# On to Feynman calculus



Recall that most of the particles we play with are unstable. Want to calculate their decay rate ($\Gamma$), the probability per unit time of decay

$$dN = -\Gamma N dt$$

$$N(t) = N(0)e^{-\Gamma t}$$

Number of particles that make it to N(t) and decay at N(t+dt)

$$N(t+dt) = N_0 e^{-\Gamma(t+dt)} = N_0 e^{-\Gamma t} e^{-\Gamma dt}$$

$$e^x \sim 1 + x \rightarrow e^{-\Gamma dt} \sim 1 - \Gamma dt$$

$$N(t+dt) = N_0 e^{-\Gamma t}(1 - \Gamma dt)$$

$$N(t) - N(t+dt) \sim N_0 e^{-\Gamma t} - N_0 e^{-\Gamma t}(1 - \Gamma dt) = \boxed{N_0 e^{-\Gamma t}\Gamma dt}$$

# What is the average lifetime?

Average lifetime

$$\tau = \int_0^\infty t\left(e^{-\Gamma t}\Gamma dt\right)$$

$$\tau = \Gamma \int_0^\infty t e^{-\Gamma t} dt$$

$$u = t, dv = e^{-\Gamma t}dt, du = dt, v = \frac{-1}{\Gamma}e^{-\Gamma t}$$

$$\tau = \Gamma\left[\frac{-t}{\Gamma}e^{-\Gamma t}\right]_0^\infty - \Gamma\int_0^\infty \frac{-1}{\Gamma}e^{-\Gamma t}dt$$

$$\tau = -\Gamma\left[\frac{1}{\Gamma^2}e^{-\Gamma t}\right]_0^\infty = \frac{1}{\Gamma}$$

Number of particles that make it to N(t) and decay at N(t+dt):

$$N_0 e^{-\Gamma t}\Gamma dt$$

So, fraction that decay at N(t+dt) is then

$$e^{-\Gamma t}\Gamma dt$$

So average lifetime = $1/\Gamma$

Typically define partial widths $\Gamma_i$ which are defined as the rates for specific decays. The total decay rate is the sum of the partial widths, and the lifetime is given by $\tau$:

$$\Gamma = \sum_{i=1}^{n} \Gamma_i$$

$$BR(i) = \frac{\Gamma_i}{\Gamma}$$

$$\tau = \frac{1}{\Gamma}$$

The branching ratio (BR) is the fraction of all decays to that go to a specific final state

We're also very often interested in the collision between two objects. Collisions can be:

A+B→A+B (elastic, no energy lost)
A+B→Other (inelastic, energy "lost" in the form of conversion to other particles)

Typically we refer to the cross section (σ) for a collision process. A natural way to think of a collision that relates to classical scattering theory. Units of area

Sometimes in particle physics we think of differential cross sections ($d\sigma/dX$), which refer to how often a process occurs per unit of X

X can be energy (ex: cross section for collision to produce a particle with a certain energy)
X can be number of objects (ex: how often does a collision produce a process with a certain number of jets)
X can be angle (ex: cross section where collision decay products travel in a certain direction)

# Cross section units



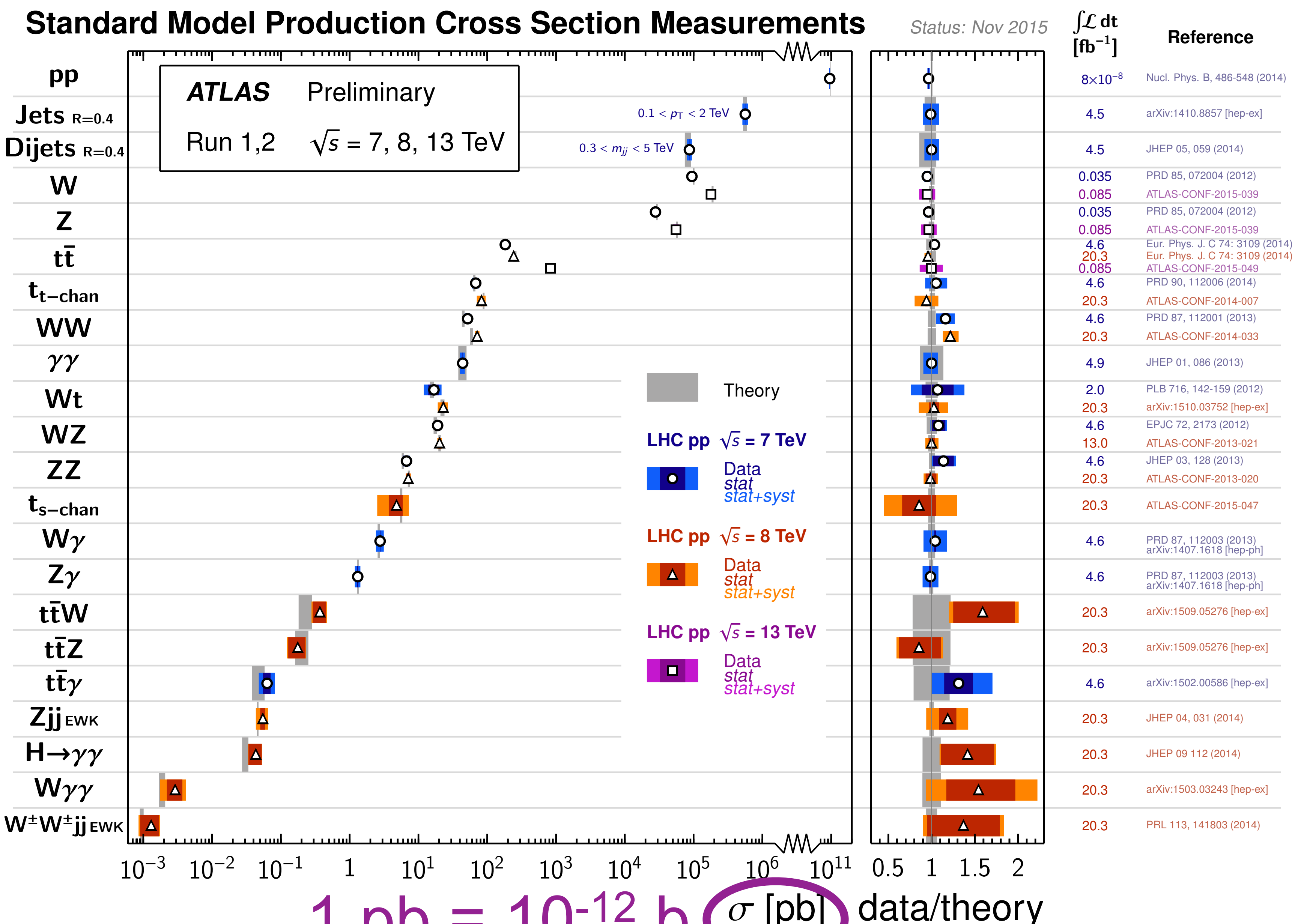
In particle physics, we typically use “barns” (b).
1 barn $=10^{-28}$ $m^2$ (typically ~area of Uranium nucleus)
If you believe Wikipedia and its references ...

## Etymology [edit]

The etymology of the unit barn is whimsical: during wartime research on the atomic bomb, American physicists at Purdue University needed a secretive unit to describe the approximate cross sectional area presented by the typical nucleus ($10^{-28}$ $m^2$) and decided on "barn." This was particularly applicable because they considered this a large target for particle accelerators that needed to have direct strikes on nuclei and the American idiom "couldn't hit the broad side of a barn"[2] refers to someone whose aim is terrible. Initially they hoped the name would obscure any reference to the study of nuclear structure; eventually, the word became a standard unit in nuclear and particle physics.[3][4]

1 barn is a huge number in particle physics!

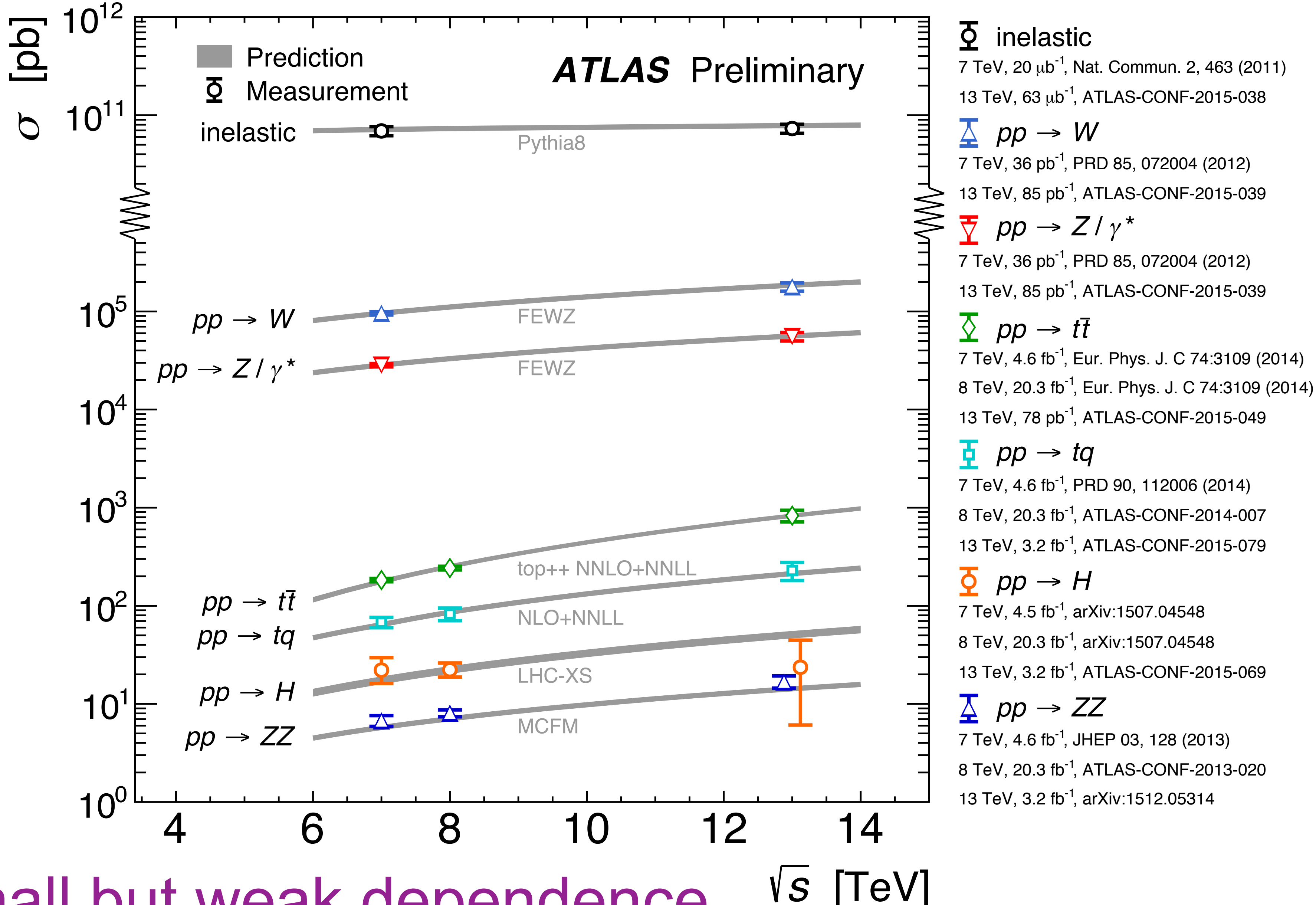
# Cross sections at the LHC




1 pb = $10^{-12}$ b

# Cross sections at the LHC






Small but weak dependence

# Limits on processes instead

## ATLAS Exotics Searches* - 95% CL Exclusion

Status: July 2015

**ATLAS** Preliminary

$\int \mathcal{L}\,dt = (4.7 - 20.3)\ \mathrm{fb}^{-1}$ $\sqrt{s} = 7, 8$ TeV

| | Model | $\ell, \gamma$ | Jets | $E_T^{miss}$ | $\int \mathcal{L}\,dt[\mathrm{fb}^{-1}]$ | Limit | | Reference |
|---|---|---|---|---|---|---|---|---|
| Extra dimensions | ADD $G_{KK} + g/q$ | – | ≥ 1 j | Yes | 20.3 | $M_D$ 5.25 TeV | $n = 2$ | 1502.01518 |
| | ADD non-resonant $\ell\ell$ | 2e, $\mu$ | – | – | 20.3 | $M_S$ 4.7 TeV | $n = 3$ HLZ | 1407.2410 |
| | ADD QBH $\to \ell q$ | 1 e, $\mu$ | 1 j | – | 20.3 | $M_{th}$ 5.2 TeV | $n = 6$ | 1311.2006 |
| | ADD QBH | – | 2 j | – | 20.3 | $M_{th}$ 5.82 TeV | $n = 6$ | 1407.1376 |
| | ADD BH high $N_{trk}$ | 2 $\mu$ (SS) | – | – | 20.3 | $M_{th}$ 4.7 TeV | $n = 6$, $M_D = 3$ TeV, non-rot BH | 1308.4075 |
| | ADD BH high $\sum p_T$ | ≥ 1 e, $\mu$ | ≥ 2 j | – | 20.3 | $M_{th}$ 5.8 TeV | $n = 6$, $M_D = 3$ TeV, non-rot BH | 1405.4254 |
| | ADD BH high multijet | – | ≥ 2 j | – | 20.3 | $M_{th}$ 5.8 TeV | $n = 6$, $M_D = 3$ TeV, non-rot BH | 1503.08988 |
| | RS1 $G_{KK} \to \ell\ell$ | 2 e, $\mu$ | – | – | 20.3 | $G_{KK}$ mass 2.68 TeV | $k/\overline{M}_{Pl} = 0.1$ | 1405.4123 |
| | RS1 $G_{KK} \to \gamma\gamma$ | 2 $\gamma$ | – | – | 20.3 | $G_{KK}$ mass 2.66 TeV | $k/\overline{M}_{Pl} = 0.1$ | 1504.05511 |
| | Bulk RS $G_{KK} \to ZZ \to qq\ell\ell$ | 2 e, $\mu$ | 2 j / 1 J | – | 20.3 | $G_{KK}$ mass 740 GeV | $k/\overline{M}_{Pl} = 1.0$ | 1409.6190 |
| | Bulk RS $G_{KK} \to WW \to qq\ell\nu$ | 1 e, $\mu$ | 2 j / 1 J | Yes | 20.3 | W′ mass 760 GeV | $k/\overline{M}_{Pl} = 1.0$ | 1503.04677 |
| | Bulk RS $G_{KK} \to HH \to b\bar{b}b\bar{b}$ | – | 4 b | – | 19.5 | $G_{KK}$ mass 500-720 GeV | $k/\overline{M}_{Pl} = 1.0$ | 1506.00285 |
| | Bulk RS $g_{KK} \to t\bar{t}$ | 1 e, $\mu$ | ≥ 1 b, ≥ 1J/2j | Yes | 20.3 | $g_{KK}$ mass 2.2 TeV | BR = 0.925 | 1505.07018 |
| | 2UED / RPP | 2 e, $\mu$ (SS) | ≥ 1 b, ≥ 1 j | Yes | 20.3 | KK mass 960 GeV | | 1504.04605 |
| Gauge bosons | SSM $Z' \to \ell\ell$ | 2 e, $\mu$ | – | – | 20.3 | Z′ mass 2.9 TeV | | 1405.4123 |
| | SSM $Z' \to \tau\tau$ | 2 $\tau$ | – | – | 19.5 | Z′ mass 2.02 TeV | | 1502.07177 |
| | SSM $W' \to \ell\nu$ | 1 e, $\mu$ | – | Yes | 20.3 | W′ mass 3.24 TeV | | 1407.7494 |
| | EGM $W' \to WZ \to \ell\nu\,\ell'\ell'$ | 3 e, $\mu$ | – | Yes | 20.3 | W′ mass 1.52 TeV | | 1406.4456 |
| | EGM $W' \to WZ \to qq\ell\ell$ | 2 e, $\mu$ | 2 j / 1 J | – | 20.3 | W′ mass 1.59 TeV | | 1409.6190 |
| | EGM $W' \to WZ \to qqqq$ | – | 2 J | – | 20.3 | W′ mass 1.3-1.5 TeV | | 1506.00962 |
| | HVT $W' \to WH \to \ell\nu bb$ | 1 e, $\mu$ | 2 b | Yes | 20.3 | W′ mass 1.47 TeV | $g_V = 1$ | 1503.08089 |
| | LRSM $W'_R \to t\bar{b}$ | 1 e, $\mu$ | 2 b, 0-1 j | Yes | 20.3 | W′ mass 1.92 TeV | | 1410.4103 |
| | LRSM $W'_R \to t\bar{b}$ | 0 e, $\mu$ | ≥ 1 b, 1 J | – | 20.3 | W′ mass 1.76 TeV | | 1408.0886 |
| CI | CI $qqqq$ | – | 2 j | – | 17.3 | $\Lambda$ 12.0 TeV | $\eta_{LL} = -1$ | 1504.00357 |
| | CI $qq\ell\ell$ | 2 e, $\mu$ | – | – | 20.3 | $\Lambda$ 21.6 TeV | $\eta_{LL} = -1$ | 1407.2410 |
| | CI $uutt$ | 2 e, $\mu$ (SS) | ≥ 1 b, ≥ 1 j | Yes | 20.3 | $\Lambda$ 4.3 TeV | $\lvert C_{LL}\rvert = 1$ | 1504.04605 |
| DM | EFT D5 operator (Dirac) | 0 e, $\mu$ | ≥ 1 j | Yes | 20.3 | $M_*$ 974 GeV | at 90% CL for $m(\chi) < 100$ GeV | 1502.01518 |
| | EFT D9 operator (Dirac) | 0 e, $\mu$ | 1 J, ≤ 1 j | Yes | 20.3 | $M_*$ 2.4 TeV | at 90% CL for $m(\chi) < 100$ GeV | 1309.4017 |
| LQ | Scalar LQ 1st gen | 2 e | ≥ 2 j | – | 20.3 | LQ mass 1.05 TeV | $\beta = 1$ | Preliminary |
| | Scalar LQ 2nd gen | 2 $\mu$ | ≥ 2 j | – | 20.3 | LQ mass 1.0 TeV | $\beta = 1$ | Preliminary |
| | Scalar LQ 3rd gen | 1 e, $\mu$ | ≥1 b, ≥3 j | Yes | 20.3 | LQ mass 640 GeV | $\beta = 0$ | Preliminary |
| Heavy quarks | VLQ $TT \to Ht + X$ | 1 e, $\mu$ | ≥ 2 b, ≥ 3 j | Yes | 20.3 | T mass 855 GeV | T in (T,B) doublet | 1505.04306 |
| | VLQ $YY \to Wb + X$ | 1 e, $\mu$ | ≥ 1 b, ≥ 3 j | Yes | 20.3 | Y mass 770 GeV | Y in (B,Y) doublet | 1505.04306 |
| | VLQ $BB \to Hb + X$ | 1 e, $\mu$ | ≥ 2 b, ≥ 3 j | Yes | 20.3 | B mass 735 GeV | isospin singlet | 1505.04306 |
| | VLQ $BB \to Zb + X$ | 2/≥3 e, $\mu$ | ≥2/≥1 b | – | 20.3 | B mass 755 GeV | B in (B,Y) doublet | 1409.5500 |
| | $T_{5/3} \to Wt$ | 1 e, $\mu$ | ≥ 1 b, ≥ 5 j | Yes | 20.3 | $T_{5/3}$ mass 840 GeV | | 1503.05425 |
| Excited fermions | Excited quark $q^* \to q\gamma$ | 1 $\gamma$ | 1 j | – | 20.3 | $q^*$ mass 3.5 TeV | only $u^*$ and $d^*$, $\Lambda = m(q^*)$ | 1309.3230 |
| | Excited quark $q^* \to qg$ | – | 2 j | – | 20.3 | $q^*$ mass 4.09 TeV | only $u^*$ and $d^*$, $\Lambda = m(q^*)$ | 1407.1376 |
| | Excited quark $b^* \to Wt$ | 1 or 2 e, $\mu$ | 1 b, 2 j or 1 j | Yes | 4.7 | $b^*$ mass 870 GeV | left-handed coupling | 1301.1583 |
| | Excited lepton $\ell^* \to \ell\gamma$ | 2 e, $\mu$, 1 $\gamma$ | – | – | 13.0 | $\ell^*$ mass 2.2 TeV | $\Lambda = 2.2$ TeV | 1308.1364 |
| | Excited lepton $\nu^* \to \ell W, \nu Z$ | 3 e, $\mu$, $\tau$ | – | – | 20.3 | $\nu^*$ mass 1.6 TeV | $\Lambda = 1.6$ TeV | 1411.2921 |
| Other | LSTC $a_T \to W\gamma$ | 1 e, $\mu$, 1 $\gamma$ | – | Yes | 20.3 | $a_T$ mass 960 GeV | | 1407.8150 |
| | LRSM Majorana $\nu$ | 2 e, $\mu$ | 2 j | – | 20.3 | $N^0$ mass 2.0 TeV | $m(W_R) = 2.4$ TeV, no mixing | 1506.06020 |
| | Higgs triplet $H^{\pm\pm} \to \ell\ell$ | 2 e, $\mu$ (SS) | – | – | 20.3 | $H^{\pm\pm}$ mass 551 GeV | DY production, BR($H_L^{\pm\pm} \to \ell\ell$)=1 | 1412.0237 |
| | Higgs triplet $H^{\pm\pm} \to \ell\tau$ | 3 e, $\mu$, $\tau$ | – | – | 20.3 | $H^{\pm\pm}$ mass 400 GeV | DY production, BR($H_L^{\pm\pm} \to \ell\tau$)=1 | 1411.2921 |
| | Monotop (non-res prod) | 1 e, $\mu$ | 1 b | Yes | 20.3 | spin-1 invisible particle mass 657 GeV | $a_{\text{non-res}} = 0.2$ | 1410.5404 |
| | Multi-charged particles | – | – | – | 20.3 | multi-charged particle mass 785 GeV | DY production, $\lvert q\rvert = 5e$ | 1504.04188 |
| | Magnetic monopoles | – | – | – | 7.0 | monopole mass 1.34 TeV | DY production, $\lvert g\rvert = 1g_D$, spin 1/2 | Preliminary |

