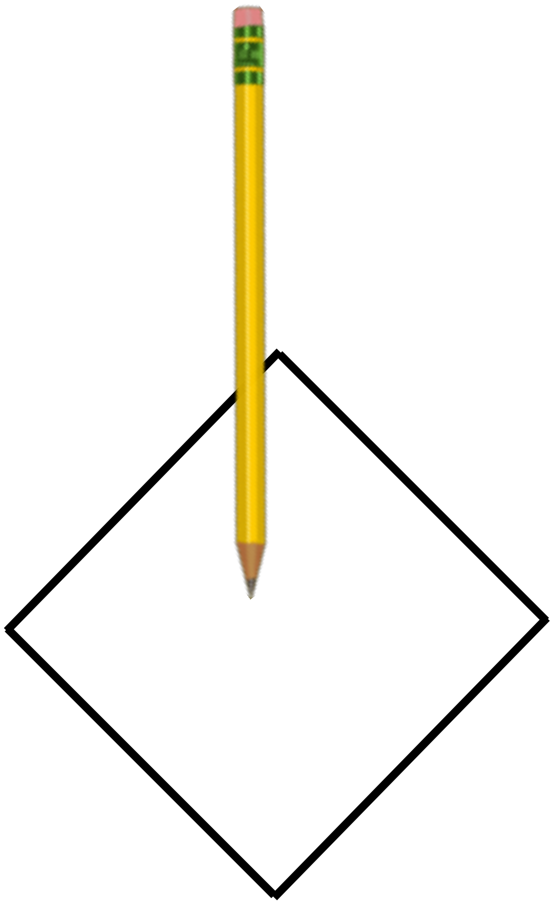
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - m^2\eta^2 \pm mk\eta^3 - \frac{k^2}{4}\eta^4$$

Original Lagrangian is symmetric in Φ→-Φ



New Lagrangian is not symmetric in this way! Selection of **specific ground state** **hides** this symmetry! We have expanded this around the minimum (otherwise perturbation theory makes no sense!)

Classical example of spontaneous symmetry breaking



Mechanical laws describing this pencil under gravity are symmetrical with respect to angle from the vertical

System choses to move to a ground state of lower energy, at the price of hiding that symmetry!

Lagrangian with two scalar fields + symmetry breaking

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) + \frac{1}{2}m^2\phi_2^2 - \frac{1}{4}k^2(\phi_1^2 + \phi_2^2)^2$$

$$U = -\frac{1}{2}m^2(\phi_1^2 + \phi_2^2) + \frac{1}{4}k^2(\phi_1^2 + \phi_2^2)^2$$

$$\frac{\partial U}{\partial \phi_1} = -m^2\phi_1 + \frac{k^2}{2}(\phi_1^2 + \phi_2^2)2\phi_1$$

$$\frac{\partial U}{\partial \phi_1} = -m^2\phi_1 + k^2(\phi_1^3 + \phi_1\phi_2^2) = 0$$

$$\frac{\partial U}{\partial \phi_2} = -m^2\phi_2 + k^2(\phi_2\phi_1^2 + \phi_2^3) = 0$$

Lagrangian with two scalar fields + symmetry breaking

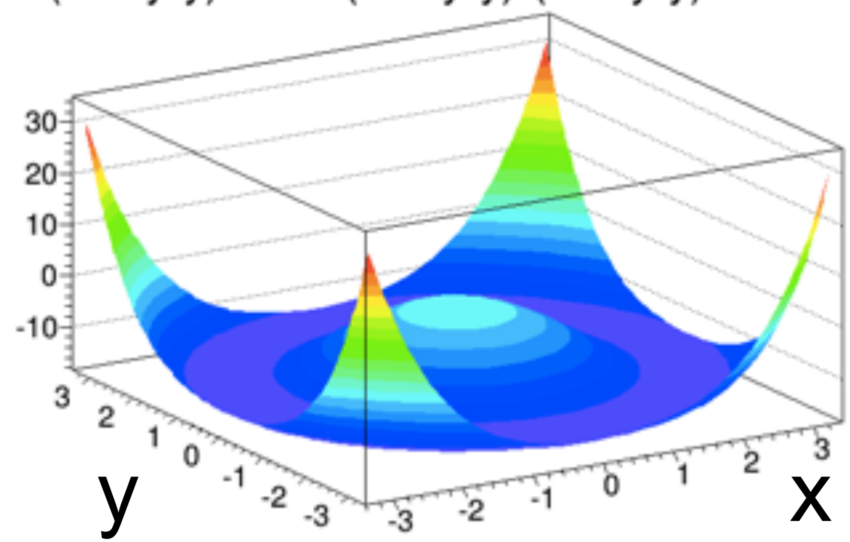
$$\frac{\partial U}{\partial \phi_1} = -m^2\phi_1 + k^2(\phi_1^3 + \phi_1\phi_2^2) = 0$$

$$\frac{\partial U}{\partial \phi_2} = -m^2\phi_2 + k^2(\phi_2\phi_1^2 + \phi_2^3) = 0$$

$$-m^2 + k^2(\phi_1^2 + \phi_2^2) = 0$$

$$\boxed{\phi_1^2 + \phi_2^2 = \frac{m^2}{k^2}}$$

A continous set of minima! But not at zero field



Lagrangian with two scalar fields + symmetry breaking

$$\phi_1^2 + \phi_2^2 = \frac{m^2}{k^2}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) + \frac{1}{2}m^2\phi_2^2 - \frac{1}{4}k^2(\phi_1^2 + \phi_2^2)^2$$

Let's pick $\Phi_1 = m/k$, $\Phi_2 = 0$, and expand around that

$$\eta = \phi_1 - m/k$$

$$\phi_1 = \eta + m/k$$

$$\theta = \phi_2$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) + \frac{1}{2}m^2(\eta^2 + \frac{m^2}{k^2} + 2\frac{m}{k}\eta + \theta^2) - \frac{1}{4}k^2(\eta^2 + \frac{m^2}{k^2} + 2\frac{m}{k}\eta + \theta^2)^2$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) + \frac{1}{2}m^2(\eta^2 + \frac{m^2}{k^2} + 2\frac{m}{k}\eta + \theta^2) -$$
$$\frac{k^2}{4}(\eta^4 + \frac{m^4}{k^4} + 4\frac{m^2}{k^2}\eta^2 + \theta^4 + 2\eta^2\frac{m^2}{k^2} + 4\eta^3\frac{m}{k} + 2\eta^2\theta^2 + 4\frac{m^3}{k^3}\eta + 2\frac{m^2}{k^2}\theta^2 + 4\frac{m}{k}\eta\theta^2)$$

Lagrangian with two scalar fields + symmetry breaking

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) + \frac{1}{2}m^2(\eta^2 + \frac{m^2}{k^2} + 2\frac{m}{k}\eta + \theta^2) -$$
$$\frac{k^2}{4}(\eta^4 + \frac{m^4}{k^4} + 4\frac{m^2}{k^2}\eta^2 + \theta^4 + 2\eta^2\frac{m^2}{k^2} + 4\eta^3\frac{m}{k} + 2\eta^2\theta^2 + 4\frac{m^3}{k^3}\eta + 2\frac{m^2}{k^2}\theta^2 + 4\frac{m}{k}\eta\theta^2)$$

Remove constant terms and combine

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) + \frac{1}{2}m^2(\eta^2 + 2\frac{m}{k}\eta + \theta^2) -$$
$$\frac{k^2}{4}(\eta^4 + \theta^4 + 6\frac{m^2}{k^2}\eta^2 + 4\eta^3\frac{m}{k} + 2\eta^2\theta^2 + 4\frac{m^3}{k^3}\eta + 2\frac{m^2}{k^2}\theta^2 + 4\frac{m}{k}\eta\theta^2)$$

Combine more terms

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) - m^2\eta^2 - \frac{k^2}{4}\eta^4 - \frac{k^2}{4}\theta^4 - km\eta^3 - \frac{k^2}{2}\theta^2\eta^2 - mk\eta\theta^2$$

Lagrangian with two scalar fields + symmetry breaking

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) - m^2\eta^2 - \frac{k^2}{4}\eta^4 - \frac{k^2}{4}\theta^4 - km\eta^3 - \frac{k^2}{2}\theta^2\eta^2 - mk\eta\theta^2$$

Let's rewrite this

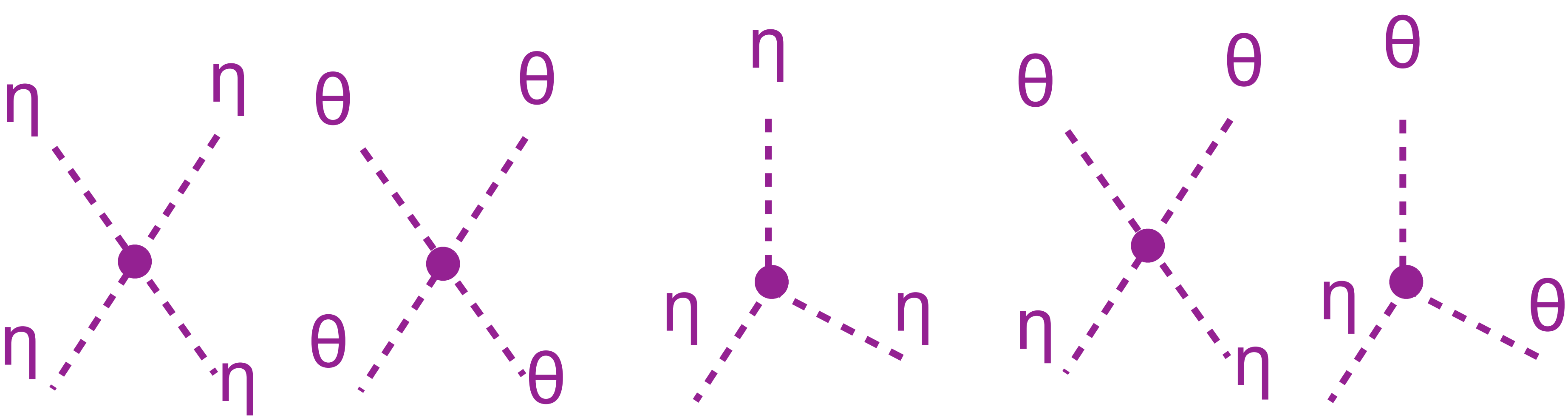
$$\mathcal{L} = \left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - m^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta)\right] - \left[\frac{k^2}{4}\eta^4 + \frac{k^2}{4}\theta^4 + km\eta^3 + \frac{k^2}{2}\theta^2\eta^2 + mk\eta\theta^2\right]$$

$$m_\eta = \sqrt{2}m$$

$$m_\theta = 0$$

One of the fields massless! Why?