$\sqrt{s}$ = 7 TeV $\sqrt{s}$ = 8 TeV

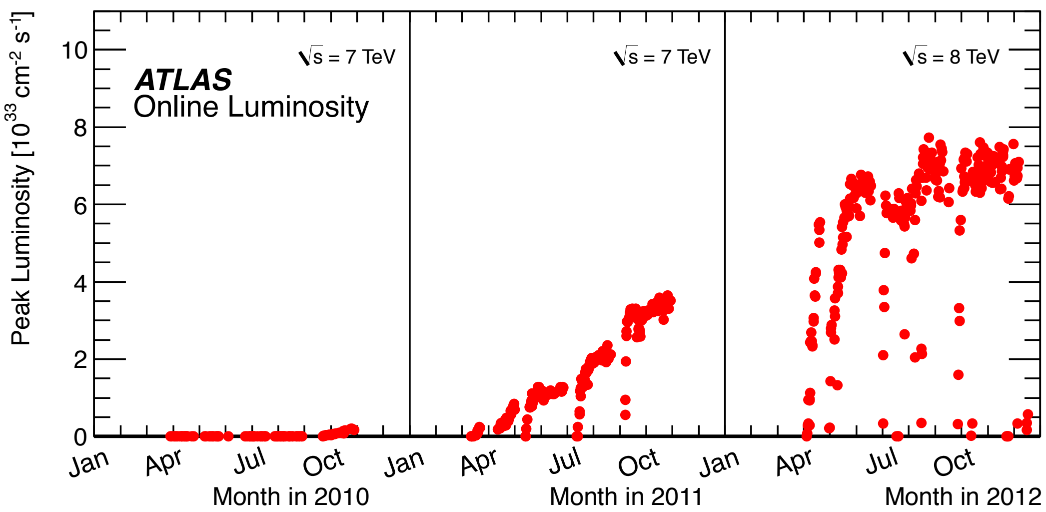
$10^{-1}$ 1 10 **Mass scale [TeV]**

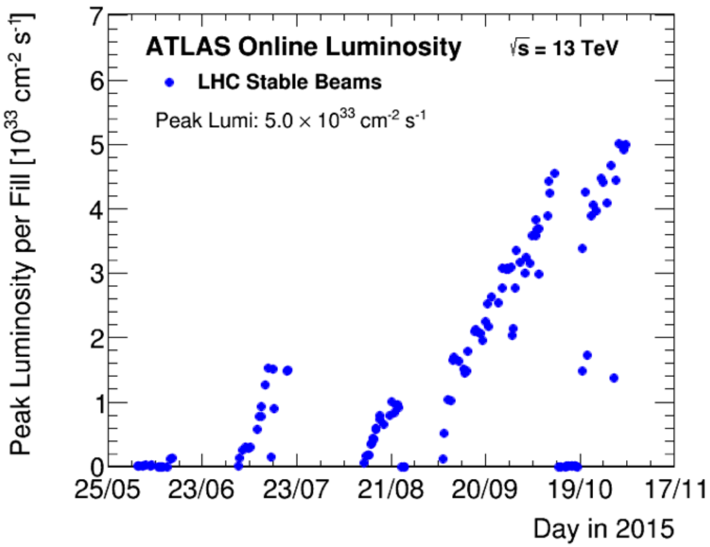
*Only a selection of the available mass limits on new states or phenomena is shown.

# Luminosity



Luminosity defines how many particles we have to collide. More specifically, the number of particles per unit time per unit area. We often think of the instantaneous luminosity, which is the luminosity at any one given time



# Luminosity

We also think of the integrated luminosity over time, which when multiplied by a cross section, tells us how many events of a certain process we expected to produce.

Example: ATLAS collected ~20 $fb^{-1}$ of data at energy of 8 TeV. The cross section for Higgs bosons at 8 TeV is ~20 pb = 20,000 fb, so ~400,000 Higgs bosons were produced in the full data set at ATLAS

The branching ratio for Higgs bosons to pairs of photons is 0.0023, so 920 Higgs bosons were produced in the diphoton final state

# Fermi's Golden Rule



Have you seen this before in Quantum Mechanics? We'll need the relativistic version of it. If not, suggest you look it up

Fermi was a smart man (hard to think of someone with more things named after him). He told us that the rate for a process to occur is equal to the square of the amplitude (aka the matrix element), multiplied by the density of states

# Aside on Dirac Delta functions

Dirac was also a smart man (maybe fewer things named after him than Fermi, but not by that much)

$$\delta(x) = \begin{matrix} 0, & x \neq 0 \\ \infty, & x = 0 \end{matrix} \qquad \delta(x-a) = \begin{matrix} 0, & x \neq a \\ \infty, & x = a \end{matrix}$$

$$f(x)\delta(x) = f(0)\delta(x)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(0)\delta(x)dx = f(0)\int_{-\infty}^{\infty} \delta(x)dx = f(0)$$

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = \int_{-\infty}^{\infty} f(a)\delta(x-a)dx = f(a)\int_{-\infty}^{\infty} \delta(x-a)dx = f(a)$$

# Aside on Dirac Delta functions



$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = ?$$

$$z = kx, dx = dz/k$$

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \frac{1}{k}\int_{-\infty}^{\infty} f(x)\delta(z)dz =$$

$$\frac{1}{k}\int_{-\infty}^{\infty} f(x)\delta(z)dz = \frac{f(0)}{k}\int_{-\infty}^{\infty} \delta(z)z = \frac{f(0)}{k}$$

Note that here, limits of integration go from -infinity to +infinity only if k is positive

# What if k is negative?



$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = ?$$

$$z = kx, dx = dz/k \text{ (k negative)}$$

$$\int_{x=-\infty}^{x=\infty} f(x)\delta(kx)dx = \frac{1}{k}\int_{z=+\infty}^{z=-\infty} f(x)\delta(z)dz = -\frac{1}{k}\int_{z=-\infty}^{z=+\infty} f(x)\delta(z)dz$$

$$-\frac{1}{k}\int_{-\infty}^{\infty} f(x)\delta(z)dz = \frac{f(0)}{|k|}\int_{-\infty}^{\infty} \delta(z)z = \frac{f(0)}{|k|}$$

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

# What about any arbitrary function?

Any arbitrary function with potentially any number of zeros.

$$\delta(g(x)), g(x_i) = 0, i = 1, 2, 3...$$

$$g(x) = g(x_i) + (x - x_i)g'(x_i) + \frac{1}{2}(x - x_i)^2 g''(x_i) + ....$$

if $x_i$ something other than zero, this just shifts the delta function

$g'(x_i)$ is "k" in the previous slide

# What about any arbitrary function?



For one zero:

$$\delta(g(x)) = \frac{1}{|g'(x_i)|}\delta(x - x_i)$$

In total:

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x - x_i)$$

# Heaviside step function

$$\theta(x) = 0(x < 0)$$
$$\theta(x) = 1(x > 0)$$
$$\delta(x) = \frac{d\theta}{dx}$$

What does he look like to you?

# Maybe it’s just me?



# Let's do

Griffiths problems A1 and A3 together

Fermi's Golden Rule: nothing to do with how you should treat others (that's a different Golden Rule). It tells us that the rate for a process (a given collision or decay) is a product of the square of the matrix element (dynamics specific to the theory of the forces at play) and the phase space (recall that things like to happen the more phase space there is for it to happen)

For example, let's begin by considering $1 \rightarrow 2+3+4+....n$

In other words, object 1 decaying to objects 2, 3, 4... (n-1 total particles)

Up to some overall normalization, consider the phase space of the $j^{th}$ object as $d^4p_j = d(p^0_j)d^3(\mathbf{p}_j)$

Hopefully that makes some intuitive sense as a definition of phase space?

But of course, the $j^{th}$ object can't just have any arbitrary value of energy and momentum

# Constraints on the phase space



**$1 \rightarrow 2+3+4+....n$**

The decay products have a definite mass. In other words, $p_j^2 = m_j^2$. Can enforce this in an integral with a delta function, $\delta(p_j^2 - m_j^2)$

Don't allow negative energy states of decay productions, so $p_j^0 > 0$. Can enforce this with Heaviside function, $\theta( p_j^0)$

Conserve energy and momentum. Can enforce this with $\delta(p_1 - p_2 - p_3 - ... p_n)$

# Golden rule for decays

**1→2+3+4+....n**

Normalization

Matrix element squared (to be worked on later)

Momentum/energy conservation

Decay rate

$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 ... - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0)\frac{d^4p_j}{(2\pi)^4}$$

On-shell final products

E>0

Phase space

# Where did that come from, though?

Recall from basic QM: What is the transition rate from state |i> to state |f>, given some interaction Hamiltonian ($\hat{H}'$)?

Given by $\Gamma_{fi}$:

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_i)$$

Matrix element ("transition")

$$|T_{fi}| = < f|\hat{H}'|i > + \sum_{j \neq i} \frac{< f|\hat{H}'|j >< j|\hat{H}'|i >}{E_i - E_j} + \ldots$$

$$\rho(E_i) = \left| \frac{dn}{dE} \right|_{E_i}$$

Density of states (phase space)

# We can rewrite phase space as:

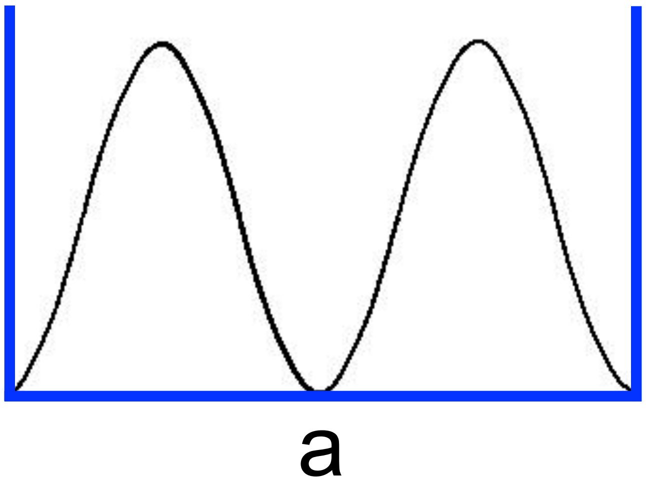
$$\rho(E_i) = \left|\frac{dn}{dE}\right|_{E_i}$$

$$\left|\frac{dn}{dE}\right|_{E_i} = \int \frac{dn}{dE}\delta(E_i - E)dE$$

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)dn$$

# 1d Boundary conditions in non-relativistic QM






Recall in a one dimensional box that boundary conditions force quantization of momentum

$$\psi(x+a) = \psi(x) \rightarrow p_x = n_x \frac{2\pi}{a}$$

# 3d Boundary conditions in non-relativistic QM



$$(p_x, p_y, p_z) = (n_x, n_y, n_z)\frac{2\pi}{a}$$

Similarly, in 3 dimensions (if in a 3D cube of length a on each side, total volume V)

What is the volume of a state in momentum space?

$$d^3\mathbf{p} = dp_x dp_y dp_z = \left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$

# Normalizing things



Common to normalize to a particle per unit volume, so that the number of states for ith particle is

$$dn_i = \frac{d^3\mathbf{p}_i}{(2\pi)^3}$$

And then the total number of states is

$$dn = \prod_{i=1}^{N-1} dn_i = \prod_{i=1}^{N-1} \frac{d^3\mathbf{p}_i}{(2\pi)^3}$$

Note that we “lost” the lost $dn_i$ because it is not independent (fixed, due to momentum conservation)

# Rewriting this a bit more nicely



$$dn = \prod_{i=1}^{N-1} \frac{d^3\mathbf{p}_i}{(2\pi)^3} = \prod_{i=1}^{N-1} \frac{d^3\mathbf{p}_i}{(2\pi)^3} \delta^3\left(\mathbf{p}_a - \sum_{i=1}^{N} \mathbf{p}_i\right) d^3\mathbf{p}_N$$

We've added in the last missing $d^3\mathbf{p}_N$ by including a delta function, which forces momentum conservation (particle a is the one decaying)

$$dn = (2\pi)^3 \prod_{i=1}^{N} \frac{d^3\mathbf{p}_i}{(2\pi)^3} \delta^3\left(\mathbf{p}_a - \sum_{i=1}^{N} \mathbf{p}_i\right)$$

# Putting it together



For particle a decaying
to particles 1 and 2...

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_a - E_1 - E_2) dn$$

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_a - E_1 - E_2) \frac{d^3\mathbf{p_1}}{(2\pi)^3} \frac{d^3\mathbf{p_2}}{(2\pi)^3} \delta^3 \left(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2\right)$$

# What about relativistic mechanics?



Recall that we normalized our transition matrix element to one particle per unit volume.

What happens in another reference frame?
Perpendicular to direction of motion, nothing.

Parallel to direction of motion, we get a Lorentz contraction of $1/\gamma = m/E$, therefore to be Lorentz invariant our normalization must be proportional to $1/E$ (we choose $1/2E$)

# Why 2E and not E?



Our new Lorentz-invariant phase space

$$\prod_{i=1}^{N} \frac{d^3\mathbf{p_i}}{2E_i(2\pi)^3}$$

$$\int \delta(E_i^2 - \mathbf{p}_i^2 - m_i^2)\, dE_i = \frac{1}{2E_i}$$

Energy-momentum relation delta function. Is this clear?

# So we can rewrite the phase space again



$$\prod_{i=1}^{N} \frac{d^3\mathbf{p_i}}{2E_i(2\pi)^3} = \prod_{i=1}^{N} \frac{d^3\mathbf{p_i}}{(2\pi)^3} \int \delta(E_i^2 - \mathbf{p}_i^2 - m_i^2)dE_i$$

$$\prod_{i=1}^{N} \frac{d^3\mathbf{p_i}}{2E_i(2\pi)^3} = \int \prod_{i=1}^{N} \frac{d^3\mathbf{p}_i dE_i}{(2\pi)^3} \delta(E_i^2 - \mathbf{p}_i^2 - m_i^2)$$

Now let's use 4-vector notation

$$\text{phase space} = \int \prod_{i=1}^{N} \frac{d^4 p_i}{(2\pi)^3} \delta(p_i^2 - m_i^2)$$

# More on phase space



Let's take another look at our Lorentz-invariant phase space and check that it really is Lorentz-invariant

$$\frac{d^3\mathbf{p}}{(2\pi)^3 2E}$$

Let's look at a transformation along the z axis. What is the phase space?

$$d^3\mathbf{p}' = dp'_x dp'_y dp'_z = dp_x dp_y \frac{dp_z'}{dp_z} dp_z = d^3\mathbf{p} \frac{dp_z'}{dp_z}$$

# More on phase space



$$d^3\mathbf{p}' = dp'_x dp'_y dp'_z = dp_x dp_y \frac{dp'_z}{dp_z} dp_z = d^3\mathbf{p} \frac{dp'_z}{dp_z}$$

$$p'_z = \gamma(p_z - \beta E), E' = \gamma(E - \beta p_z), E^2 = p_x^2 + p_y^2 + p_z^2 + m^2$$

$$\frac{dp'_z}{dp_z} = \gamma(1 - \beta \frac{\partial E}{\partial p_z})$$

$$\frac{\partial E}{\partial p_z} = \frac{1}{2} \cdot 2p_z \left(\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}\right)^{-1} = \frac{p_z}{E}$$

$$\frac{dp'_z}{dp_z} = \gamma(1 - \beta \frac{p_z}{E}) = \frac{\gamma}{E}(E - \beta p_z) = \frac{E'}{E}$$

So ...

$$d^3\mathbf{p}' = d^3\mathbf{p} \frac{E'}{E} \rightarrow \frac{d^3\mathbf{p}'}{E'} = \frac{d^3\mathbf{p}}{E}$$

# Aside on your final presentation



Many of you already proposed topics to me over email already - thanks!

Note 0: It's not first-come/first-served, but instead we will flip coins or play rock-paper-scissors for who gets which topic. There's one especially popular paper out there, as you might guess

Note 1: If you dropped by my office to discuss the paper, that does not count as fulfilling your homework assignment

Note 2: This really counts as a homework assignment. So don't miss the deadline! If you do, you get points off - and I get to pick a topic for you :)

# Aside on your final presentation



Note 3: Only one topic per person, so you might want to have 1-2 backups in mind. I'll let you know in class on Wednesday where we have duplicates (and alternates due then next Monday)

Note 4: You are not covering an "experiment", but rather a single analysis/limit/measurement. So I want you to have the paper you will be reading in mind before Wednesday. If there is no physics result, then this does not count

# Scheduling the final presentation



There are ~9 of you, so if we have two presentations per class @ 20-25 + 10-15 minutes each, that would be 4.5 classes. We'll aim to fit them into 4 classes (I expect 20 minute presentations, but perhaps some discussion can go short, or one night can go a few minutes long if all are OK with that).