And have new vertices:



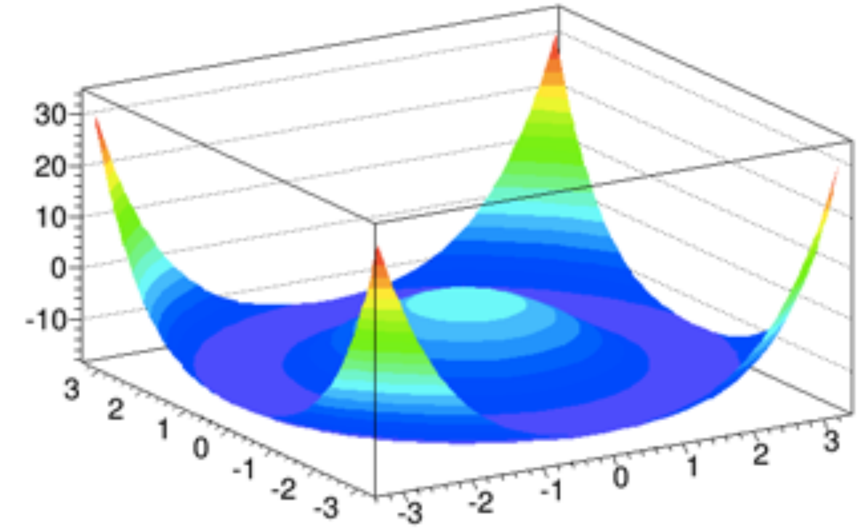
Massless field

$$\mathcal{L} = \left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - m^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta)\right] - \left[\frac{k^2}{4}\eta^4 + \frac{k^2}{4}\theta^4 + km\eta^3 + \frac{k^2}{2}\theta^2\eta^2 + mk\eta\theta^2\right]$$

$$m_\eta = \sqrt{2}m$$
$$m_\theta = 0$$

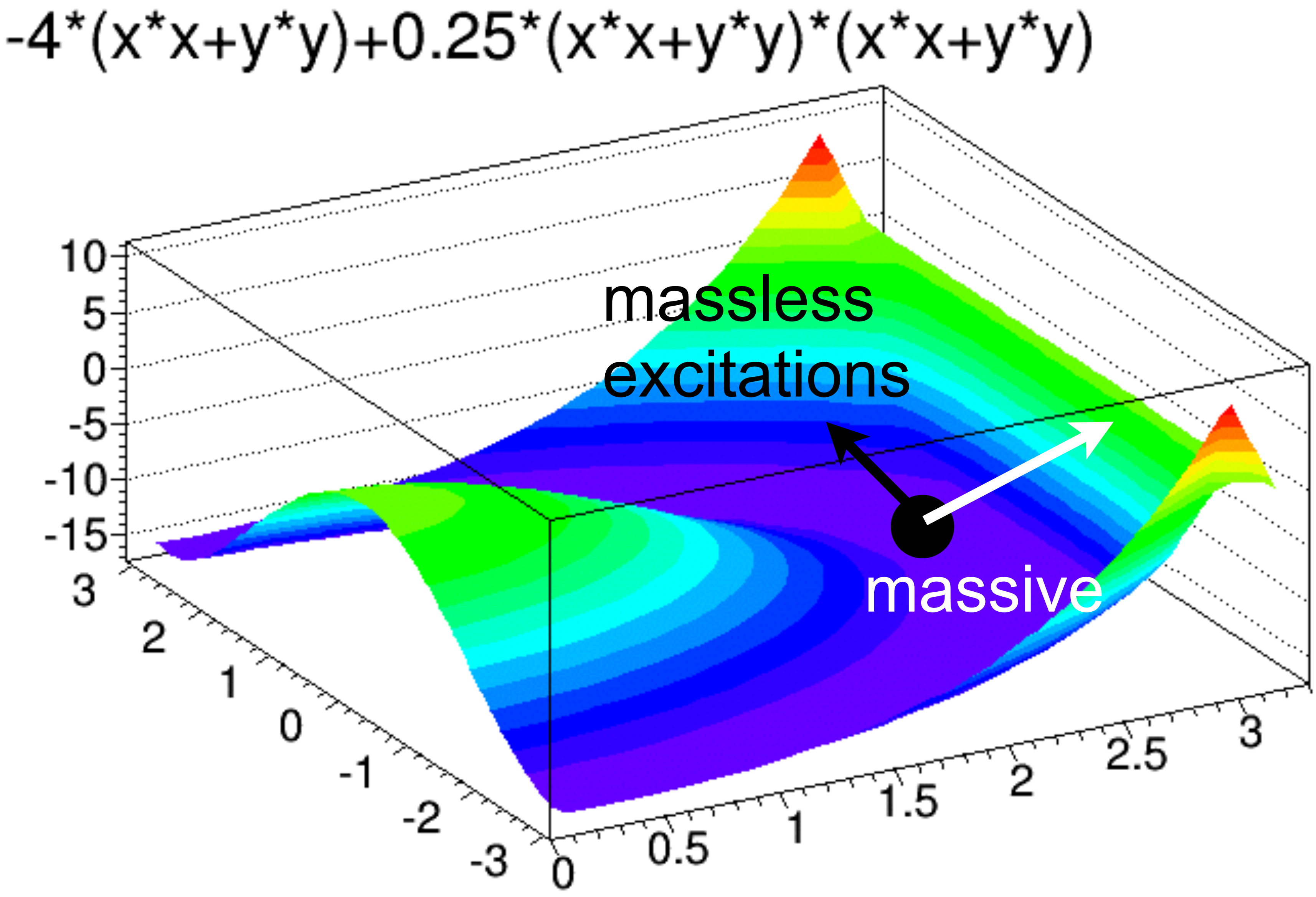
One of the fields massless! Why?

Continuous global symmetry! Goldstone's theorem - have you seen this?



Massless in direction where potential does not change, massive in orthogonal direction

Massless and massive field



Two scalar fields with spontaneous symmetry breaking

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) + \frac{1}{2}m^2\phi_2^2 - \frac{1}{4}k^2(\phi_1^2 + \phi_2^2)^2$$

Rewrite this as a single complex field

$$\phi = \phi_1 + i\phi_2$$

$$\phi^*\phi = \phi_1^2 + \phi_2^2$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^*(\partial^\mu \phi) + \frac{1}{2}m^2(\phi^*\phi) - \frac{1}{4}k^2(\phi^*\phi)^2$$

Now we want to apply local gauge transformations

$$\phi = \phi_1 + i\phi_2$$

$$\phi^*\phi = \phi_1^2 + \phi_2^2$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^*(\partial^\mu \phi) + \frac{1}{2}m^2(\phi^*\phi) - \frac{1}{4}k^2(\phi^*\phi)^2$$

Can get this: $\phi \rightarrow e^{i\theta(x)}\phi$

If we make this change: $\partial_\mu \rightarrow \mathcal{D}_\mu = \partial_\mu + iqA_\mu$

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu + iqA_\mu)\phi\right]^*\left[(\partial^\mu + iqA^\mu)\phi\right] + \frac{1}{2}m^2(\phi^*\phi) - \frac{1}{4}k^2(\phi^*\phi)^2 - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}$$

Interaction terms for new massless field

Writing out our Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu - iqA_\mu)\phi^*\right]\left[(\partial^\mu + iqA^\mu)\phi\right] + \frac{1}{2}m^2(\phi^*\phi) - \frac{1}{4}k^2(\phi^*\phi)^2 - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}$$

Again expand about real minimum (has not changed due to adding gauge symmetry):

$$\eta = \phi_1 - m/k$$
$$\phi_1 = \eta + m/k$$
$$\psi = \phi_2$$

The F terms do not change via this transformation. And the m^2 and k^2 terms were calculated by us a few slides ago. So let's work out the nasty first two terms

The ugly pieces

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu - iqA_\mu)\phi^*\right]\left[(\partial^\mu + iqA^\mu)\phi\right] + \frac{1}{2}m^2(\phi^*\phi) - \frac{1}{4}k^2(\phi^*\phi)^2 - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}$$

$$\eta = \phi_1 - m/k$$

$$\phi_1 = \eta + m/k$$

$$\psi = \phi_2$$

$$\frac{1}{2}\left[(\partial_\mu - iqA_\mu)\phi^*\right]\left[(\partial^\mu + iqA^\mu)\phi\right]$$

$$\frac{1}{2}\left[(\partial_\mu - iqA_\mu)(\phi_1 - i\phi_2)\right]\left[(\partial^\mu + iqA^\mu)(\phi_1 + i\phi_2)\right]$$

Can already get apprehensive about the algebra!

The ugly pieces

$$\frac{1}{2}\left[(\partial_\mu - iqA_\mu)(\phi_1 - i\phi_2)\right]\left[(\partial^\mu + iqA^\mu)(\phi_1 + i\phi_2)\right]$$

$$\frac{1}{2}\left[\partial_\mu\phi_1 - i\partial_\mu\phi_2 - iqA_\mu\phi_1 - qA_\mu\phi_2\right]\left[\partial^\mu\phi_1 + i\partial^\mu\phi_2 + iqA^\mu\phi_1 - qA^\mu\phi_2\right]$$

$$\frac{1}{2}[\partial_\mu\phi_1\partial^\mu\phi_1 + i\partial_\mu\phi_1\partial^\mu\phi_2 + iqA^\mu\phi_1\partial_\mu\phi_1 - qA^\mu\phi_2\partial_\mu\phi_1 +$$

$$-i\partial_\mu\phi_2\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 + qA^\mu\phi_1\partial_\mu\phi_2 + iq\phi_2A^\mu\partial_\mu\phi_2$$

$$-iqA_\mu\phi_1\partial^\mu\phi_1 + qA_\mu\phi_1\partial^\mu\phi_2 + q^2A_\mu A^\mu\phi_1^2 + iq^2A_\mu A^\mu\phi_1\phi_2 +$$

$$-qA_\mu\phi_2\partial^\mu\phi_1 - iqA_\mu\phi_2\partial^\mu\phi_2 - iq^2\phi_2A_\mu A^\mu\phi_1 + q^2A_\mu A^\mu\phi_2^2]$$

Simplifying things

$$\frac{1}{2}[\partial_\mu\phi_1\partial^\mu\phi_1 + i\partial_\mu\phi_1\partial^\mu\phi_2 + iqA^\mu\phi_1\partial_\mu\phi_1 - qA^\mu\phi_2\partial_\mu\phi_1 +$$

$$-i\partial_\mu\phi_2\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 + qA^\mu\phi_1\partial_\mu\phi_2 + iq\phi_2 A^\mu\partial_\mu\phi_2$$

$$-iqA_\mu\phi_1\partial^\mu\phi_1 + qA_\mu\phi_1\partial^\mu\phi_2 + q^2A_\mu A^\mu\phi_1^2 + iq^2A_\mu A^\mu\phi_1\phi_2 +$$

$$-qA_\mu\phi_2\partial^\mu\phi_1 - iqA_\mu\phi_2\partial^\mu\phi_2 - iq^2\phi_2 A_\mu A^\mu\phi_1 + q^2A_\mu A^\mu\phi_2^2]$$

$$\frac{1}{2}[\partial_\mu\phi_1\partial^\mu\phi_1 - qA^\mu\phi_2\partial_\mu\phi_1 +$$