That would mean using the last two weeks of classes, April 25, April 27, May 2 and May 4. The sessions devoted to finals (May 9 and May 11) are for overflow. I expect you all to show up and participate, to listen, and to ask good questions

# Order for presentations



You should be ready on the first day of presentations! That being said, we need to pick an order for them. So time for some random numbers (I will write them down so I know when you are presenting). But if someone doesn't show you may be called to go, so... be ready

# OK, back to the golden rule... for decays

**1→2+3+4+....n**

$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0)\frac{d^4 p_j}{(2\pi)^4}$$

Start with this delta function. We know that $p_j^2 = (p^0)^2 - \mathbf{p}^2$

$$\delta(p_j^2 - m_j^2) = \delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m^2\right)$$

$$\rightarrow (p^0)_j = \pm\sqrt{\mathbf{p_j}^2 + m^2}$$

# Golden rule for decays



$$\delta(p_j^2 - m_j^2) = \delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m^2\right)$$

$$\rightarrow (p^0)_j = \pm\sqrt{\mathbf{p_j}^2 + m^2}$$

$$\frac{d}{dp_j^0}\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = 2p_j^0$$

$$\rightarrow \delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = \frac{1}{2p_j^0}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m_j^2}\right) + \delta\left(p_j^0 + \sqrt{\mathbf{p_j}^2 + m_j^2}\right)\right]$$

# Golden rule for decays



$$\delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = \frac{1}{2p_j^0}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m_j^2}\right) + \delta\left(p_j^0 + \sqrt{\mathbf{p_j}^2 + m_j^2}\right)\right]$$

$$\theta(p_j^0)\delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = \frac{\theta(p_j^0)}{2p_j^0}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m_j^2}\right) + \delta\left(p_j^0 + \sqrt{\mathbf{p_j}^2 + m_j^2}\right)\right]$$

Heaviside forces $p_j^0$ to always be greater than 0

$$\theta(p_j^0)\delta\left((p^0)_j^2 - \mathbf{p_j}^2 - m_j^2\right) = \frac{1}{2p_j^0}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m_j^2}\right)\right]$$

# Golden rule for decays

**1→2+3+4+....n**

$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0)\frac{d^4 p_j}{(2\pi)^4}$$

Delta picks out specific value for $p_j^0$, no need to integrate

So...

$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi \frac{\delta\left(p_j^0 - \sqrt{\mathbf{p_j}^2 + m^2}\right)}{2p_j^0}\frac{d^4 p_j}{(2\pi)^4}$$

# Golden rule for decays



$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 (p_1 - p_2 - p_3 \ldots - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi \frac{\delta \left( p_j^0 - \sqrt{\mathbf{p_j}^2 + m^2} \right)}{2p_j^0} \frac{d^4 p_j}{(2\pi)^4}$$

Make delta function substitution

$$\Gamma = \frac{1}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 (p_1 - p_2 - p_3 \ldots - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^4}$$

# Golden rule for decays



$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times$$

$$\prod_{j=2}^{n} 2\pi \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^4}$$

And rearrange a bit

$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times$$

$$\prod_{j=2}^{n} \frac{1}{2\sqrt{\mathbf{p_j}^2 + m^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

# Let's assume only two particles in final state



1→2+3

$$\Gamma = \frac{1}{2m_1}\int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 ... - p_n)\times$$

$$\frac{1}{2\sqrt{\mathbf{p_2}^2 + m_2^2}}\frac{d^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p_3}^2 + m_3^2}}\frac{d^3\mathbf{p}_3}{(2\pi)^3}$$

Rearrange...

$$\Gamma = \frac{1}{32\pi^2 m_1}\int |\mathcal{M}|^2 \delta^4(p_1 - p_2 - p_3 ... - p_n)\times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}}\frac{1}{\sqrt{\mathbf{p_3}^2 + m_3^2}} d^3\mathbf{p}_2 d^3\mathbf{p}_3$$

# More delta functions (a common theme)



$$\delta^4(p_1 - p_2 - p_3) = \delta(p_1^0 - p_2^0 - p_3^0)\delta^3(\mathbf{p_1} - \mathbf{p_2} - \mathbf{p_3})$$

Let's choose reference frame where $p_1$ is at rest, so $\mathbf{p_1}=\mathbf{0}$ and $p_1 = (m_1, \mathbf{0})$

$$\delta^4(p_1 - p_2 - p_3) = \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_3}^2 + m_3^2})\delta^3(\mathbf{p_2} + \mathbf{p_3})$$

$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_3}^2 + m_3^2})\delta^3(\mathbf{p_2} + \mathbf{p_3}) \times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}} \frac{1}{\sqrt{\mathbf{p_3}^2 + m_3^2}} d^3\mathbf{p}_2 d^3\mathbf{p}_3$$

# Continuing to work this out

**$\mathbf{p_3}$=-$\mathbf{p_2}$** (had to be, due to conservation of momentum if **$\mathbf{p_1}$**=0)

$$\Gamma = \frac{1}{32\pi^2 m_1}\int |\mathcal{M}|^2 \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_3}^2 + m_3^2})\delta^3(\mathbf{p_2} + \mathbf{p_3})\times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}}\frac{1}{\sqrt{\mathbf{p_3}^2 + m_3^2}} d^3\mathbf{p}_2 d^3\mathbf{p}_3$$

$$\Gamma = \frac{1}{32\pi^2 m_1}\int |\mathcal{M}|^2 \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2})\times$$

$$\frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}}\frac{1}{\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p}_2$$

# Almost there



$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \delta(p_1^0 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2}) \times \frac{1}{\sqrt{\mathbf{p_2}^2 + m_2^2}} \frac{1}{\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p}_2$$

Let's rearrange again

$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta(m_1 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2})}{\sqrt{\mathbf{p_2}^2 + m_2^2}\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p}_2$$

Stepping back, this is impressive but not surprising. Only have integral over momentum of one particle left for phase space (why not surprising?)!

# Almost there



$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta(m_1 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2})}{\sqrt{\mathbf{p_2}^2 + m_2^2}\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p}_2$$

Let's go to spherical coordinates, $\mathbf{p_2} = (r,\theta,\phi)$ and $d^3\mathbf{p_2} = r^2 \sin\theta \, dr \, d\theta \, d\phi$

Matrix element squared cannot be a function of anything but $|\mathbf{p_2}|$ anymore since object 1 was at rest and $\mathbf{p_3}$ is just $-\mathbf{p_2}$ so angular integrals can be easily done

$$\int \sin\theta d\theta d\phi = 4\pi$$

# Almost there



$$\Gamma = \frac{1}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta(m_1 - \sqrt{\mathbf{p_2}^2 + m_2^2} - \sqrt{\mathbf{p_2}^2 + m_3^2})}{\sqrt{\mathbf{p_2}^2 + m_2^2}\sqrt{\mathbf{p_2}^2 + m_3^2}} d^3\mathbf{p}_2$$

Do substitution and angular integrals

$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \frac{\delta(m_1 - \sqrt{r^2 + m_2^2} - \sqrt{r^2 + m_3^2})}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}} r^2 dr$$

Let's make another substitution...

$$u = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

# Almost there



$$u = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \frac{\delta(m_1 - \sqrt{r^2 + m_2^2} - \sqrt{r^2 + m_3^2})}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}} r^2 dr$$

$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \frac{\delta(m_1 - u)}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}} r^2 dr$$

$$\frac{du}{dr} = \frac{r}{\sqrt{r^2 + m_2^2}} + \frac{r}{\sqrt{r^2 + m_3^2}}$$

$$\frac{du}{dr} = \frac{r\left(\sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}\right)}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}}$$

$$\frac{du}{dr} = \frac{ru}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}}$$

# Some more substitutions



$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \frac{\delta(m_1 - u)}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}} r^2 dr$$

$$\frac{du}{dr} = \frac{ru}{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}}$$

$$dr = du \frac{\sqrt{r^2 + m_2^2}\sqrt{r^2 + m_3^2}}{ru}$$

$$\Gamma = \frac{1}{8\pi m_1} \int |\mathcal{M}(r)|^2 \delta(m_1 - u) \frac{r}{u} du$$

$u=m_1$

# More on that delta function



$$\Gamma = \frac{1}{8\pi m_1}\int |\mathcal{M}(r)|^2 \delta(m_1 - u)\frac{r}{u}du$$

u=$m_1$

What does SR tell us?

$$u = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

$$m_1 = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$$

$$m_1^2 = r^2 + m_2^2 + r^2 + m_3^2 + 2\sqrt{(r^2 + m_3^2)(r^2 + m_2^2)}$$

$$m_1^2 - 2r^2 - m_2^2 - m_3^2 = 2\sqrt{(r^2 + m_3^2)(r^2 + m_2^2)}$$

$$m_1^4 + 4r^4 + m_2^4 + m_3^4 - 4r^2m_1^2 + 4r^2m_2^2 + 4r^2m_3^2 - 2m_1^2m_2^2 - 2m_1^2m_3^2 + 2m_2^2m_3^2 = 4r^4 + 4m_2^2m_3^2 + 4r^2m_2^2 + 4r^2m_3^2$$

$$-4r^2m_1^2 + m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2 = 0$$

$$r = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2}$$

# Finalizing it



$$r = |\mathbf{p_2}| = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2}$$

$$\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2$$

Note that matrix element factorizes (not always possible, but a pretty nice result!)

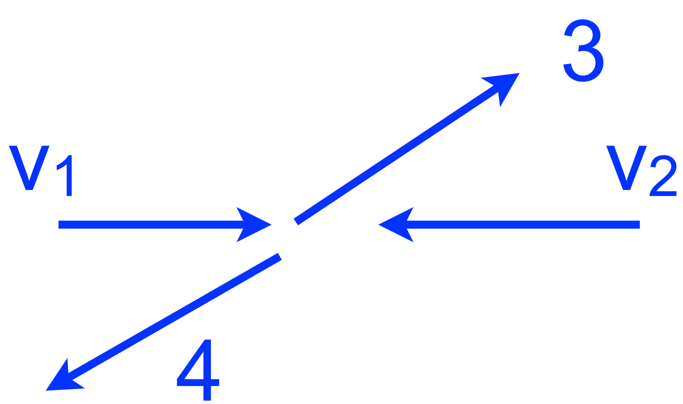
# What about scattering?

**1+2→3+4**

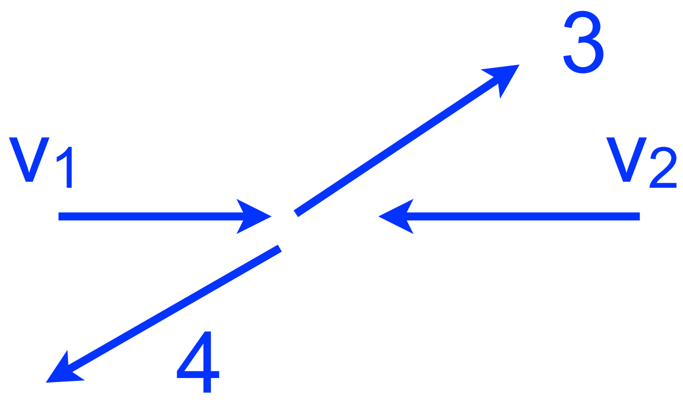
$$\Gamma = \frac{1}{4E_1E_2}\int |\mathcal{M}|^2(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times$$

$$\prod_{j=3}^{4} 2\pi\delta(p_j^2-m_j^2)\theta(p_j^0)\frac{d^4p_j}{(2\pi)^4}$$

This is the rate, though not quite what we're looking for. We are interested in the **cross section (σ)**



# Scattering




If we assume one particle per unit volume, then rate = $(v_1+v_2)\sigma$, ie the faster the set of objects 1 and 2 pass through each other, the larger the rate

Be careful ($v_1$ and $v_2$ minus signs)

$$\Gamma = (v_1 + v_2)\sigma$$

$$\sigma = \frac{\Gamma}{v_1 + v_2}$$

# So the cross section is ...

$$\sigma = \frac{1}{4E_1E_2}\frac{1}{v_1+v_2}\int |\mathcal{M}|^2(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times$$

$$\prod_{j=3}^{4} 2\pi\delta(p_j^2-m_j^2)\theta(p_j^0)\frac{d^4p_j}{(2\pi)^4}$$

Looks a bit odd to have velocity there!
Can that at all be Lorentz invariant?

$$F = 4E_1E_2(v_1+v_2) = 4E_1E_2\left(\frac{|\mathbf{p_1}|}{E_1}+\frac{|\mathbf{p_2}|}{E_2}\right) = 4\left(E_2|\mathbf{p_1}|+E_1|\mathbf{p_2}|\right)$$

$$F^2 = 16\left(E_2^2|\mathbf{p_1}|^2+E_1^2|\mathbf{p_2}|^2+2E_1E_2|\mathbf{p_1}||\mathbf{p_2}|\right)$$

# So the cross section is ...

In case where particles 1 and 2 are collinear

$$(p_1 \cdot p_2) = E_1 E_2 + \mathbf{p_1 p_2}$$

Remember the extra minus sign here

$$(p_1 \cdot p_2)^2 = E_1^2 E_2^2 + \mathbf{p_1}^2 \mathbf{p_2}^2 + 2E_1 E_2 \mathbf{p_1 p_2}$$

$$F^2 = 16\left(E_2^2|\mathbf{p_1}|^2 + E_1^2|\mathbf{p_2}|^2 + 2E_1E_2|\mathbf{p_1}||\mathbf{p_2}|\right)$$

$$F^2 = 16\left(E_2^2|\mathbf{p_1}|^2 + E_1^2|\mathbf{p_2}|^2 + (p_1 \cdot p_2)^2 - E_1^2E_2^2 - \mathbf{p_1}^2\mathbf{p_2}^2\right)$$

$$F^2 = 16\left[(p_1 \cdot p_2)^2 - (E_1^2 - \mathbf{p_1}^2)(E_2^2 - \mathbf{p_2}^2)\right]$$

$$F^2 = 16\left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2\right]$$

Lorentz invariant!

$$\boxed{F = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}$$

# So the cross section is ...

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4}$$

As before, 1d delta function is easy (and rearranges some $2\pi$'s). Heaviside enforces only one solution

$$p_j^0 = E_j = \sqrt{\mathbf{p}_j^2 + m_j^2}$$

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^3}$$

# Moving along with cross sections



$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

$$p_j^0 = E_j = \sqrt{\mathbf{p}_j^2 + m_j^2}$$

Let's put back the earlier form for F

$$\sigma = \frac{1}{4E_1 E_2} \frac{1}{v_1 + v_2} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

# Moving along with cross sections



$$F = \frac{1}{4E_1E_2}\frac{1}{v_1 + v_2} = \frac{1}{4E_1E_2}\frac{1}{p_1/E_1 + p_2/E_2}$$

$$F = \frac{1}{4}\frac{1}{E_2p_1 + E_1p_2}$$

In Center of Mass frame, $|\mathbf{p}_2| = |\mathbf{p}_1|$

$$F = \frac{1}{4|p_1|}\frac{1}{E_1 + E_2}$$

$$\sigma = \frac{1}{4|p_1|(E_1 + E_2)}\int |\mathcal{M}|^2(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)\times$$

$$\prod_{j=3}^{4}\frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}}\frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

# Let's combine some factors



$$\sigma = \frac{1}{64\pi^2|p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta^4(p_1 + p_2 - p_3 - p_4) \times$$

$$\prod_{j=3}^{4} \frac{1}{\sqrt{\mathbf{p}_j^2 + m_j^2}} d^3\mathbf{p}_j$$

$$p_j^0 = E_j = \sqrt{\mathbf{p}_j^2 + m_j^2}$$

Let's split up the 4d delta function

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$

# Delta functions in action



$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$

$$p_j^0 = E_j = \sqrt{\mathbf{p}_j^2 + m_j^2}$$

In CoM frame, we know that **$\mathbf{p}_1$+$\mathbf{p}_2$** = 0

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_3 + \mathbf{p}_4)$$

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1 + E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_3 + \mathbf{p}_4) \times \frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2}} \frac{1}{\sqrt{\mathbf{p}_4^2 + m_4^2}} d^3\mathbf{p}_3 d^3\mathbf{p}_4$$

# Delta functions in action



$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\mathbf{p}_3+\mathbf{p}_4)\times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}}\frac{1}{\sqrt{\mathbf{p}_4^2+m_4^2}}d^3\mathbf{p}_3 d^3\mathbf{p}_4$$

The 3D delta function enforces $\mathbf{p}_3 = -\mathbf{p}_4$

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2 \delta(E_1+E_2-E_3-E_4)\times$$

$$\frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}}\frac{1}{\sqrt{\mathbf{p}_3^2+m_4^2}}d^3\mathbf{p}_3$$

# Delta functions in action



$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2\delta(E_1+E_2-E_3-E_4)\times$$
$$\frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}}\frac{1}{\sqrt{\mathbf{p}_3^2+m_4^2}}d^3\mathbf{p}_3$$

Recall that we had delta functions on $E_3$ and $E_4$ (the mass relations)

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2\delta(E_1+E_2-\sqrt{\mathbf{p}_3^2+m_3^2}-\sqrt{\mathbf{p}_4^2+m_4^2})\times$$
$$\frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}}\frac{1}{\sqrt{\mathbf{p}_3^2+m_4^2}}d^3\mathbf{p}_3$$

# Delta functions in action



$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2\delta(E_1+E_2-\sqrt{\mathbf{p}_3^2+m_3^2}-\sqrt{\mathbf{p}_4^2+m_4^2})\times$$
$$\frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}}\frac{1}{\sqrt{\mathbf{p}_3^2+m_4^2}}d^3\mathbf{p}_3$$

The 3D delta function enforced $\mathbf{p}_3 = -\mathbf{p}_4$

$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2\delta(E_1+E_2-\sqrt{\mathbf{p}_3^2+m_3^2}-\sqrt{\mathbf{p}_3^2+m_4^2})\times$$
$$\frac{1}{\sqrt{\mathbf{p}_3^2+m_3^2}}\frac{1}{\sqrt{\mathbf{p}_3^2+m_4^2}}d^3\mathbf{p}_3$$

# As before, we change coordinate systems



$$\sigma = \frac{1}{64\pi^2|p_1|(E_1+E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - \sqrt{\mathbf{p}_3^2 + m_3^2} - \sqrt{\mathbf{p}_3^2 + m_4^2}) \times \frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2}} \frac{1}{\sqrt{\mathbf{p}_3^2 + m_4^2}} d^3\mathbf{p}_3$$

$$d^3\mathbf{p}_3 = r^2 dr d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2|p_1|(E_1+E_2)} \int |\mathcal{M}|^2 \delta(E_1 + E_2 - \sqrt{r^2 + m_3^2} - \sqrt{r^2 + m_4^2}) \times \frac{1}{\sqrt{r^2 + m_3^2}} \frac{1}{\sqrt{r^2 + m_4^2}} r^2 dr$$

# Evaluating the integral



$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2|p_1|(E_1+E_2)}\int |\mathcal{M}|^2\delta(E_1+E_2-\sqrt{r^2+m_3^2}-\sqrt{r^2+m_4^2})\times \frac{1}{\sqrt{r^2+m_3^2}}\frac{1}{\sqrt{r^2+m_4^2}}r^2dr$$

Same exact form as last ugly integral, so nothing new here

# Differential cross section



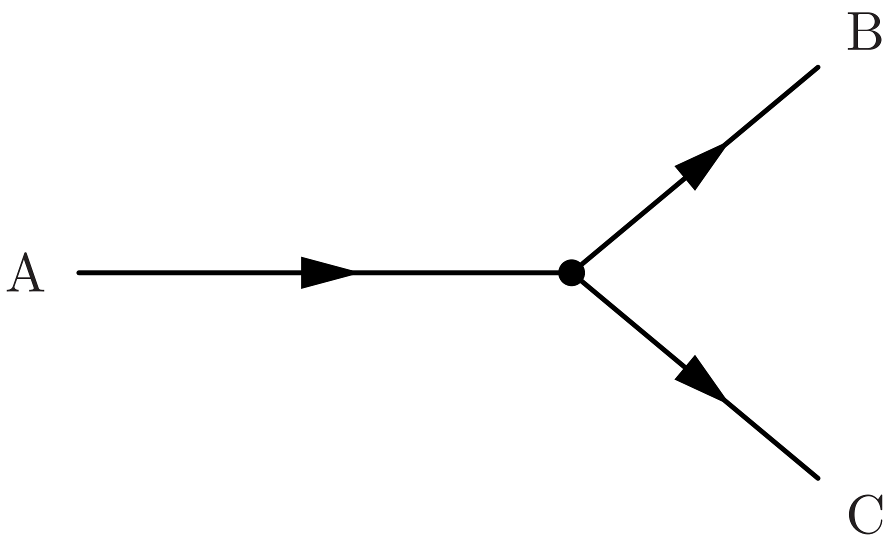
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|M|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

# On to (toy) theories of particle physics!

I think this is where Griffiths does a really nice job. We won't dive into QED, but will instead start with a simpler theory.