$$\partial_\mu\phi_2\partial^\mu\phi_2 + qA^\mu\phi_1\partial_\mu\phi_2 +$$

$$qA_\mu\phi_1\partial^\mu\phi_2 + q^2A_\mu A^\mu\phi_1^2 +$$

$$-qA_\mu\phi_2\partial^\mu\phi_1 + q^2A_\mu A^\mu\phi_2^2]$$

Simplifying things

$$\frac{1}{2}[\partial_\mu\phi_1\partial^\mu\phi_1 - qA^\mu\phi_2\partial_\mu\phi_1 +$$
$$\partial_\mu\phi_2\partial^\mu\phi_2 + qA^\mu\phi_1\partial_\mu\phi_2 +$$
$$qA_\mu\phi_1\partial^\mu\phi_2 + q^2A_\mu A^\mu\phi_1^2 +$$
$$-qA_\mu\phi_2\partial^\mu\phi_1 + q^2A_\mu A^\mu\phi_2^2]$$

$$\frac{1}{2}[\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 + q^2A^2(\phi_1^2 + \phi_2^2) + 2qA^\mu(\phi_1\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_1)]$$

Now let's apply this and remember that we don't care about derivatives of constants

$$\eta = \phi_1 - m/k$$
$$\phi_1 = \eta + m/k$$
$$\psi = \phi_2$$

Putting it all together

$$\frac{1}{2}[\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 + q^2A^2(\phi_1^2 + \phi_2^2) + 2qA^\mu(\phi_1\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_1)]$$

Now let's apply this and remember that we don't care about derivatives of constants

$$\eta = \phi_1 - m/k$$

$$\phi_1 = \eta + m/k$$

$$\psi = \phi_2$$

$$\frac{1}{2}[\partial_\mu\eta\partial^\mu\eta + \partial_\mu\psi\partial^\mu\psi + q^2A^2(\psi^2 + \eta^2 + \frac{m^2}{k^2} + \eta\frac{2m}{k}) + 2qA^\mu(\eta\partial_\mu\psi + \frac{m}{k}\partial_\mu\psi - \psi\partial_\mu\eta)]$$

Now let's add in our missing pieces from the full Lagrangian

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu - iqA_\mu)\phi^*][(\partial^\mu + iqA^\mu)\phi] + \frac{1}{2}m^2(\phi^*\phi) - \frac{1}{4}k^2(\phi^*\phi)^2 - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}$$

Putting it all together

$$\left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - m^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu \psi)(\partial^\mu \psi)\right] +$$

$$+\frac{q^2A^2}{2}(\psi^2 + \eta^2 + \frac{m^2}{k^2} + \eta\frac{2m}{k}) +$$

$$qA^\mu(\eta\partial_\mu\psi + \frac{m}{k}\partial_\mu\psi - \psi\partial_\mu\eta)] +$$

$$-\left[\frac{k^2}{4}\eta^4 + \frac{k^2}{4}\psi^2 + km\eta^3 + \frac{k^2}{2}\eta^2\psi^2 + mk\eta\psi^2\right] +$$

$$-\frac{F^{\mu\nu}F_{\mu\nu}}{16\pi}$$

Let’s rewrite all of that

Putting it all together

Massive field as before

Massless field as before

$$\left[\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - m^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu\psi)(\partial^\mu\psi)\right] +$$

Gauge field description

$$+\frac{q^2A^2}{2}(\psi^2 + \eta^2 + \frac{m^2}{k^2} + \eta\frac{2m}{k})+$$

Interactions between scalars and vectors

$$qA^\mu(\eta\partial_\mu\psi + \frac{m}{k}\partial_\mu\psi - \psi\partial_\mu\eta)]+$$

$$-\left[\frac{k^2}{4}\eta^4 + \frac{k^2}{4}\psi^4 + km\eta^3 + \frac{k^2}{2}\eta^2\psi^2 + mk\eta\psi^2\right] +$$

$$-\frac{F^{\mu\nu}F_{\mu\nu}}{16\pi}$$

Most importantly

Mass of scalar field, scalar field self-interaction strength and interactions between scalars and vectors not all independent!

$$\left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - m^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu \psi)(\partial^\mu \psi)\right] +$$

$$+\frac{q^2A^2}{2}(\psi^2 + \eta^2 + \frac{m^2}{k^2} + \eta\frac{2m}{k})+$$

$$qA^\mu(\eta\partial_\mu\psi + \frac{m}{k}\partial_\mu\psi - \psi\partial_\mu\eta)]+$$

$$-\left[\frac{k^2}{4}\eta^4 + \frac{k^2}{4}\psi^4 + km\eta^3 + \frac{k^2}{2}\eta^2\psi^2 + mk\eta\psi^2\right] +$$

$$-\frac{F^{\mu\nu}F_{\mu\nu}}{16\pi}$$

Vector field now has a mass!!!!

Scalar field self-interactions

Finally we get to the Higgs Lagrangian

$$\mathcal{L} = \frac{1}{2}(\mathcal{D}_\mu \phi)^\dagger (\mathcal{D}^\mu \phi) - V(\phi)$$

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$$

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

It’s a complex scalar doublet - one field has electric charge (will be giving mass to the W bosons) and the other is neutral (giving mass to the Z boson)

Minimum of potential

$$\phi^\dagger\phi = \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = \frac{v^2}{2} = -\frac{\mu^2}{2\lambda}$$

We know that in the ground state, after symmetry breaking, photon remains massless. So ground state should only contain electrically neutral piece

Note original symmetry among all 4 directions. Now we are going to choose one direction, breaking the symmetry! (Other 3 directions are going to give mass to W^+, W^- and Z) Compare with our Mexican hat previous examples

$$\phi(vacuum) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$\phi \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

Higgs potential

$$\phi^\dagger\phi = \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = \frac{v^2}{2} = -\frac{\mu^2}{2\lambda}$$