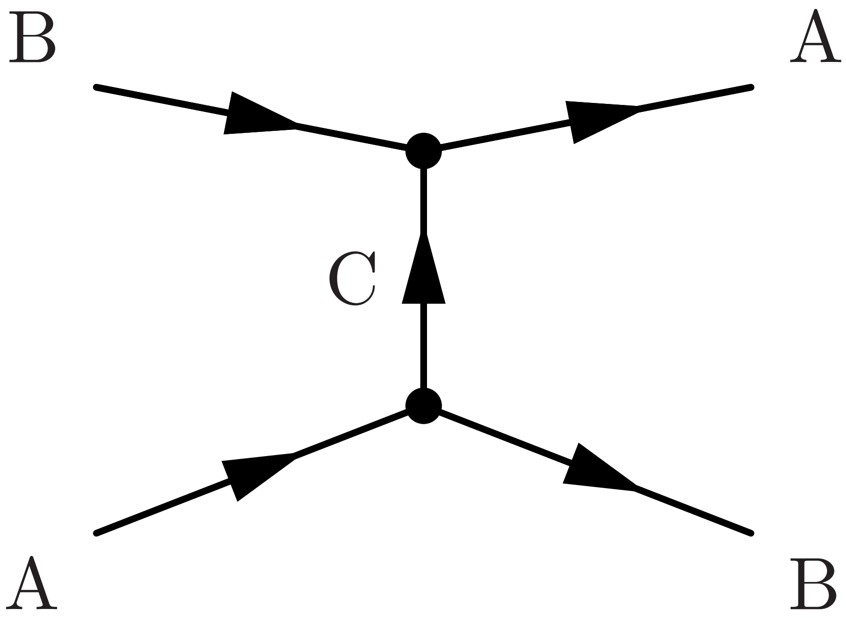
Feynman's calculus/his rules tell us how to calculate the matrix elements (why? we won't be diving into QFT, so for now please just accept them, as unappealing as that might be)

# Our toy theory



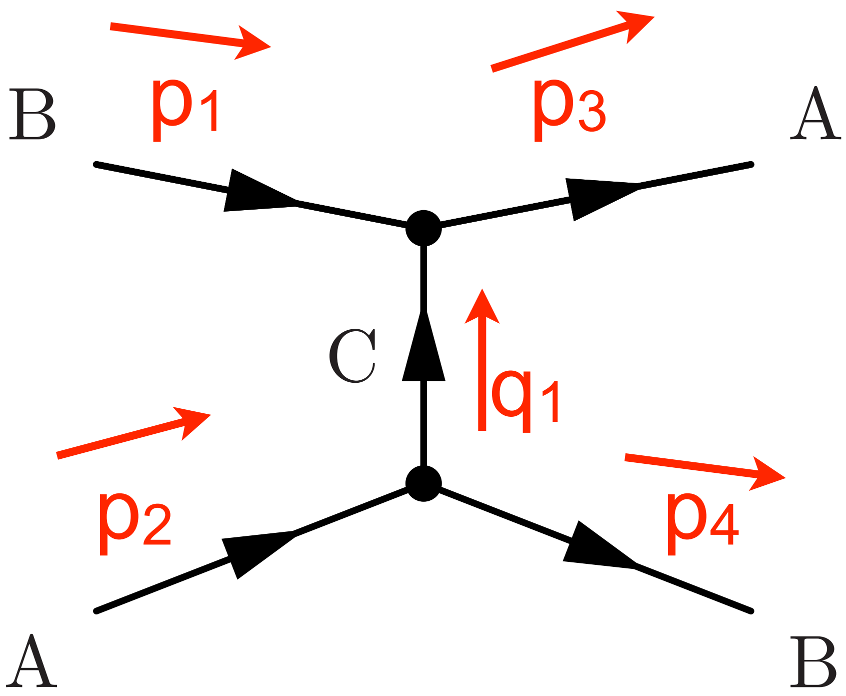

Our toy theory has 3 types of spin-0 particles, A, B and C. Let’s assume that $m_A > m_B + m_C$. Here, A is incoming, and B and C are outgoing. This is a decay vertex

# Our toy theory




Here we have scattering A+B→A+B (one example diagram)

# Feynman rules (1)




p vs q is pure convention!

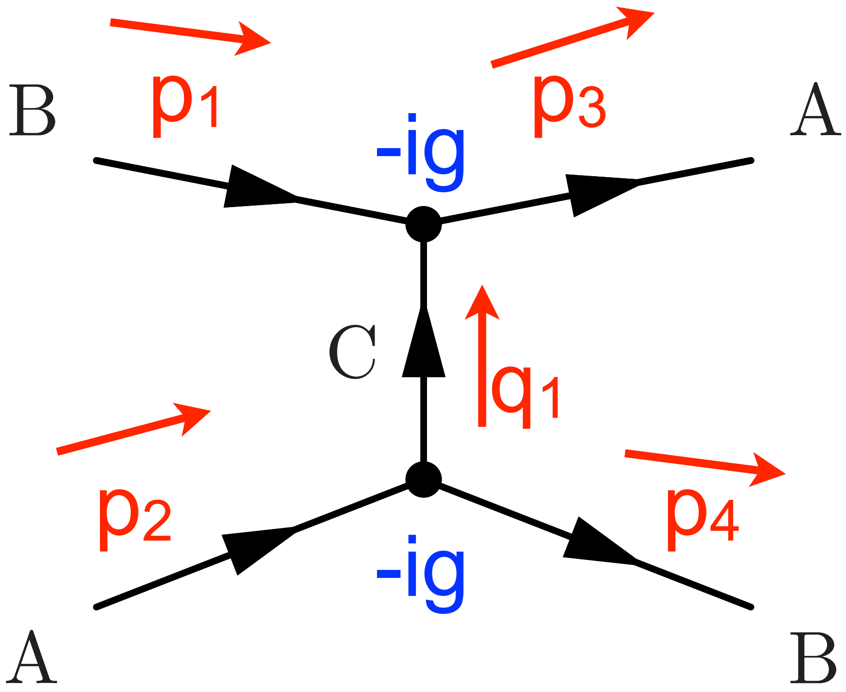
Label all incoming and outgoing lines with $p_1$, $p_2$, ... $p_n$

Internal lines can go either way

Use arrows to keep track of what is going in and out (here this looks trivial, but can be more tricky with anti-particles)

# Feynman rules (2)

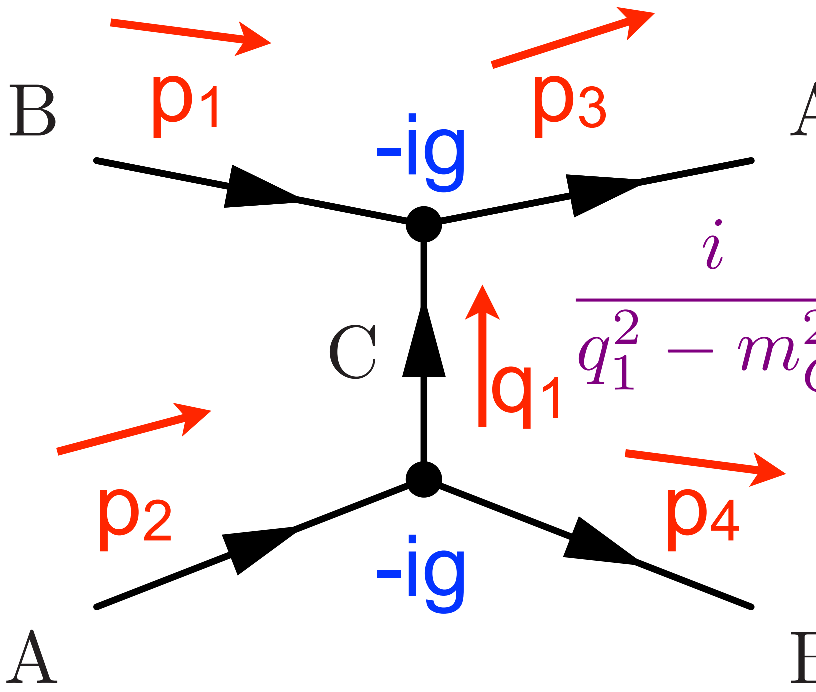





Add factors of -ig for each vertex, specifying the coupling constants

# Feynman rules (3)



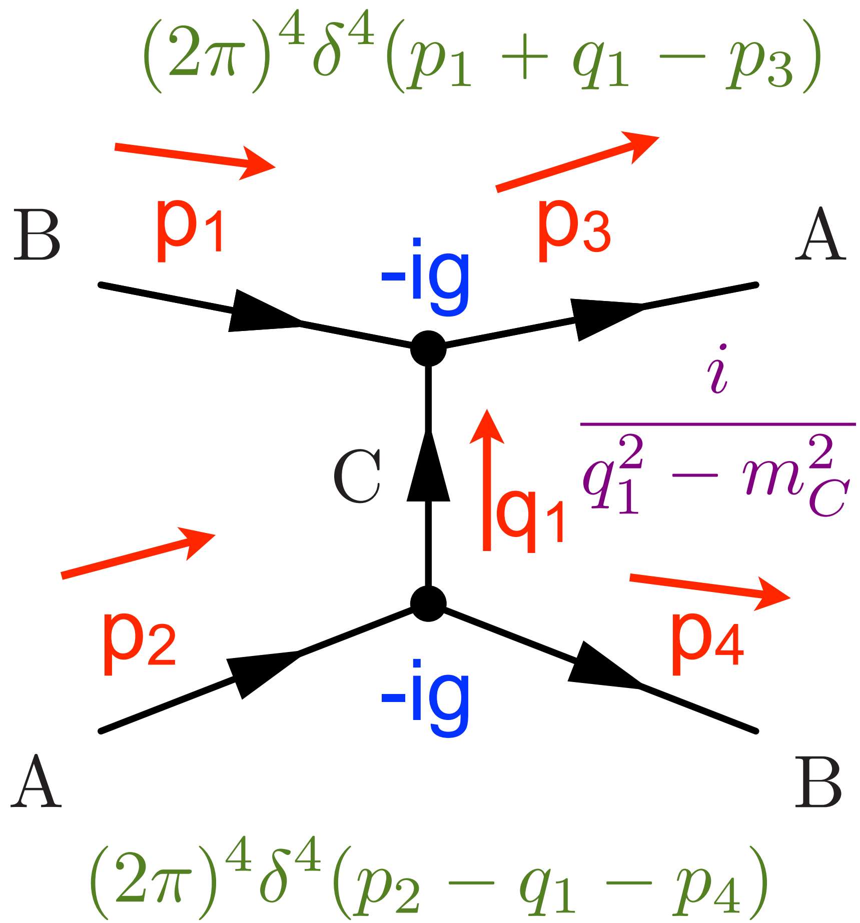



$$\frac{i}{q_1^2 - m_C^2}$$

For each internal line add a factor for the propagator (note that we don't have to be on-shell here!)

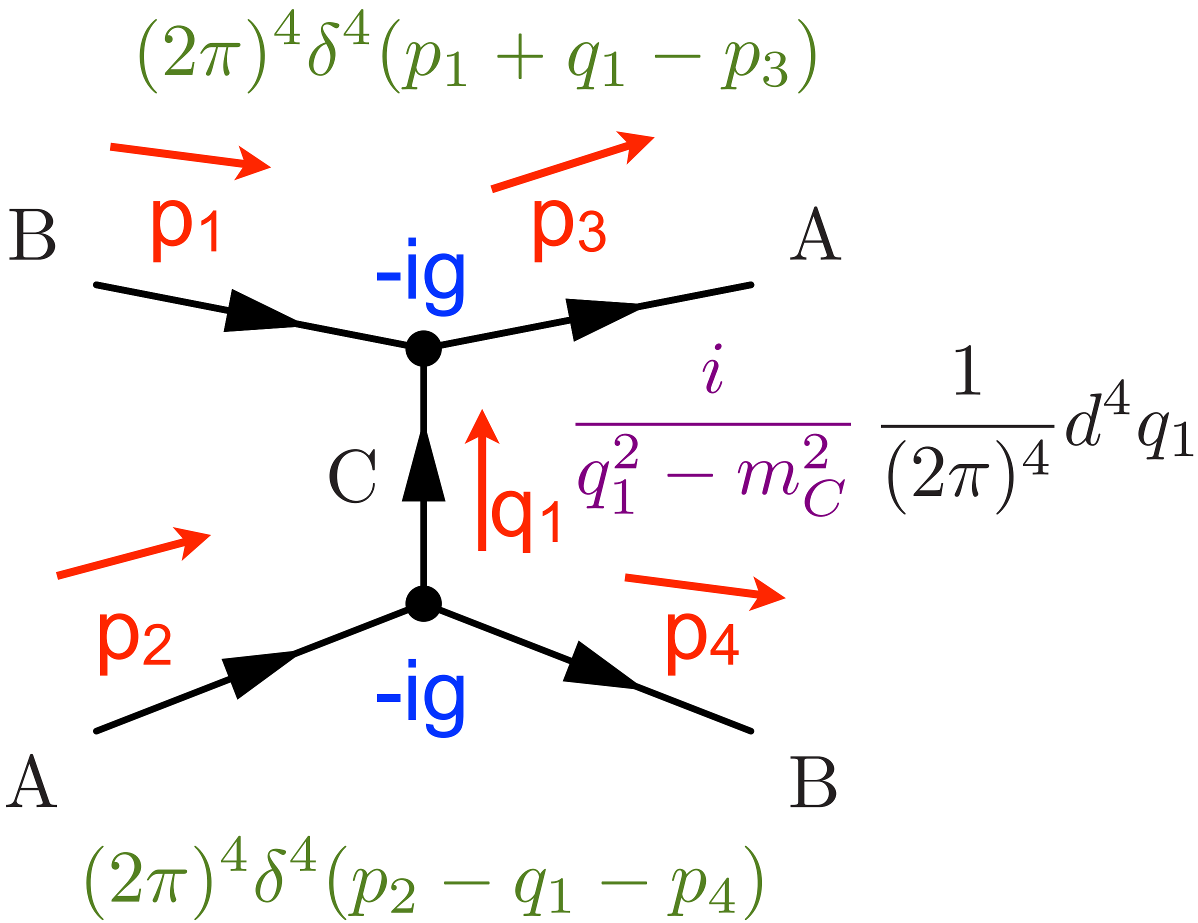
# Feynman rules (4)






Impose conservation of energy and momentum at each vertex with 4d Dirac Delta function (with appropriate 2pi normalization)

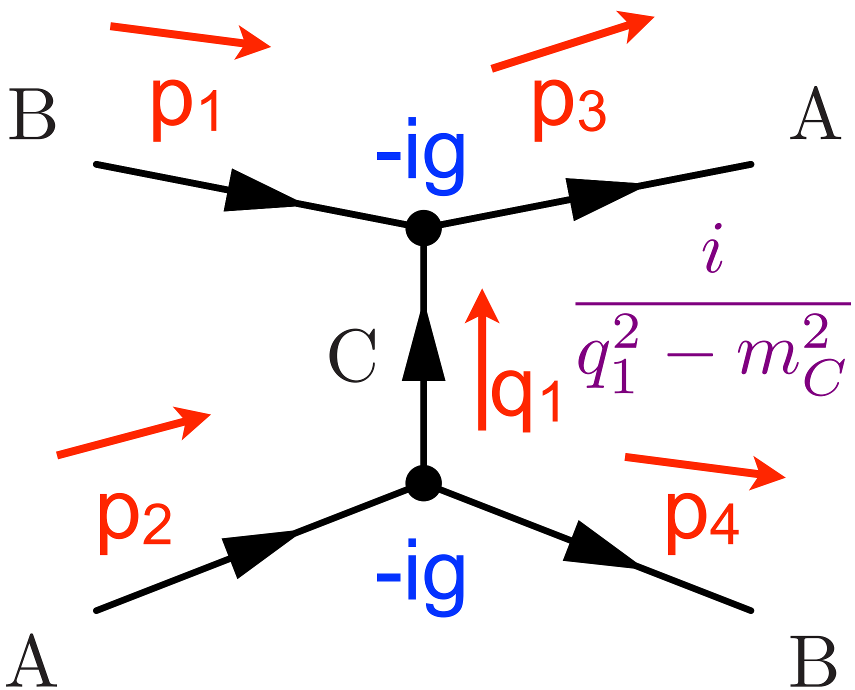
# Feynman rules (5)




Integrate over 4-momentum of internal lines with appropriate 2pi normalization factor