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$$

$$\phi(vacuum) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$\phi \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

$$V(\phi) = \frac{\mu^2}{2}\left(v^2 + h^2 + 2vh\right) + \frac{\lambda}{4}\left(v^4 + h^4 + 4v^2h^2 + 2v^2h^2 + 4v^3h + 4vh^3\right)$$

Higgs potential

$$V(\phi) = \frac{\mu^2}{2}\left(v^2 + h^2 + 2vh\right) + \frac{\lambda}{4}\left(v^4 + h^4 + 4v^2h^2 + 2v^2h^2 + 4v^3h + 4vh^3\right)$$

Keeping terms only up to second order in h and ignoring constants

$$V(\phi) = \frac{\mu^2}{2}\left(h^2 + 2vh\right) + \frac{\lambda}{4}\left(6v^2h^2 + 4v^3h\right)$$

$$\frac{v^2}{2} = -\frac{\mu^2}{2\lambda}$$

$$V(\phi) = \frac{-\lambda v^2}{2}\left(h^2 + 2vh\right) + \frac{\lambda}{4}\left(6v^2h^2 + 4v^3h\right)$$

$$V(\phi) = \lambda v^2 h^2$$ Higgs boson mass term!

On the vacuum

You might hear about the Higgs field's non-zero field, or "vacuum expectation value" (or "vev"). Find a very simple relation between W boson mass, weak coupling g_W and v... calculate v = 246 GeV!

$$\phi(vacuum) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$\phi \to \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

After applying local gauge symmetry

Previous symmetry that we demanded was a U(1) local gauge symmetry. Now we ask for the Lagrangian to respect the symmetry of the electroweak theory, namely [$SU(2)_L \times U(1)_Y$]

Find that the $W^{\pm}$ and Z bosons obtain mass through the breaking of the symmetry. We just found that the vacuum has massive excitations (Higgs boson!)

Full Higgs Lagrangian

$$\mathcal{L} = \frac{1}{2}(\mathcal{D}_\mu \phi)^\dagger (\mathcal{D}^\mu \phi) - V(\phi)$$

Want to pick the covariant derivative so that it also respects [SU(2)$_L$ x U(1)$_Y$] symmetry

$$\mathcal{D}_\mu = \partial_\mu + i g_W \vec{\tau} \cdot \vec{W}_\mu + i g' B_\mu$$

$$\frac{1}{2\sqrt{2}}(D_\mu \phi) = \frac{1}{2}\begin{pmatrix} i g_W W_{3\mu} + i g' B_\mu + \partial_\mu & i g_W W_{1\mu} + g_W W_{2\mu} \\ i g_W W_{1\mu} - g_W W_{2\mu} & -i g_W W_{3\mu} + i g' B_\mu + \partial_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v + h \end{pmatrix}$$

$$\frac{1}{2\sqrt{2}}(D_\mu \phi) = \frac{1}{2\sqrt{2}}\begin{bmatrix} (v+h)(i g_W W_{1\mu} + g_W W_{2\mu}) \\ (-i g_W W_{3\mu} + i g' B_\mu + \partial_\mu)(v+h) \end{bmatrix}$$

Full Higgs Lagrangian

$$\frac{1}{2\sqrt{2}}(D_\mu \phi) = \frac{1}{2\sqrt{2}} \left[\begin{array}{c} (v+h)(ig_W W_{1\mu} + g_W W_{2\mu}) \\ (-ig_W W_{3\mu} + ig' B_\mu + \partial_\mu)(v+h) \end{array} \right]$$

$$\frac{1}{2}(D_\mu \phi)^\dagger)(D^\mu \phi) =$$

$$\frac{1}{8}\left[(v+h)(-ig_W W_{1\mu} + g_W W_{2\mu}) \quad (ig_W W_{3\mu} - ig' B_\mu)(v+h)\right] \times$$

$$\left[\begin{array}{c} (v+h)(ig_W W_1^\mu + g_W W_2^\mu) \\ (-ig_W W_3^\mu + ig' B^\mu)(v+h) \end{array} \right]$$

$$\frac{1}{2}(D_\mu \phi)^\dagger)(D^\mu \phi) =$$

$$\frac{1}{8}\left((v+h)^2(g_W^2 W_{1\mu} W_1^\mu + g_W^2 W_{2\mu} W_2^\mu) + (v+h)^2(g_W^2 W_3 \mu W_3^\mu - g' g_W B_\mu W_3^\mu - g' g_W B^\mu W_{3\mu} + g'^2 B_\mu B^\mu\right)$$

Recall that:

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \pm iW_\mu^2)$$

Full Higgs Lagrangian

$$\frac{1}{2}(D_\mu\phi)^\dagger)(D^\mu\phi) =$$

$$\frac{1}{8}\left((v+h)^2(g_W^2 W_{1\mu}W_1^\mu + g_W^2 W_{2\mu}W_2^\mu) + (v+h)^2(g_W^2 W_3\mu W_3^\mu - g' g_W B_\mu W_3^\mu - g' g_W B^\mu W_{3\mu} + g'^2 B_\mu B^\mu\right)$$

Recall that:

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \pm iW_\mu^2)$$

Mass of W1, W2 given by:

$$\frac{1}{2}m_{W1,2}^2 W_{\mu,1,2}W_{1,2}^\mu \text{ term: } \frac{1}{8}v^2 g_W^2 W_{\mu,1,2}W_{1,2}^\mu$$

$$m_W = \frac{1}{2}g_W v$$

Full Higgs Lagrangian

$$\frac{1}{2}(D_\mu\phi)^\dagger)(D^\mu\phi) =$$

$$\frac{1}{8}\left((v+h)^2(g_W^2 W_{1\mu}W_1^\mu + g_W^2 W_{2\mu}W_2^\mu) + (v+h)^2(g_W^2 W_{3}\mu W_3^\mu - g' g_W B_\mu W_3^\mu - g' g_W B^\mu W_{3\mu} + g'^2 B_\mu B^\mu\right)$$

Quadratic terms for spin-1 fields:

Mass matrix M

$$\frac{v^2}{8}(W_{\mu,3}\ B_\mu)\begin{pmatrix} g_W^2 & -g_W g' \\ -g_W g' & g'^2 \end{pmatrix}\begin{pmatrix} W_3^\mu \\ B^\mu \end{pmatrix}$$

$$\det(\mathbf{M}-\lambda I) = 0 \rightarrow (g_W^2-\lambda)(g'^2-\lambda) - g_W^2 g'^2 = 0$$

$$\lambda = 0, \lambda = g_W^2 + g'^2$$

Eigenvalues for neutral gauge bosons

$$\frac{v^2}{8}\left(W_{\mu,3}\;B_\mu\right)\begin{pmatrix} g_W^2 & -g_W g' \\ -g_W g' & g'^2 \end{pmatrix}\begin{pmatrix} W_3^\mu \\ B^\mu \end{pmatrix}$$

$$\lambda = 0, \lambda = g_W^2 + g'^2$$

Mass terms diagonalized:

$$\frac{v^2}{8}\left(A_\mu Z_\mu\right)\begin{pmatrix} 0 & 0 \\ 0 & g'^2 + g_W^2 \end{pmatrix}\begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix}$$

$$m_A = 0, m_Z = \frac{1}{2}v\sqrt{g'^2 + g_W^2}$$

Photon remains massless, but have a massive Z!

Eigenvectors for neutral gauge bosons

$$\frac{v^2}{8}\left(W_{\mu,3}\;B_\mu\right)\begin{pmatrix} g_W^2 & -g_W g' \\ -g_W g' & g'^2 \end{pmatrix}\begin{pmatrix} W_3^\mu \\ B^\mu \end{pmatrix}$$

$$\lambda = 0, \lambda = g_W^2 + g'^2$$

First eigenvector:

$$\begin{pmatrix} g_W^2 & -g_W g' \\ -g_W g' & g'^2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$p g_W^2 - g_W g' q = 0 \rightarrow p = \frac{g'}{g_W} q \rightarrow A_\mu = g' W_\mu^3 + g_W B_\mu$$

Normalize:

$$A_\mu = \frac{g' W_\mu^3 + g_W B_\mu}{\sqrt{g_W^2 + g'^2}}$$

Eigenvectors for neutral gauge bosons

$$\frac{v^2}{8}\left(W_{\mu,3}\ B_\mu\right)\begin{pmatrix} g_W^2 & -g_W g' \\ -g_W g' & g'^2 \end{pmatrix}\begin{pmatrix} W_3^\mu \\ B^\mu \end{pmatrix}$$

$$\lambda = 0, \lambda = g_W^2 + g'^2$$

Second eigenvector:

$$\begin{pmatrix} g_W^2 - g_W^2 - g'^2 & -g_W g' \\ -g_W g' & g'^2 - g_W^2 - g'^2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$-pg'^2 - g_W g' q = 0 \rightarrow p = \frac{-g_W}{g'} q \rightarrow Z_\mu = g_W W_\mu^3 - g' B_\mu$$

Normalize:

$$Z_\mu = \frac{g_W W_\mu^3 - g' B_\mu}{\sqrt{g_W^2 + g'^2}}$$

Neutral gauge bosons

$$A_\mu = \frac{g' W^3_\mu + g_W B_\mu}{\sqrt{g_W^2 + g'^2}} \qquad Z_\mu = \frac{g_W W^3_\mu - g' B_\mu}{\sqrt{g_W^2 + g'^2}}$$

$$\frac{g'}{g_W} = \tan\theta_W \rightarrow \sqrt{g_W^2 + g'^2} = \sqrt{g_W^2 + g_W^2 \tan^2\theta}$$

$$\sqrt{g_W^2 + g'^2} = \sqrt{g_W^2 (1 + \tan^2\theta)} = \frac{g_W}{\cos\theta}$$

$$A_\mu = \frac{g' W^3_\mu + g_W B_\mu}{\sqrt{g_W^2 + g'^2}} = \frac{\cos\theta (g' W^3_\mu + g_W B_\mu)}{g_W}$$

$$A_\mu = \cos\theta (\frac{g'}{g_W} W^3_\mu + B_\mu) = \cos\theta (\tan\theta W^3_\mu + B_\mu)$$

Exactly what we proposed earlier!

$$A_\mu = \sin\theta W^3_\mu + \cos\theta B_\mu$$

Neutral gauge bosons

$$A_\mu = \frac{g' W_\mu^3 + g_W B_\mu}{\sqrt{g_W^2 + g'^2}} \qquad Z_\mu = \frac{g_W W_\mu^3 - g' B_\mu}{\sqrt{g_W^2 + g'^2}}$$

$$\frac{g'}{g_W} = \tan\theta_W \rightarrow \sqrt{g_W^2 + g'^2} = \sqrt{g_W^2 + g_W^2 \tan^2\theta}$$

$$\sqrt{g_W^2 + g'^2} = \sqrt{g_W^2 (1 + \tan^2\theta)} = \frac{g_W}{\cos\theta}$$

$$Z_\mu = \frac{g_W W_\mu^3 - g' B_\mu}{\sqrt{g_W^2 + g'^2}} = \frac{\cos\theta}{g_W}(g_W W_\mu^3 - g' B_\mu)$$

$$Z_\mu = \cos\theta W_\mu^3 - \frac{g' \cos\theta}{g_W} B_\mu = \cos\theta W_\mu^3 - \cos\theta \tan\theta B_\mu$$