# Feynman rules (6)

$$(2\pi)^4 \delta^4 (p_1 + q_1 - p_3)$$



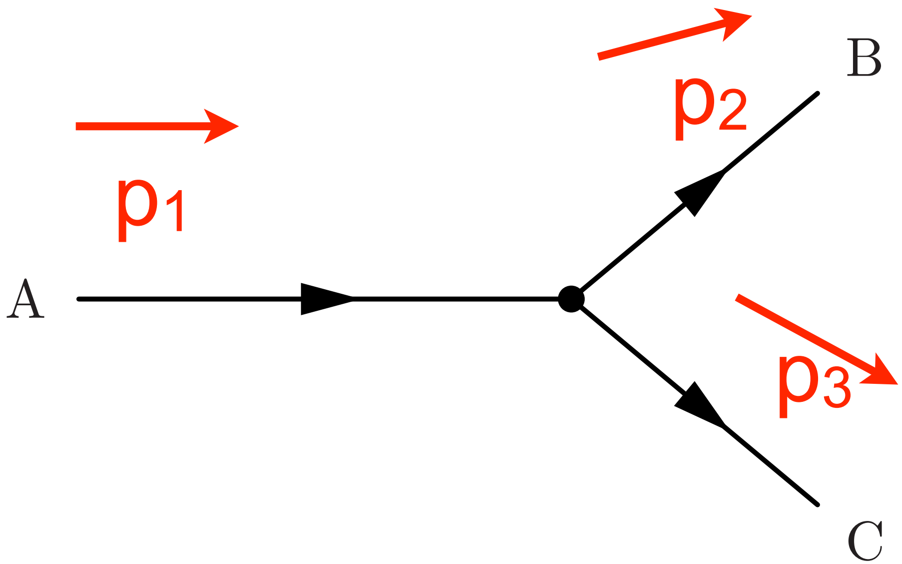

$$i$$

$$\frac{i}{q_1^2 - m_C^2} \frac{1}{(2\pi)^4} d^4 q_1$$

$$(2\pi)^4 \delta^4 (p_2 - q_1 - p_4)$$

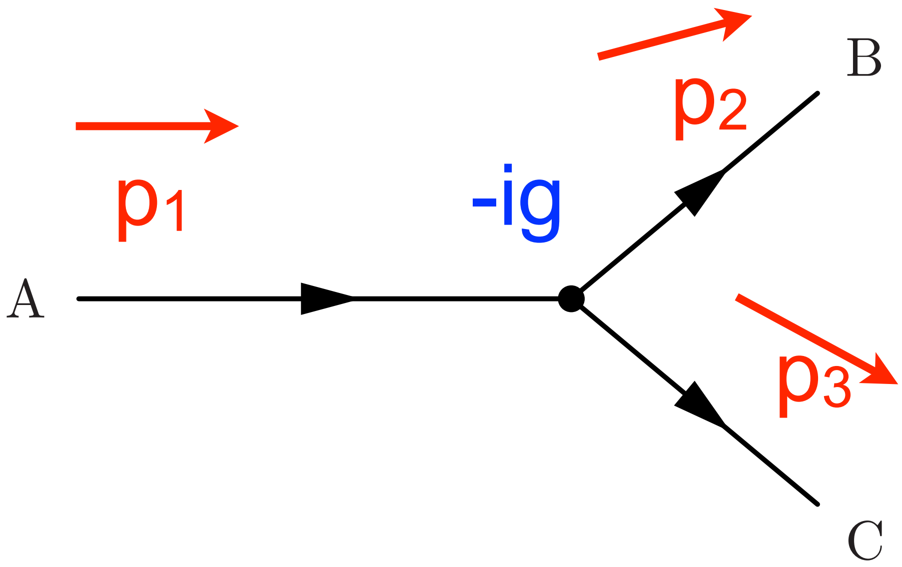
Cancel remaining delta function and add a factor of i, and you have the matrix element

# Feynman rules for decay of object A (1)




Label external (p) and internal lines (q) and draw arrows. Here we have no internal lines

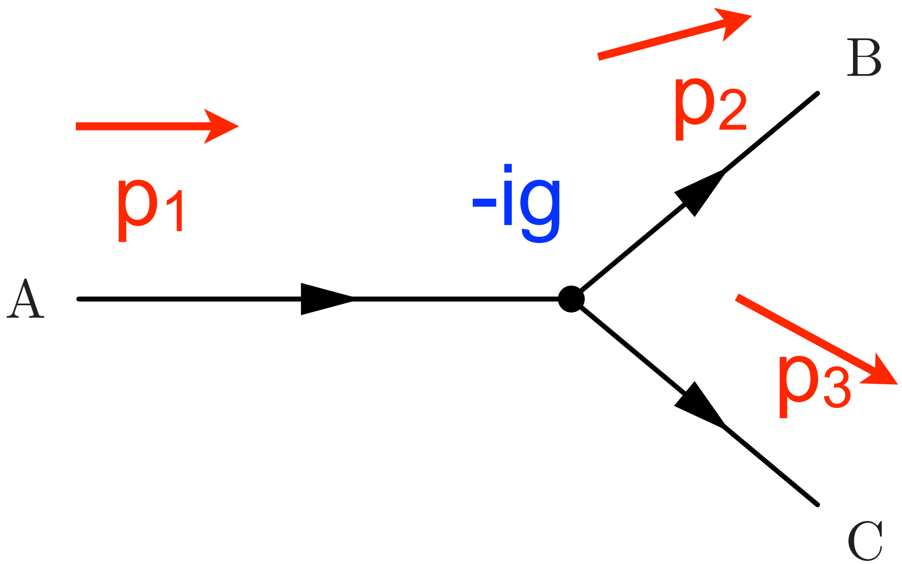
# Feynman rules for decay of object A (2)




Single factor of -ig for our one vertex

# Feynman rules for decay of object A (3)

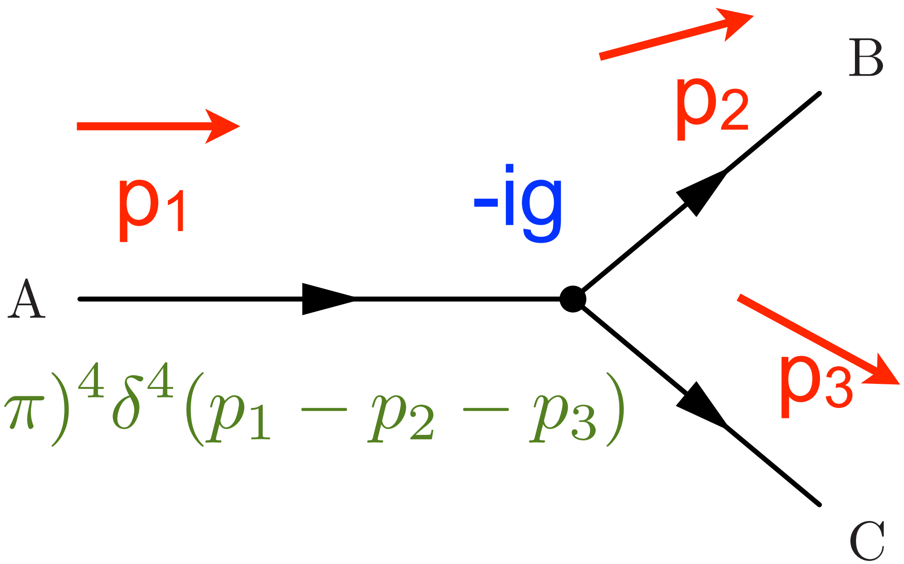





For each internal line add a factor for the propagator, but we don't have one here! (It's nice when things are simple)

# Feynman rules for decay of object A (4)



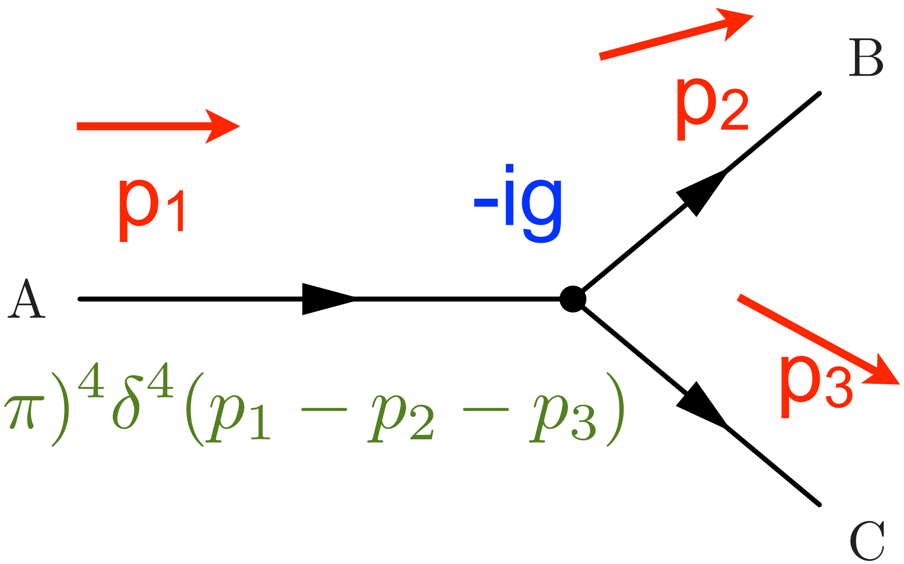



$$(2\pi)^4\delta^4(p_1 - p_2 - p_3)$$

Impose conservation of energy and momentum at each vertex with 4d Dirac Delta function (with appropriate 2pi normalization)

# Feynman rules for decay of object A (5)



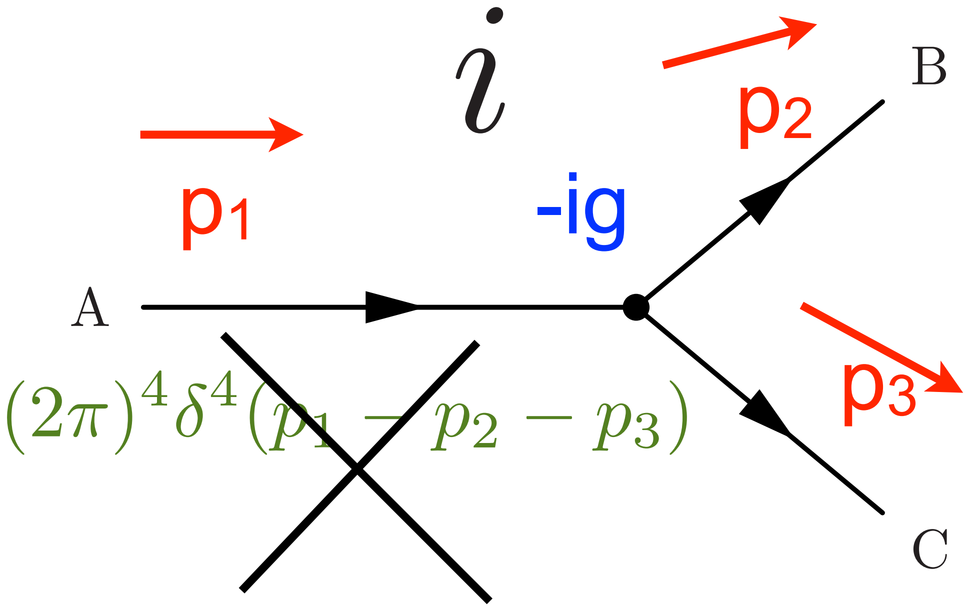



$$(2\pi)^4 \delta^4(p_1 - p_2 - p_3)$$

Integrate over 4-momentum of internal lines with appropriate 2pi normalization factor (here, none)

# Feynman rules for decay of object A (6)

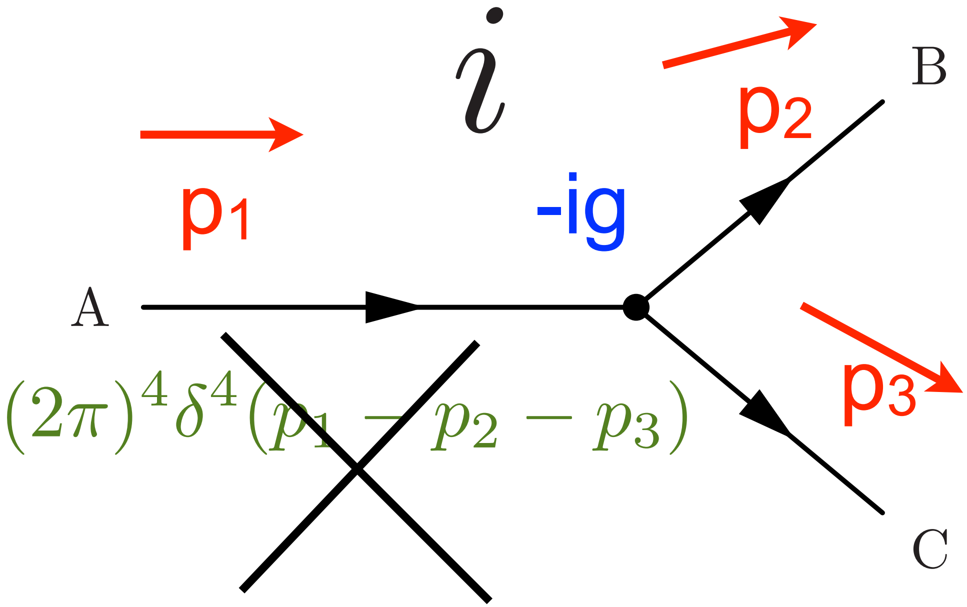





Add factor of i and cancel remaining delta function

# Feynman rules for decay of object A






We're left only with M = i(-ig) = g

$$\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2 = \frac{g^2|\mathbf{p}|}{8\pi m_A^2}$$

$$|\mathbf{p}| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$$

# Lifetime of object A



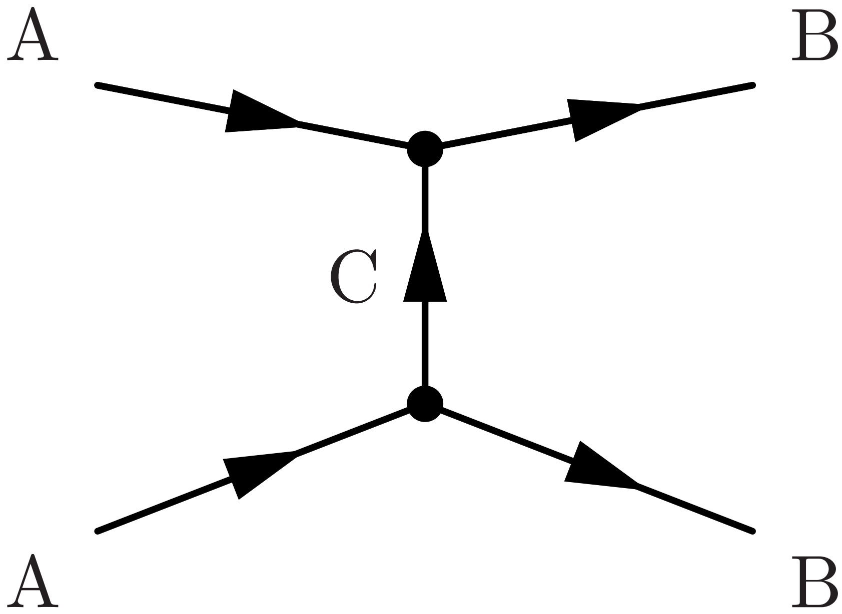
$$\Gamma = \frac{|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2 = \frac{g^2|\mathbf{p}|}{8\pi m_A^2}$$

$$|\mathbf{p}| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$$

$$\tau = \frac{1}{\Gamma} = \frac{8\pi m_A^2}{g^2|\mathbf{p}|}$$

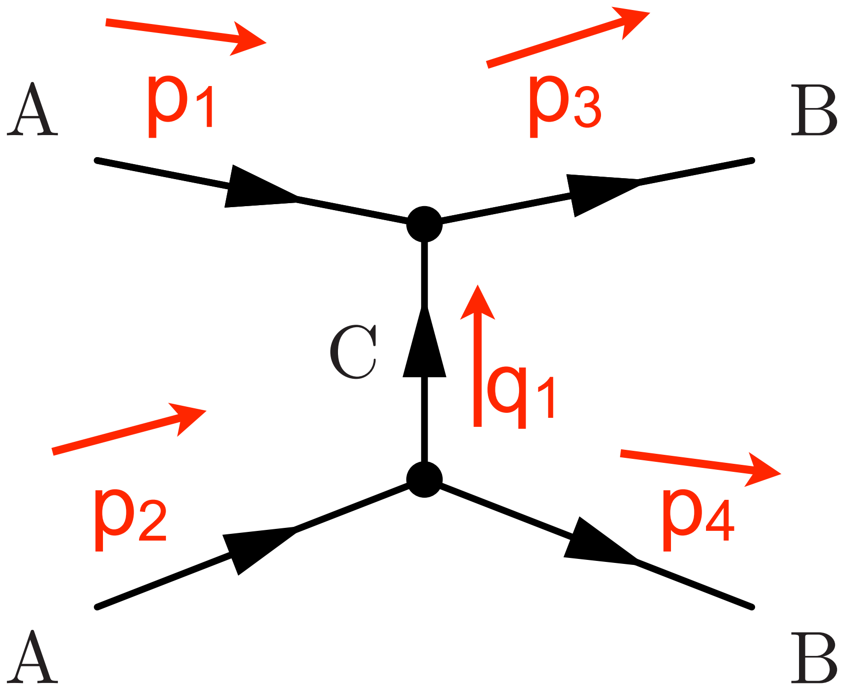
# Let's calculate matrix element for AA→BB

Leading order diagram (there are others at higher order, as we'll see)



# AA→BB Rule 1

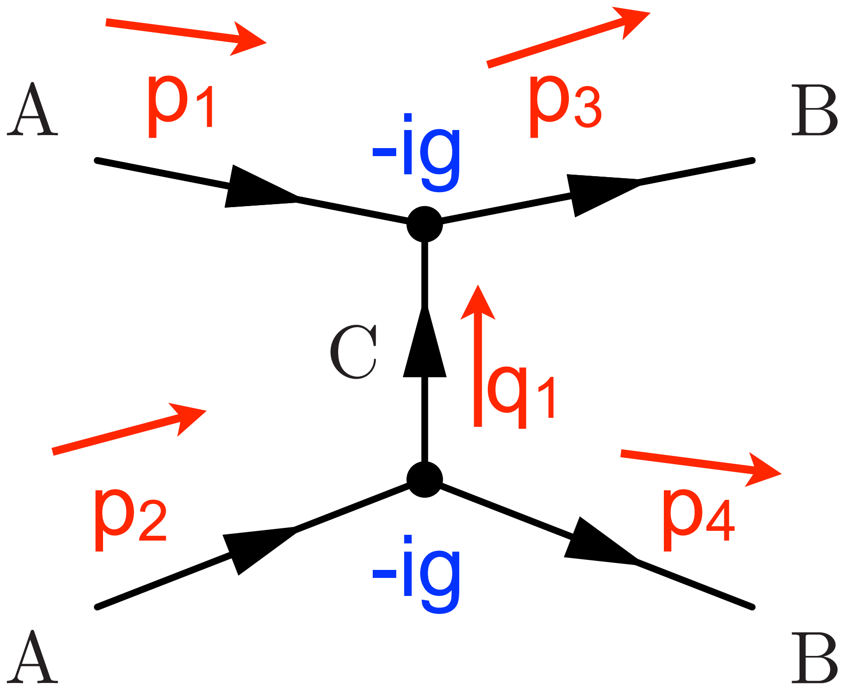





Label all incoming and outgoing lines with $p_1$, $p_2$, ... $p_n$
Internal lines can go either way
Use arrows to keep track of what is going in and out (here this looks trivial, but can be more tricky with anti-particles)