Exactly what we proposed earlier!

$$Z_\mu = \cos\theta W_\mu^3 - \sin\theta B_\mu$$

In addition

$$\sqrt{g_W^2 + g'^2} = \frac{g_W}{\cos\theta} \qquad m_A = 0, m_Z = \frac{1}{2} v \sqrt{g'^2 + g_W^2}$$

So that $$m_Z = \frac{v g_W}{2\cos\theta}$$ **From before** $$m_W = \frac{1}{2} g_W v$$

$$\frac{m_W}{m_Z} = \cos\theta_w$$

Plug in measured values and find $v = 246$ GeV

Similarly

Find that mass terms for the fermions do not observe the [SU(2)$_L$ x U(1)$_Y$] symmetry. But interactions with the Higgs field allow them to obtain mass in the same way!

$$\mathcal{L}_{\text{fermion mass}} = -\lambda_f [\overline{\psi}_L \phi \psi_R + \overline{\psi}_R \overline{\phi} \psi_L]$$

$$\phi(vacuum) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$\phi \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

Fermion mass term for electrons (example)

$$\mathcal{L}_{\text{fermion mass}} = -\lambda_f[\overline{\psi}_L\phi\psi_R + \overline{\psi}_R\overline{\phi}\psi_L]$$

$$\mathcal{L}_{\text{fermion mass}} = -\lambda_f\left[(\overline{\nu},\overline{e})_L\begin{pmatrix}0\\ v+h\end{pmatrix}e_R + \overline{e}_R(0,v+h)\begin{pmatrix}\nu\\ e\end{pmatrix}_L\right]$$

Fermion-specific Higgs Yukawa coupling

$$\mathcal{L}_{\text{fermion mass}} = -\lambda_f\left[\overline{e}_L(v+h)e_r + \overline{e}_R(v+h)e_L\right]$$

$$\mathcal{L}_{\text{fermion mass}} = -\lambda_f(v+h)\left[\overline{e}_Le_R + \overline{e}_Re_L\right] = -\lambda_f(v+h)(\overline{e}e)$$

Fermion mass term for electrons (example)

$$\mathcal{L}_{\text{fermion mass}} = -\lambda_f (v + h) \left[\bar{e}_L e_R + \bar{e}_R e_L \right] = -\lambda_f (v + h)(\bar{e}e)$$

$$\mathcal{L}_{\text{fermion mass}} = -\lambda_f v \bar{e}e + -\lambda_f h \bar{e}e$$

Electron mass term

Electron-Higgs boson vertex, with interaction strength proportional to Higgs mass (very small)

Note something - where did the neutrino mass term go?

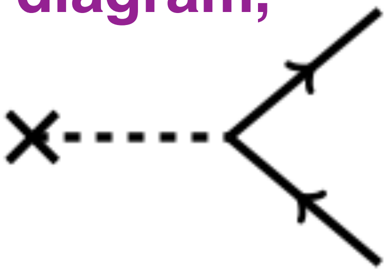
Some thoughts on Fermion masses

We have a new idea of mass in particle physics - how much the object has a well-defined helicity. This is measured by chirality

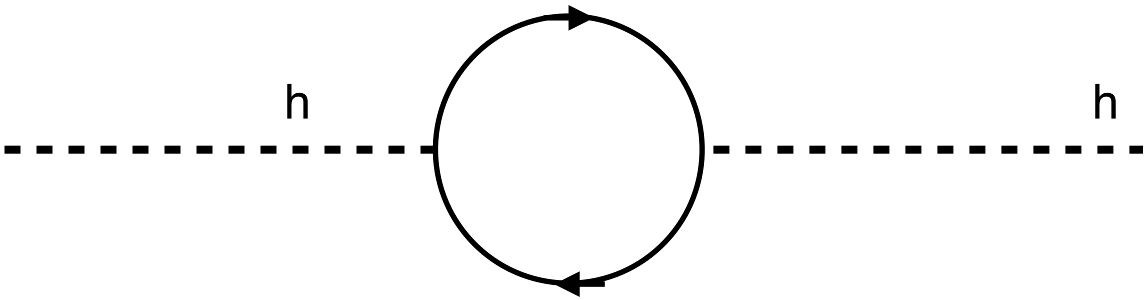
We've seen that the left-handed electron and right-handed electron are different objects (the first interacts with the W boson, the second does not)

Similarly, the left-handed positron and right-handed positron are different objects (the second interacts with the W boson, the first does not)

The Higgs field contains weak charge, and the non-zero value of the field allows a left-handed electron to convert to a right-handed electron. It's just like any other Feynman diagram, converting one object into another



The Hierarchy problem



Find that diagrams like this (with any objects running through the loop) contribute to the Higgs boson mass and self-energy. Naively, these can have energies up through the Planck scale (10^{19} GeV). But the Higgs boson mass is only 125 GeV - why?!?! Some diagrams cancel with opposite sign, but getting a ~10^2 number from integrals involving 10^{19} is: 1) lucky, 2) coincidence, 3) pointing to some hidden, deep symmetry.

One strong motivation for supersymmetry (SUSY), among other theories. In SUSY, every SM boson has a SUSY boson counterpart (bosino), and every SM fermion has a SUSY fermion counterpart (sfermion). This symmetry is obviously broken, but these then have opposite signs in cancellations and can explain the small Higgs boson mass (and thus the small electroweak mass scale).