# AA→BB Rule 2

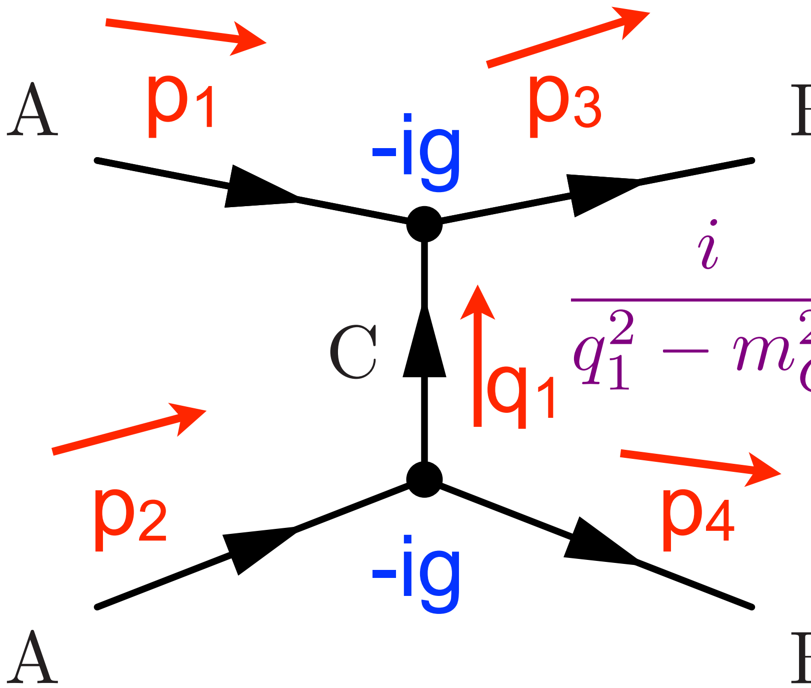





Add factors of -ig for each vertex, specifying the coupling constants

# AA→BB Rule 3

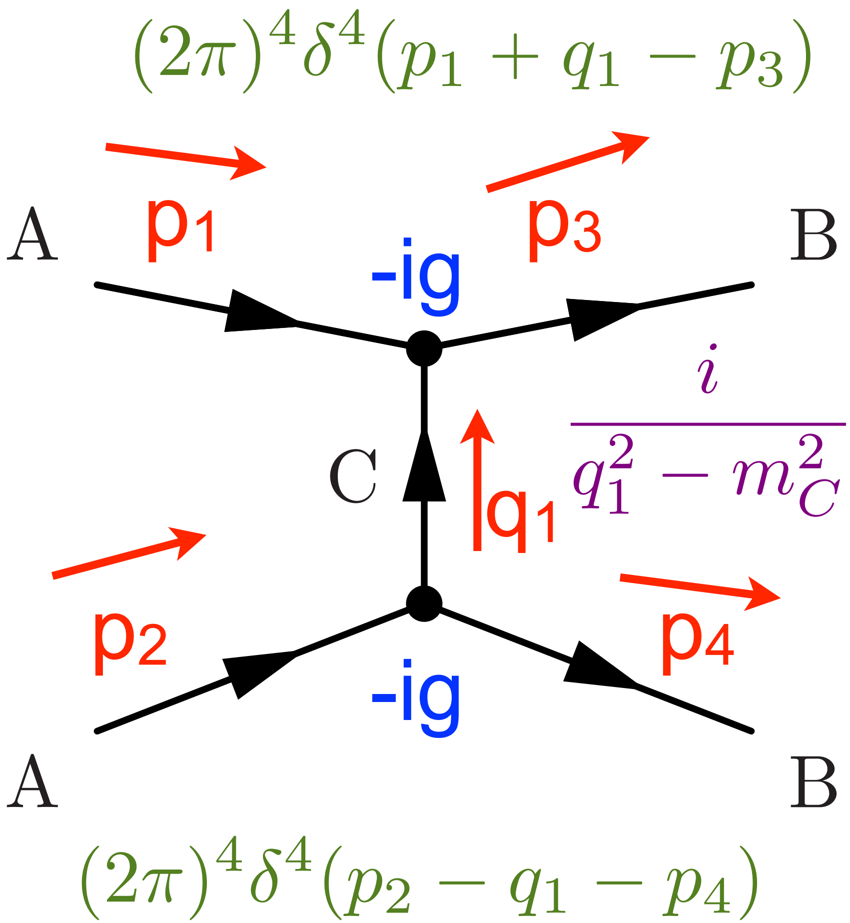





$$\frac{i}{q_1^2 - m_C^2}$$

For each internal line add a factor for the propagator (note that we don't have to be on-shell here!)

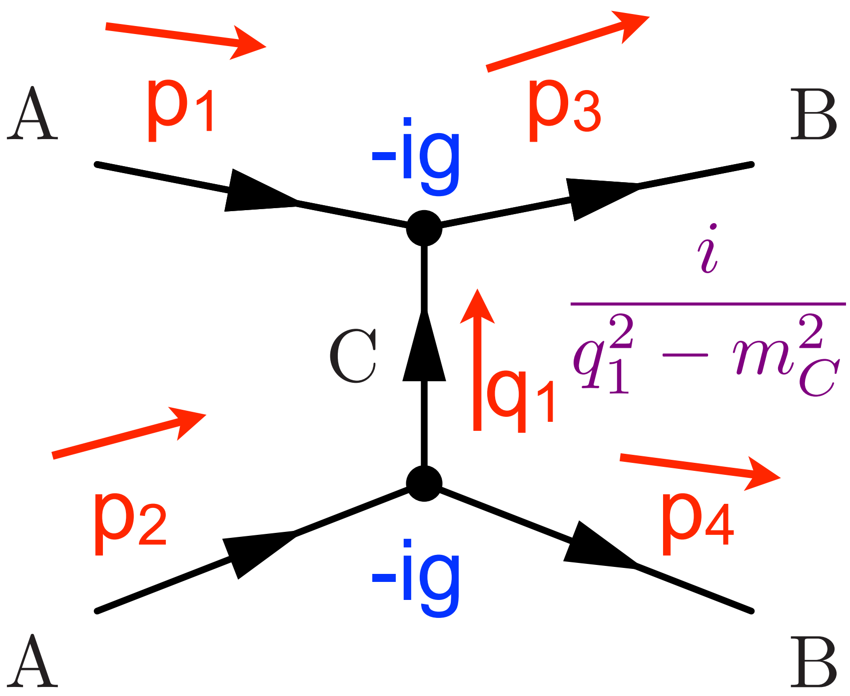
# AA→BB Rule 4




Impose conservation of energy and momentum at each vertex with 4d Dirac Delta function (with appropriate 2pi normalization)

# AA→BB Rule 5

$$(2\pi)^4 \delta^4(p_1 + q_1 - p_3)$$



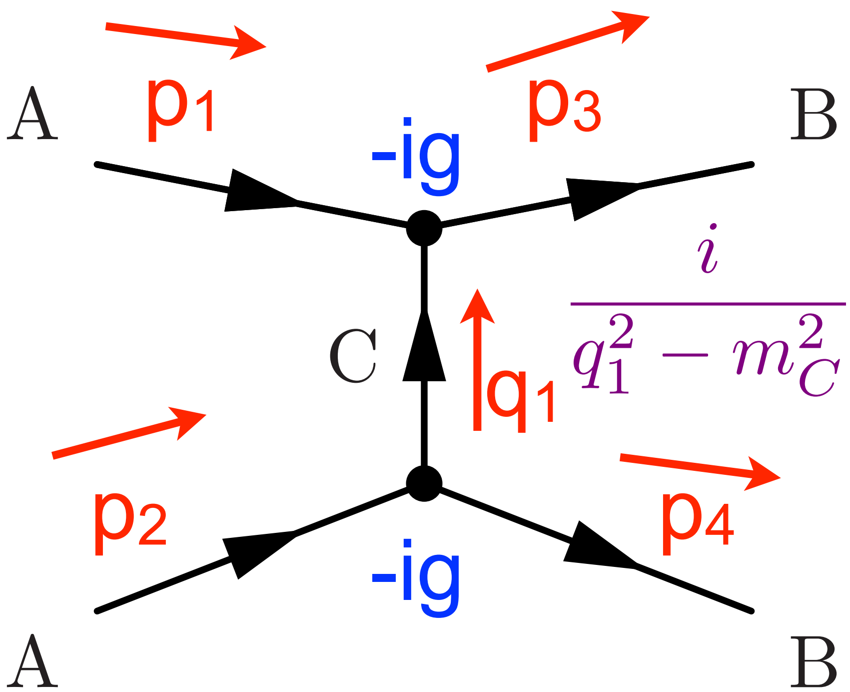

$$\frac{i}{q_1^2 - m_C^2} \frac{1}{(2\pi)^4} d^4 q_1$$

$$(2\pi)^4 \delta^4(p_2 - q_1 - p_4)$$

Integrate over 4-momentum of internal lines with appropriate 2pi normalization factor

# AA→BB Rule 5, putting it together

$$(2\pi)^4\delta^4(p_1 + q_1 - p_3)$$




$$\frac{i}{q_1^2 - m_C^2} \; \frac{1}{(2\pi)^4} d^4q_1$$

$$(2\pi)^4\delta^4(p_2 - q_1 - p_4)$$

$$\int (-ig)(-ig)\frac{i}{q_1^2 - m_C^2}(2\pi)^4\delta^4(p_1 + q_1 - p_3)(2\pi)^4\delta^4(p_2 - q_1 - p_4)\frac{d^4q_1}{(2\pi)^4}$$

# AA→BB Rule 5, simplification



$$\int (-ig)(-ig)\frac{i}{q_1^2 - m_C^2}(2\pi)^4\delta^4(p_1 + q_1 - p_3)(2\pi)^4\delta^4(p_2 - q_1 - p_4)\frac{d^4q_1}{(2\pi)^4}$$

$$(2\pi)^4(-g^2)\int \frac{i}{q_1^2 - m_C^2}\delta^4(p_1 + q_1 - p_3)\delta^4(p_2 - q_1 - p_4)d^4q_1$$

Integral is over 4-momentum of $q_1$ but this gets picked up by the Delta function. Let's use the first one, so $q_1 = p_3 - p_1$

# AA→BB Rule 5, simplification

$$(2\pi)^4(-g^2)\int \frac{i}{q_1^2 - m_C^2}\delta^4(p_1 + q_1 - p_3)\delta^4(p_2 - q_1 - p_4)d^4q_1$$

Integral is over 4-momentum of $q_1$ but this gets picked up by the Delta function. Let's use the first one, so $q_1 = p_3 - p_1$ (note that this is true for each component of $q_1$, as we're using some notation shorthand here)

$$(2\pi)^4(-g^2)\frac{i}{(p_3 - p_1)^2 - m_C^2}\delta^4(p_2 + p_1 - p_3 - p_4)$$

Conservation of **p**/E for total system

# AA→BB Rule 6

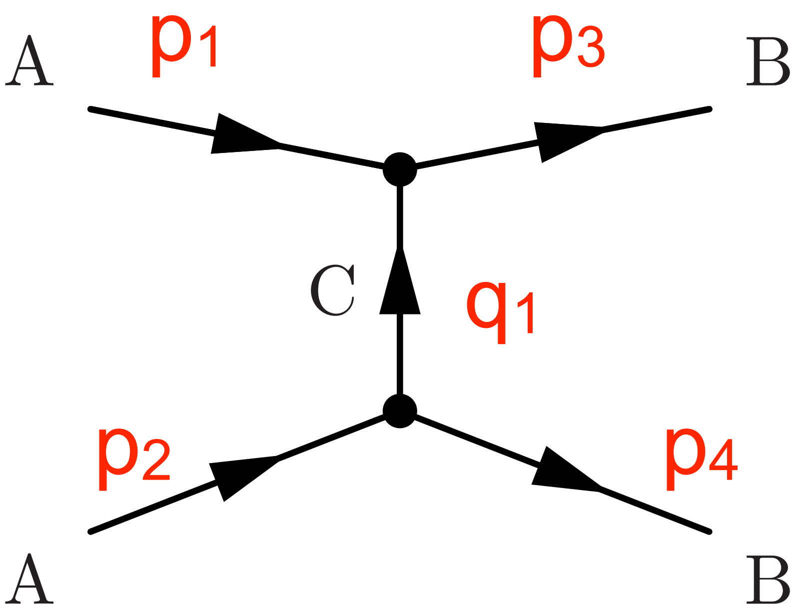
$$(2\pi)^4(-g^2)\frac{i}{(p_3-p_1)^2-m_C^2}\delta^4(p_2+p_1-p_3-p_4)$$

Rule 6: cancel delta function (and 2pi)$^4$ and multiply by i to get Matrix Element

$$\mathcal{M} = i(2\pi)^4(-g^2)\frac{i}{(p_3-p_1)^2-m_C^2}$$

$$\mathcal{M} = \frac{g^2}{(p_3-p_1)^2-m_C^2}$$

# First complication for AA→BB scattering



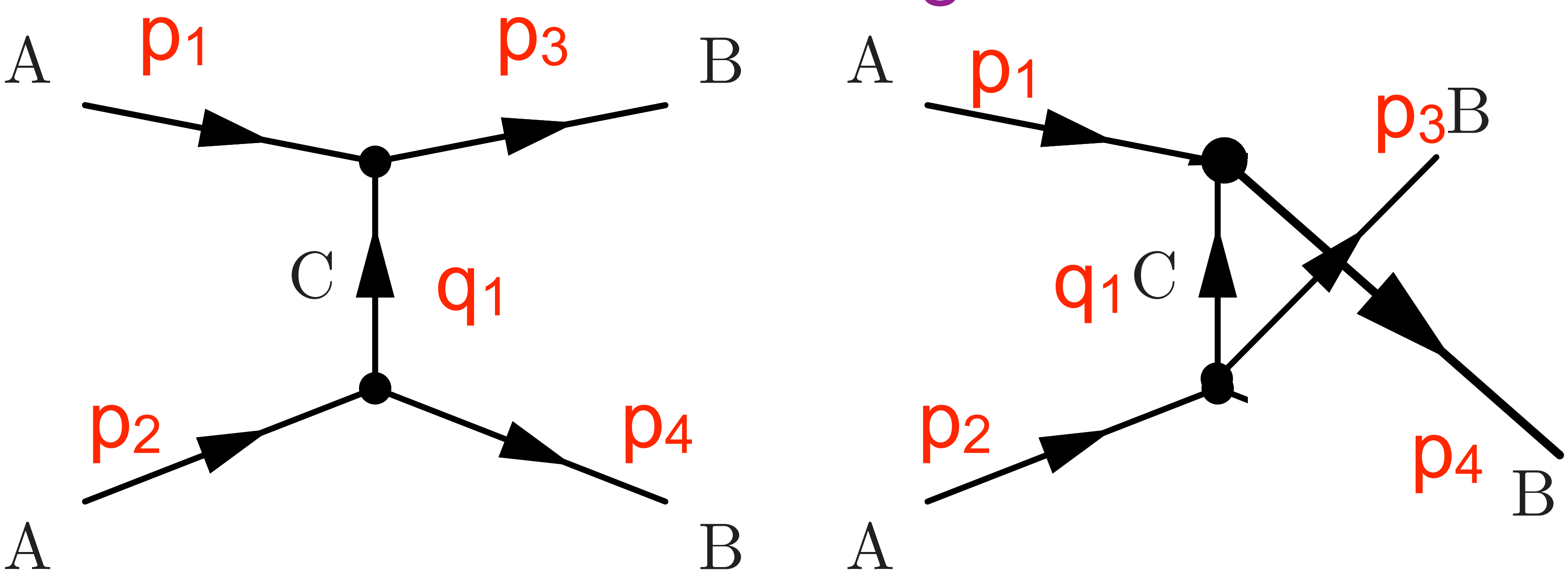

$$\mathcal{M} = \frac{g^2}{(p_3 - p_1)^2 - m_C^2}$$

This is not the only diagram. There is another similar one with the same initial state and the same final state. So they must be added together, as they interfere!

# Second AA→BB diagram



Let's work this out together on the board

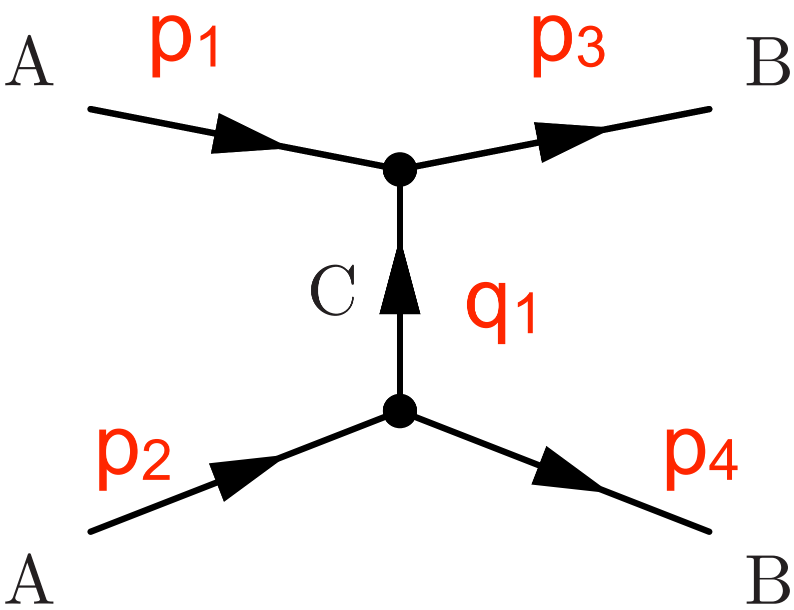


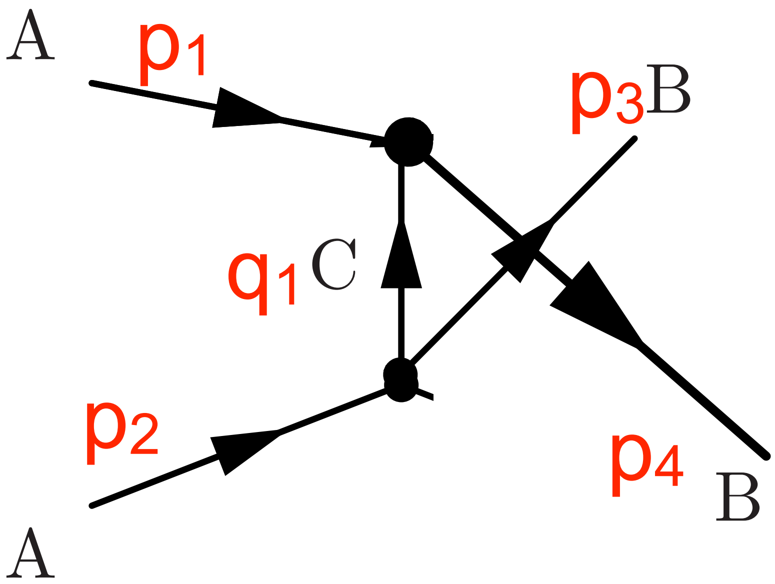

$$\mathcal{M} = \frac{g^2}{(p_3 - p_1)^2 - m_C^2}$$

Same diagram except that $p_1$ connects to $p_4$, not to $p_3$

# Second AA→BB diagram






$$\mathcal{M} = \frac{g^2}{(p_3 - p_1)^2 - m_C^2}$$

$$\mathcal{M} = \frac{g^2}{(p_4 - p_1)^2 - m_C^2}$$

# So total AA→BB ME is



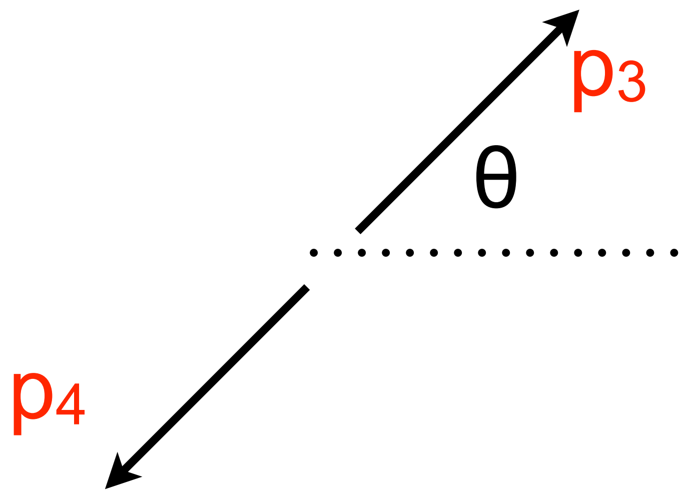
$$\mathcal{M} = g^2 \left[ \frac{1}{(p_3 - p_1)^2 - m_C^2} + \frac{1}{(p_4 - p_1)^2 - m_C^2} \right]$$

As in Griffiths, let's assume $m_C$ is zero to simplify things, $m_A=m_B=m$

$$\mathcal{M} = g^2 \left[ \frac{1}{(p_3 - p_1)^2} + \frac{1}{(p_4 - p_1)^2} \right]$$

# AA→BB ME for toy with massless $m_C$

$$\mathcal{M} = g^2 \left[ \frac{1}{(p_3 - p_1)^2} + \frac{1}{(p_4 - p_1)^2} \right]$$


Use center of mass reference frame, where $|\mathbf{p}_1|=|\mathbf{p}_2|$, $|\mathbf{p}_3|=|\mathbf{p}_4|$

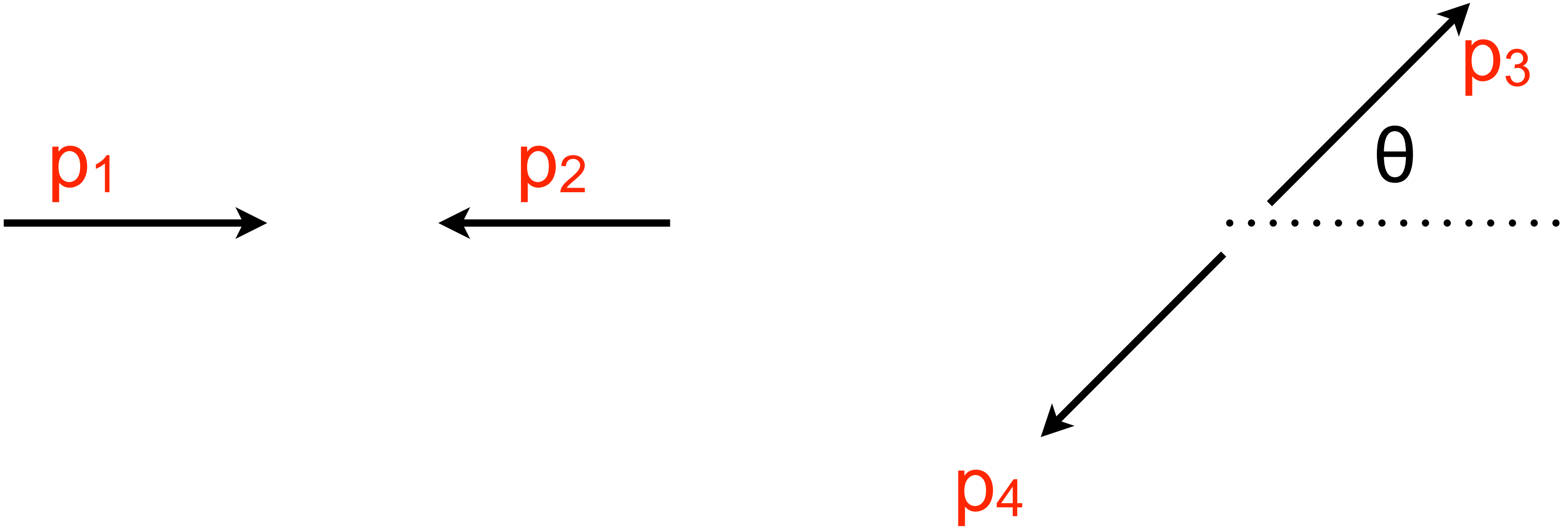
Since $m_A=m_B$ this means

$|\mathbf{p}_1|=|\mathbf{p}_2|=|\mathbf{p}_3|=|\mathbf{p}_4|=p$ $\quad E_1=E_2=E_3=E_4=E$

That clear?

# AA→BB ME for toy with massless mC






$$(p_4 - p_1)^2 = p_1^2 + p_4^2 - 2p_4 \cdot p_1 = 2m^2 - 2E_4E_1 + 2\mathbf{p}_4 \cdot \mathbf{p}_1$$

$$(p_4 - p_1)^2 = 2m^2 - 2E^2 + 2\mathbf{p}_4 \cdot \mathbf{p}_1 = 2(E^2 - \mathbf{p}^2) - 2E^2 + 2\mathbf{p}_4 \cdot \mathbf{p}_1$$

$$(p_4 - p_1)^2 = 2(E^2 - \mathbf{p}^2) - 2E^2 + 2\mathbf{p}_4 \cdot \mathbf{p}_1 = -2\mathbf{p}^2 + 2\mathbf{p}_4 \cdot \mathbf{p}_1$$

$$(p_4 - p_1)^2 = -2\mathbf{p}^2 - 2\mathbf{p}^2 \cos\theta = -2\mathbf{p}^2(1 + \cos\theta)$$

# AA→BB ME for toy with massless $m_C$



$$\mathcal{M} = g^2 \left[ \frac{1}{(p_3 - p_1)^2} + \frac{1}{(p_4 - p_1)^2} \right]$$

$$(p_4 - p_1)^2 = -2\mathbf{p}^2(1 + \cos\theta)$$

$$(p_3 - p_1)^2 = -2\mathbf{p}^2(1 - \cos\theta)$$

$$\mathcal{M} = g^2 \left[ \frac{1}{-2\mathbf{p}^2(1 - \cos\theta)} + \frac{1}{-2\mathbf{p}^2(1 + \cos\theta)} \right]$$

$$\mathcal{M} = \frac{g^2}{-2\mathbf{p}^2} \left[ \frac{1}{(1 - \cos\theta)} + \frac{1}{(1 + \cos\theta)} \right]$$

$$\mathcal{M} = \frac{g^2}{-2\mathbf{p}^2} \left[ \frac{(1 + \cos\theta) + (1 - \cos\theta)}{(1 - \cos\theta)(1 + \cos\theta)} \right]$$

$$\mathcal{M} = \frac{g^2}{-2\mathbf{p}^2} \left[ \frac{2}{1 - \cos^2\theta} \right]$$

$$\boxed{\mathcal{M} = \frac{-g^2}{\mathbf{p}^2 \sin^2\theta}}$$

# Differential cross section here



$$\mathcal{M} = \frac{-g^2}{\mathbf{p}^2 \sin^2\theta}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|M|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

$$|\mathbf{p}_f| = |\mathbf{p}_i| = |\mathbf{p}|$$

$$E_1 = E_2 = E$$

$$|\mathcal{M}|^2 = \frac{g^4}{\mathbf{p}^4 \sin^4\theta}$$

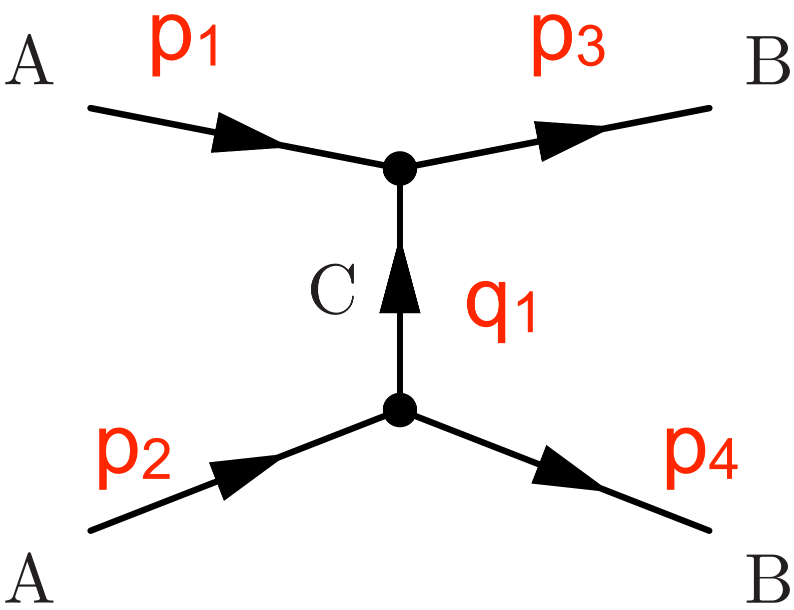
Infinite cross section as θ→0

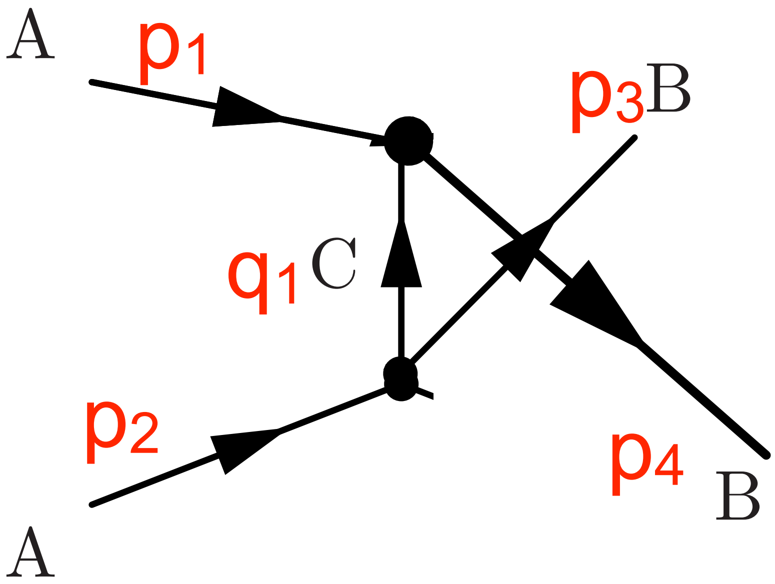
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{g^4}{(2E)^2 \mathbf{p}^4 \sin^4\theta}$$

# We are very close except for one final thing



Here we have two instances of particle B in the final state. For every s identical particles, we add factor of 1/(s!) to account for this, or else we have overcounted the phase space



# The final differential cross section

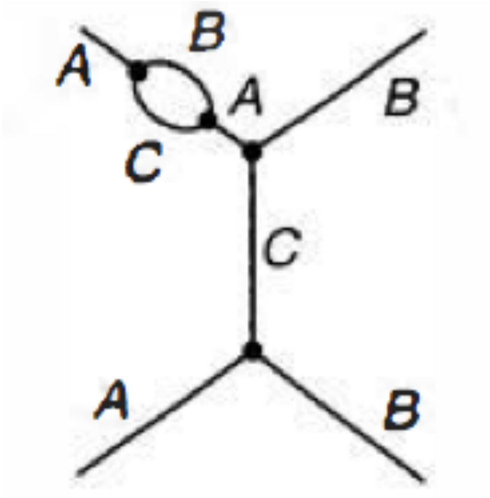


$$\frac{d\sigma}{d\Omega} = \frac{1}{2!}\frac{1}{64\pi^2}\frac{g^4}{(2E)^2 2 * \mathbf{p}^4 \sin^4\theta}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{1024\pi^2}\frac{g^4}{E^2\mathbf{p}^4 \sin^4\theta}$$

# What about beyond leading order

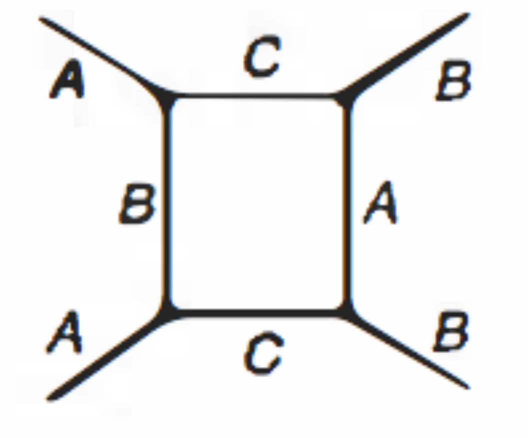
(From Griffiths)



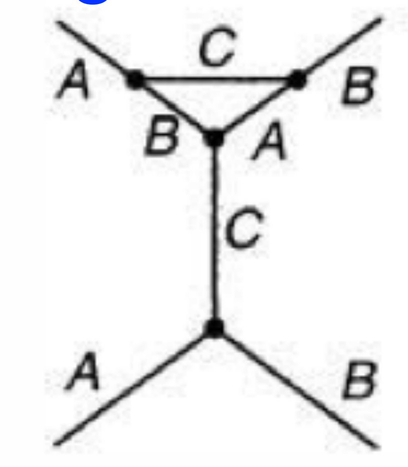

"Self-energy" diagrams

There are many of these. And even more at higher order

Box diagram

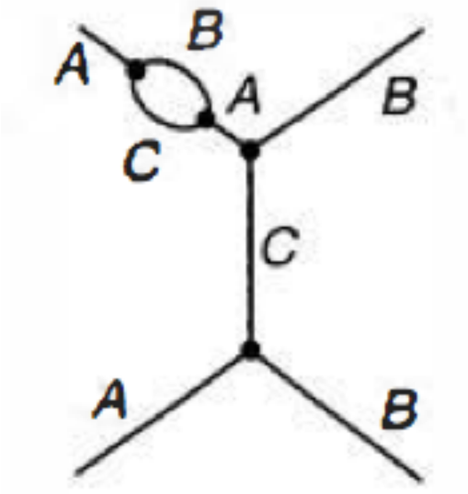



Vertex correction diagrams



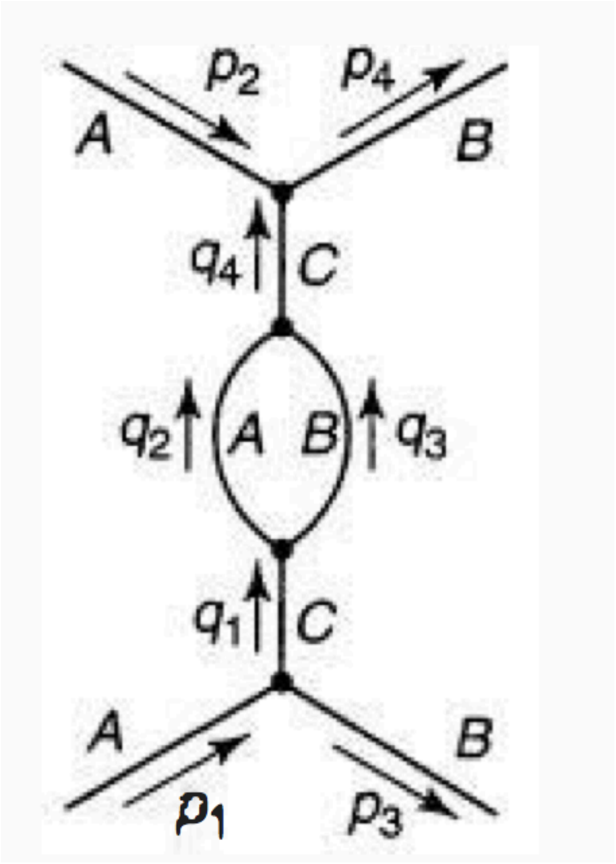
# The good thing...






Each vertex carries a factor of "g" in the matrix element, so $g^2$ for physical quantities. So diagrams with extra vertices should be sub-dominant corrections, or so we hope

# Griffiths' suggested diagram to calculate




1. Label your p's and q's
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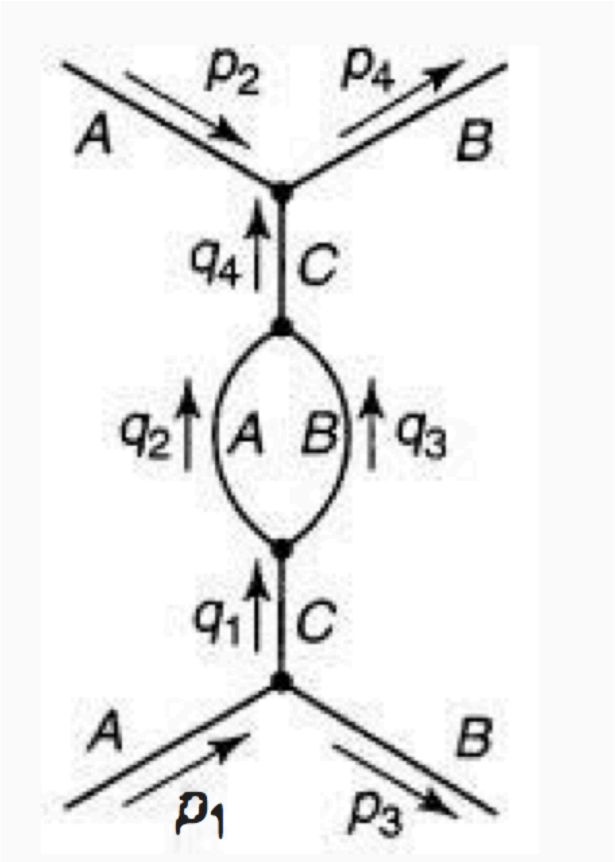
5. Internal momentum integration
6. Cancel delta function and add extra 'i'

# Let’s try this one

$$(-ig)^4$$

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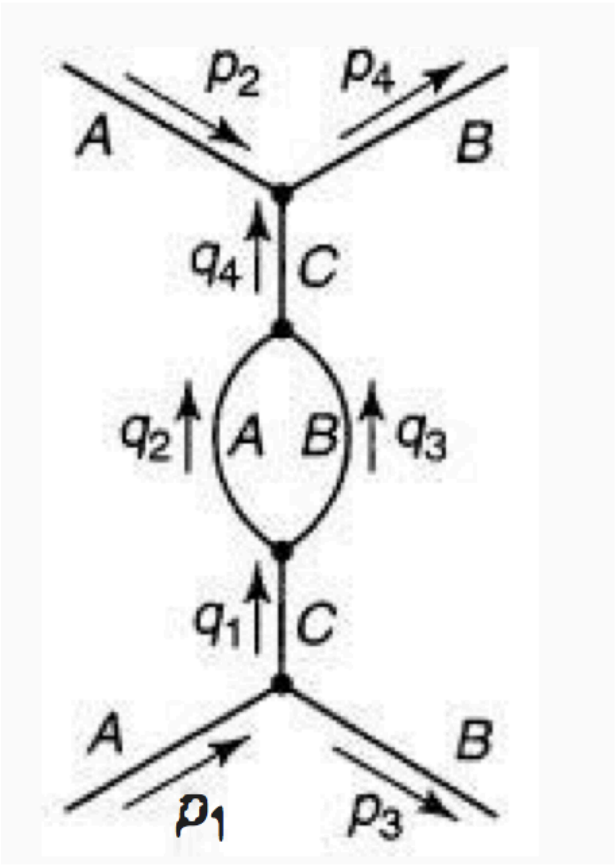
# Let's try this one



$$(-ig)^4 \quad \frac{i}{q_4^2 - m_C^2} \frac{i}{q_2^2 - m_A^2} \frac{i}{q_3^2 - m_B^2} \frac{i}{q_1^2 - m_C^2}$$

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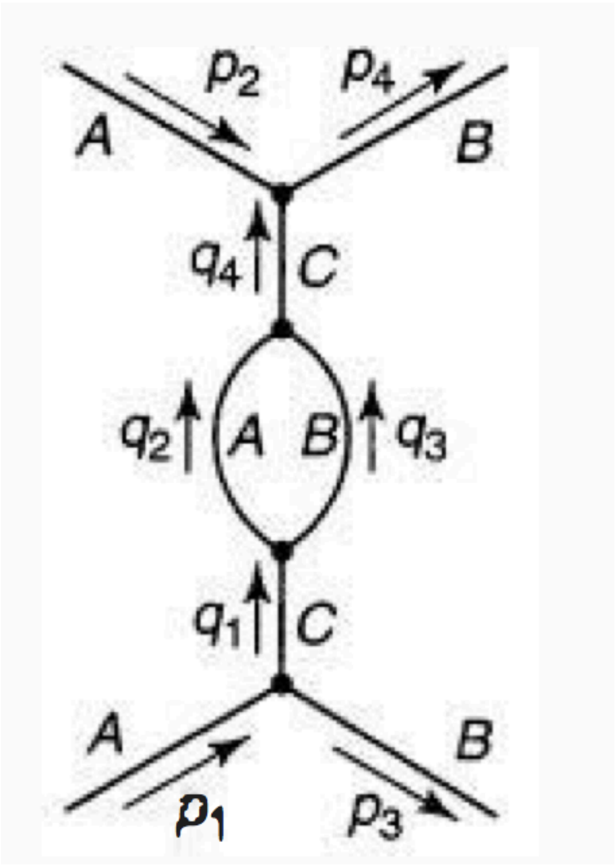
$$(2\pi)^4\delta^4(p_2 + q_4 - p_4)(2\pi)^4\delta^4(q_2 + q_3 - q_4)(2\pi)^4\delta^4(q_1 - q_2 - q_3)(2\pi)^4\delta^4(p_1 - q_1 - p_3)$$

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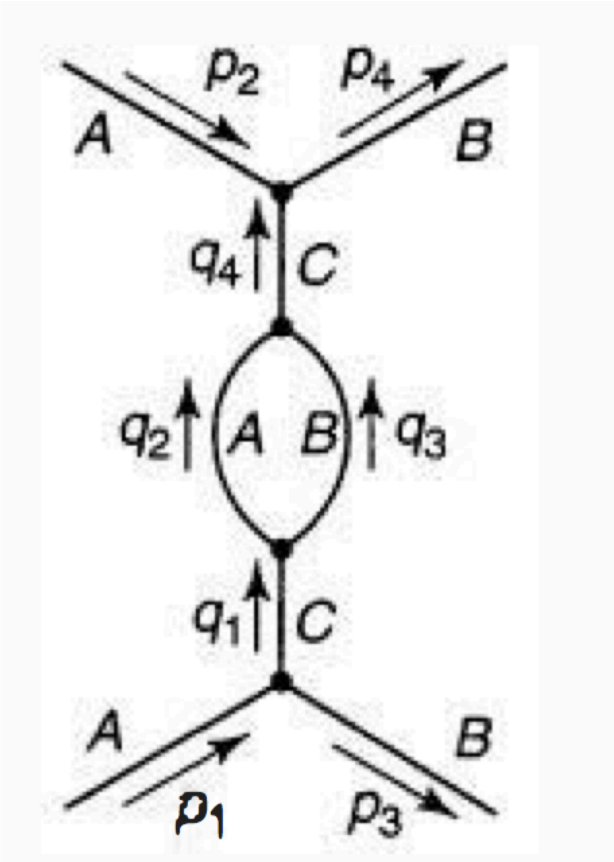
# Let's try this one

$$\int (-ig)^4 \quad \frac{i}{q_4^2 - m_C^2} \frac{i}{q_2^2 - m_A^2} \frac{i}{q_3^2 - m_B^2} \frac{i}{q_1^2 - m_C^2}$$

$$(2\pi)^4\delta^4(p_2 + q_4 - p_4)(2\pi)^4\delta^4(q_2 + q_3 - q_4)(2\pi)^4\delta^4(q_1 - q_2 - q_3)(2\pi)^4\delta^4(p_1 - q_1 - p_3)$$

$$\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\frac{d^4q_4}{(2\pi)^4}$$

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# Let's combine / cancel terms

$$\int (-ig)^4 \quad \frac{i}{q_4^2 - m_C^2} \frac{i}{q_2^2 - m_A^2} \frac{i}{q_3^2 - m_B^2} \frac{i}{q_1^2 - m_C^2}$$

$$(2\pi)^4 \delta^4(p_2 + q_4 - p_4)(2\pi)^4 \delta^4(q_2 + q_3 - q_4)(2\pi)^4 \delta^4(q_1 - q_2 - q_3)(2\pi)^4 \delta^4(p_1 - q_1 - p_3)$$

$$\frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{d^4 q_3}{(2\pi)^4} \frac{d^4 q_4}{(2\pi)^4}$$

$$g^4 \int \frac{1}{q_4^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{q_1^2 - m_C^2}$$

$$\delta^4(p_2 + q_4 - p_4)\delta^4(q_2 + q_3 - q_4)\delta^4(q_1 - q_2 - q_3)\delta^4(p_1 - q_1 - p_3) d^4 q_1 d^4 q_2 d^4 q_3 d^4 q_4$$

Now let's use those delta functions

# Delta functions to the rescue



$$g^4 \int \frac{1}{q_4^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{q_1^2 - m_C^2}$$

$$\delta^4(p_2 + q_4 - p_4)\delta^4(q_2 + q_3 - q_4)\delta^4(q_1 - q_2 - q_3)\delta^4(p_1 - q_1 - p_3)d^4q_1 d^4q_2 d^4q_3 d^4q_4$$

## First eliminate $q_1$

$$g^4 \int \frac{1}{q_4^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2}$$

$$\delta^4(p_2 + q_4 - p_4)\delta^4(q_2 + q_3 - q_4)\delta^4(p_1 - p_3 - q_2 - q_3)d^4q_2 d^4q_3 d^4q_4$$

## Now $q_4$

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2}$$

$$\delta^4(q_2 + q_3 + p_2 - p_4)\delta^4(p_1 - p_3 - q_2 - q_3)d^4q_2 d^4q_3$$

# Delta functions to the rescue



$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{q_2^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2}$$

$$\delta^4(q_2 + q_3 + p_2 - p_4)\delta^4(p_1 - p_3 - q_2 - q_3)d^4q_2 d^4q_3$$

Now eliminate $q_2$

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2}$$

$$\delta^4(p_1 - p_3 - q_3 + q_3 + p_2 - p_4)d^4q_3$$

Rearrange

$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2} \delta^4(p_1 + p_2 - p_3 - p_4)d^4q_3$$

# A bit more rearranging



$$g^4 \int \frac{1}{(p_4 - p_2)^2 - m_C^2} \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2} \delta^4(p_1 + p_2 - p_3 - p_4) d^4 q_3$$

Delta function implies $(p_4 - p_2) = (p_1 - p_3)$

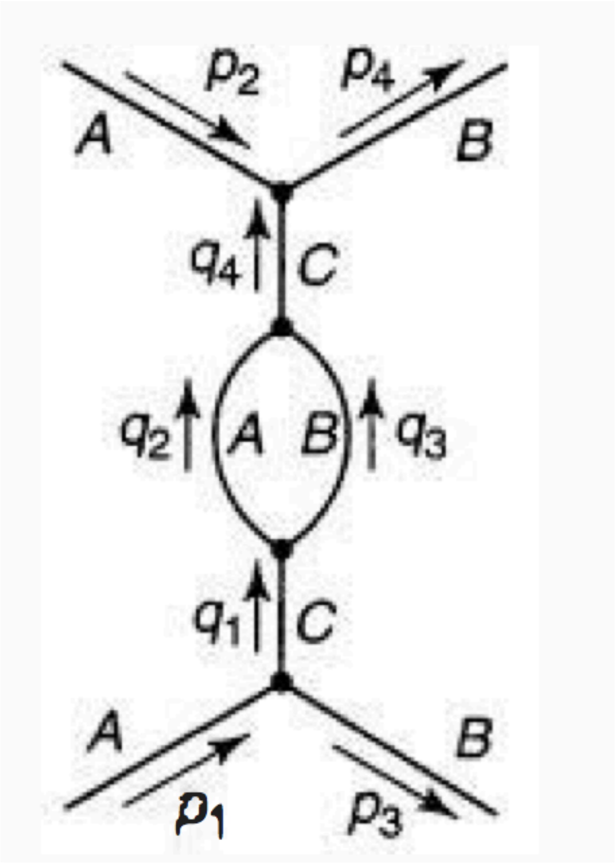
$$g^4 \int \frac{1}{(p_1 - p_3)^2 - m_C^2} \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \frac{1}{(p_1 - p_3)^2 - m_C^2} \delta^4(p_1 + p_2 - p_3 - p_4) d^4 q_3$$

$$g^4 \int \left( \frac{1}{(p_1 - p_3)^2 - m_C^2} \right)^2 \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \delta^4(p_1 + p_2 - p_3 - p_4) d^4 q_3$$

# We just finished step 5



$$g^4 \int \left( \frac{1}{(p_1 - p_3)^2 - m_C^2} \right)^2 \frac{1}{(p_1 - p_3 - q_3)^2 - m_A^2} \frac{1}{q_3^2 - m_B^2} \delta^4(p_1 + p_2 - p_3 - p_4) d^4 q_3$$

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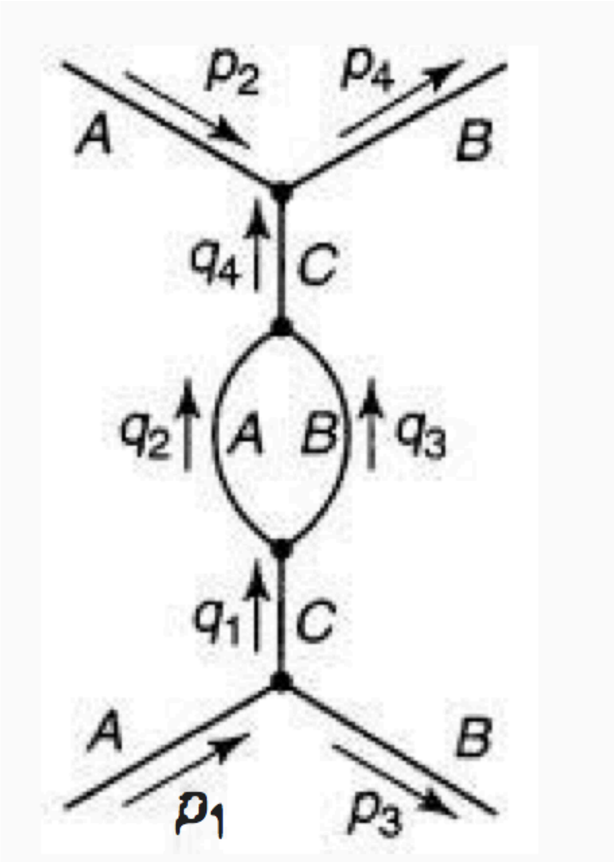
# Last step



$$i\left(\frac{g}{2\pi}\right)^4\left(\frac{1}{(p_1-p_3)^2-m_C^2}\right)^2\int\frac{1}{\left((p_1-p_3-q)^2-m_A^2\right)\left(q^2-m_B^2\right)}d^4q$$

(2pi)$^{-4}$ from canceling delta

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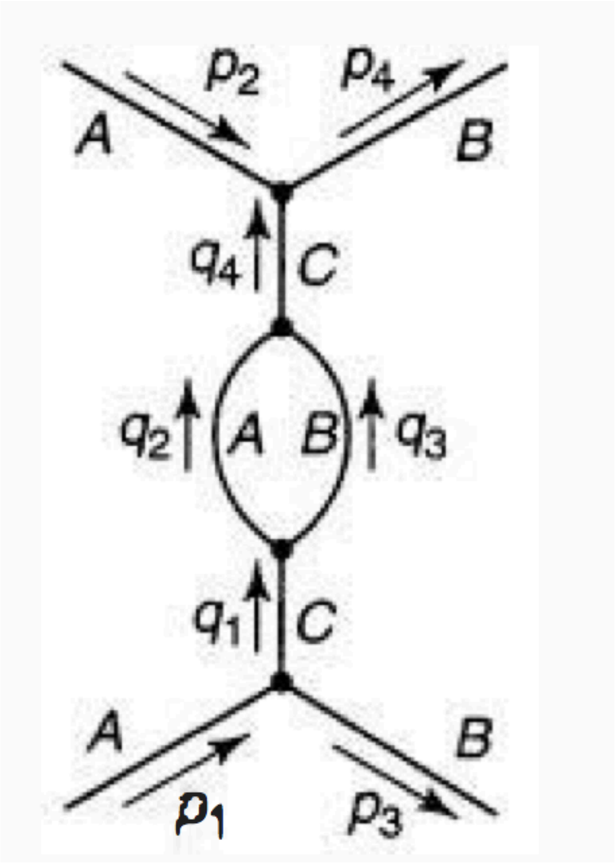
# Phew

$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \left(\frac{1}{(p_1 - p_3)^2 - m_C^2}\right)^2 \int \frac{1}{\left((p_1 - p_3 - q)^2 - m_A^2\right)\left(q^2 - m_B^2\right)} d^4q$$

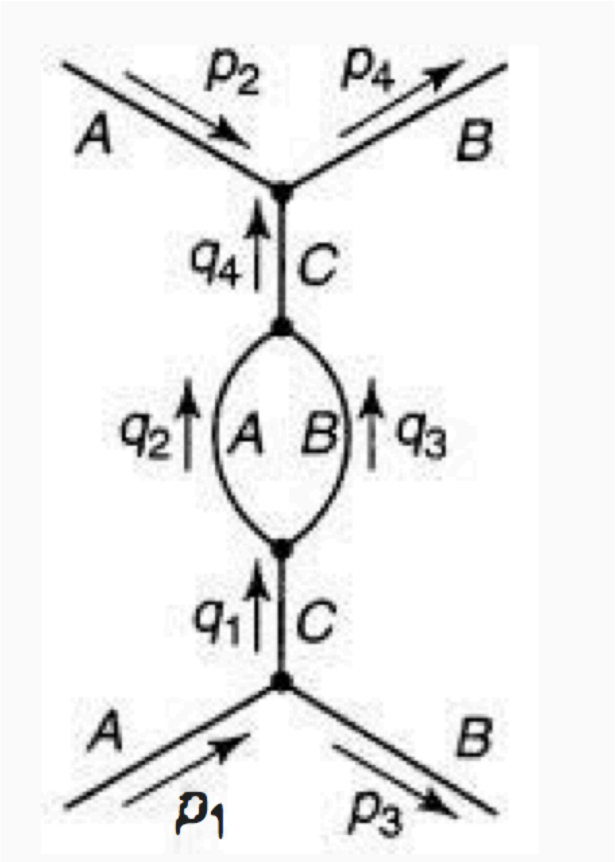



Can try and evaluate that ugly integral, but we find that it diverges at large momentum for internal q!

# How to deal with infinities?!?!



$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \left(\frac{1}{(p_1 - p_3)^2 - m_C^2}\right)^2 \int \frac{1}{\left((p_1 - p_3 - q)^2 - m_A^2\right)\left(q^2 - m_B^2\right)} d^4q$$

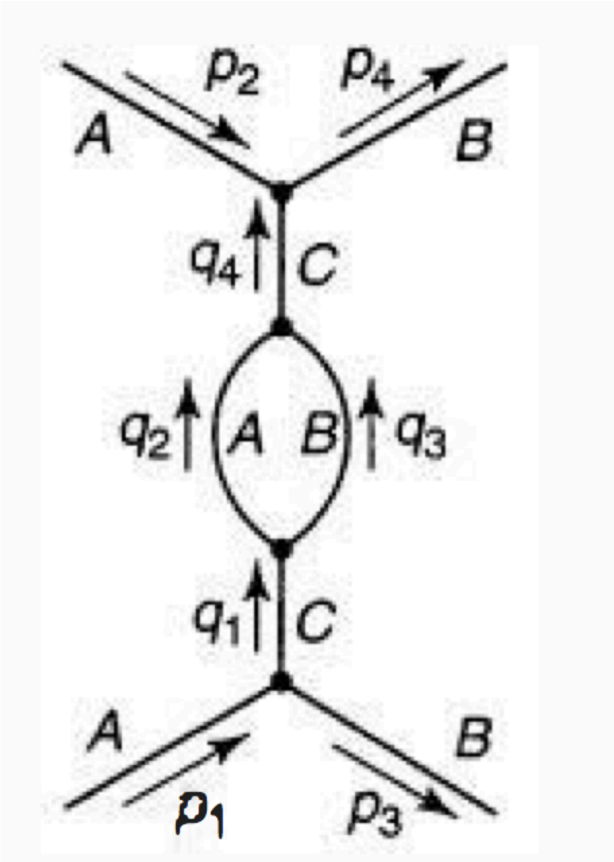



More thorough investigation finds that the infinities are really just affecting the masses of objects and the coupling constants - but these are measured quantities anyway, so we can "use" the measured values. Such theories are **renormalizable**

# How to deal with infinities?!?!



$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \left(\frac{1}{(p_1 - p_3)^2 - m_C^2}\right)^2 \int \frac{1}{((p_1 - p_3 - q)^2 - m_A^2)\,(q^2 - m_B^2)} d^4q$$

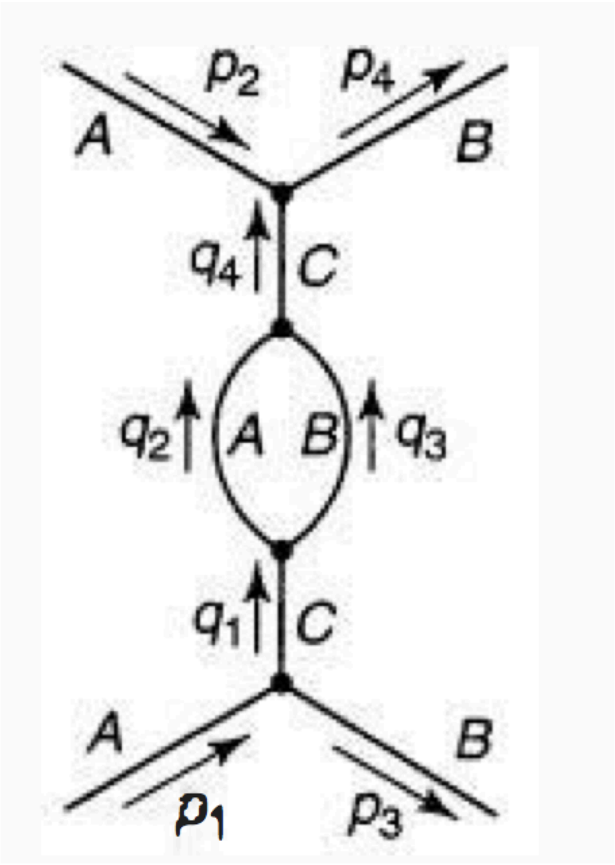



We can't assume that physics inside those loops can have infinite momentum - there must be a "cutoff" at which new physics appears (of course, this is just a toy theory, but the same thing happens in real theories!)

# Beyond the infinities



$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \left(\frac{1}{(p_1 - p_3)^2 - m_C^2}\right)^2 \int \frac{1}{((p_1 - p_3 - q)^2 - m_A^2)\,(q^2 - m_B^2)} d^4q$$




There **are** modifications to the matrix element that are not infinite that do provide corrections to m and g. Implies that there is energy dependence to masses and couplings! (See running of the couplings from earlier in the semester)