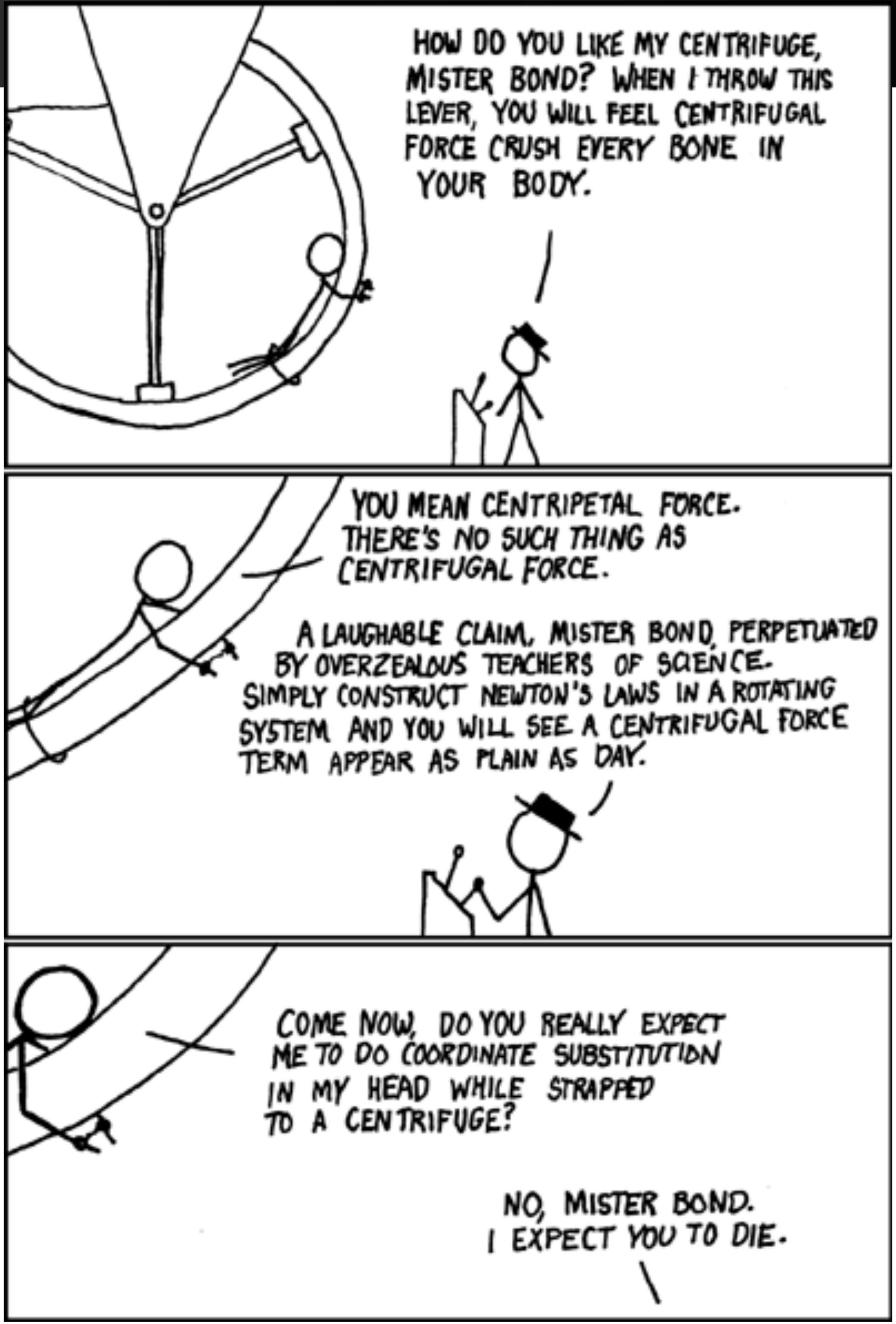
Non-inertial frames

On to a new topic

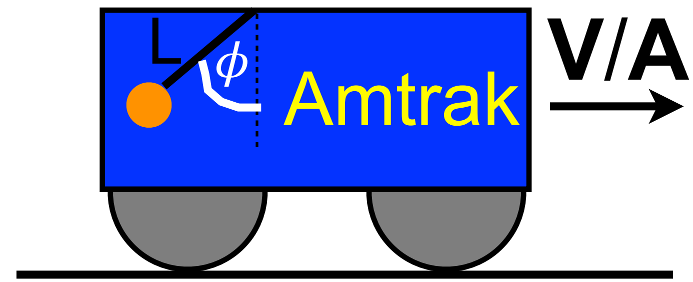
For fans of xkcd and 007



Starting from non-rotating non-inertial frames

Standing on the ground
(in inertial frame):

$$m\ddot{\mathbf{r}}_0 = \mathbf{F}$$



The velocity $\mathbf{v}_0$ of the pendulum
as seen from the ground:

$$\mathbf{v}_0 = \mathbf{v} + \mathbf{V}$$

$$\dot{\mathbf{v}}_0 = \dot{\mathbf{v}} + \dot{\mathbf{V}}$$

$$\dot{\mathbf{v}} = \dot{\mathbf{v}}_0 - \mathbf{A}$$

v is velocity as seen in the train

$$m\dot{\mathbf{v}} = m\dot{\mathbf{v}}_0 - m\mathbf{A}$$

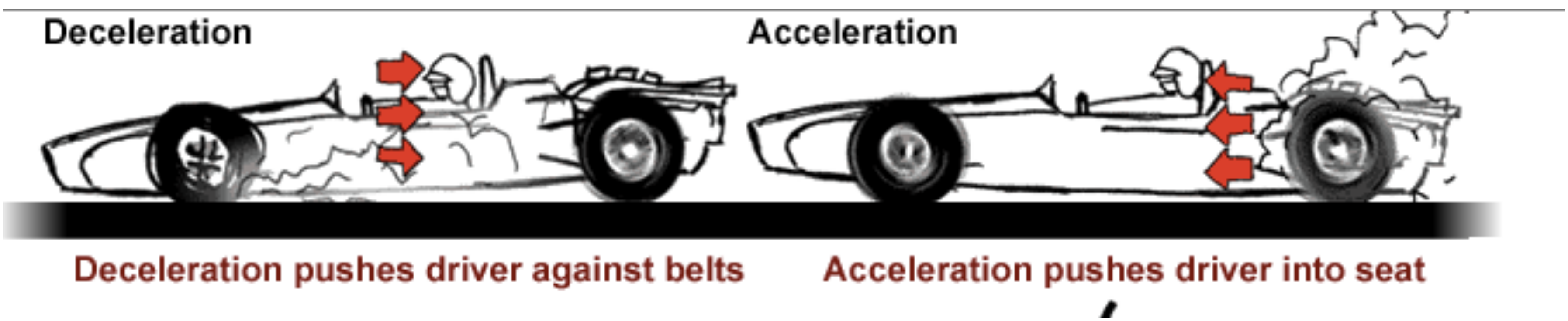
$$m\dot{\mathbf{v}} = \mathbf{F} - m\mathbf{A}$$

$$\longrightarrow$$

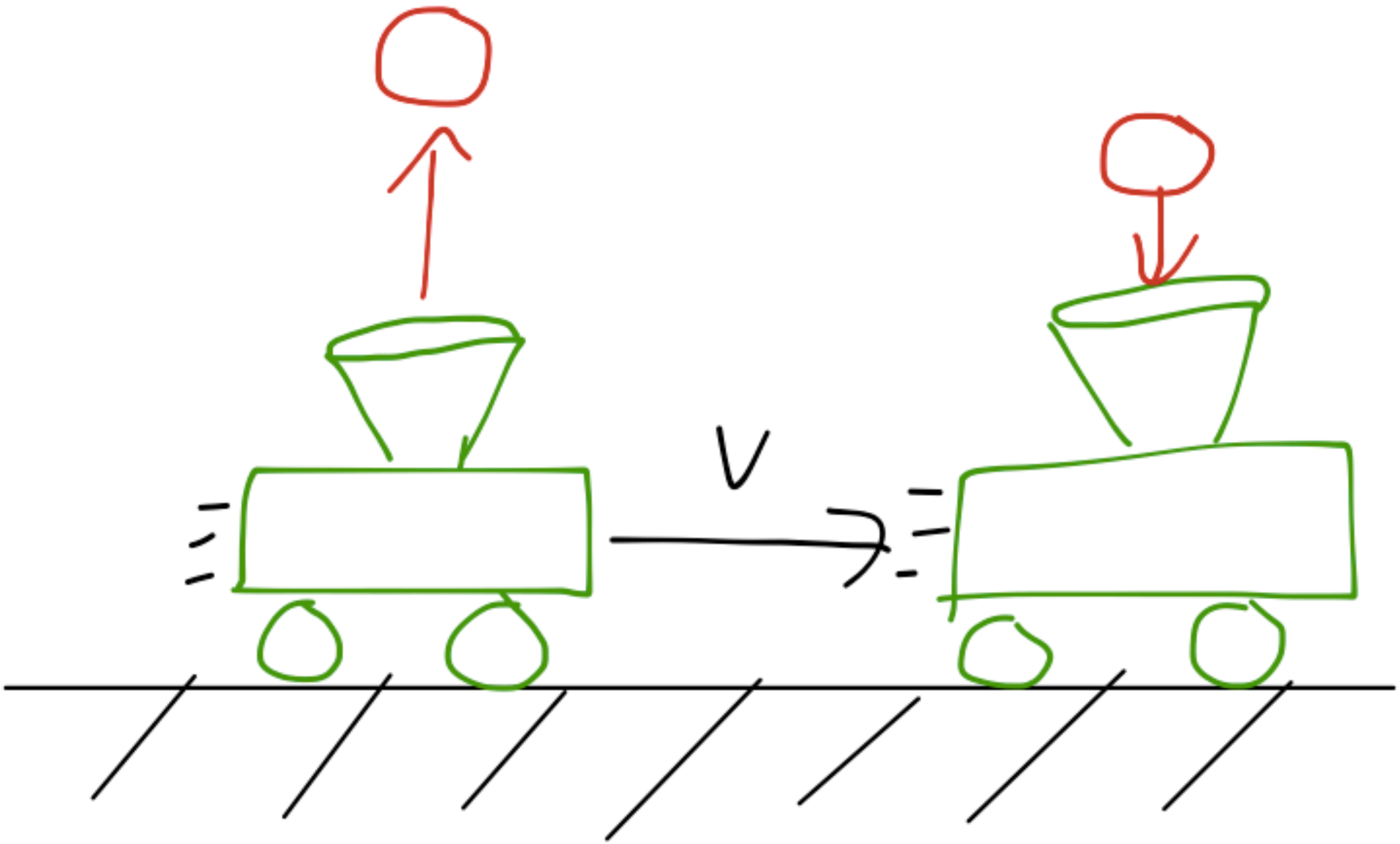
Note minus sign

$$\mathbf{F}_{inertial} = -m\mathbf{A}$$

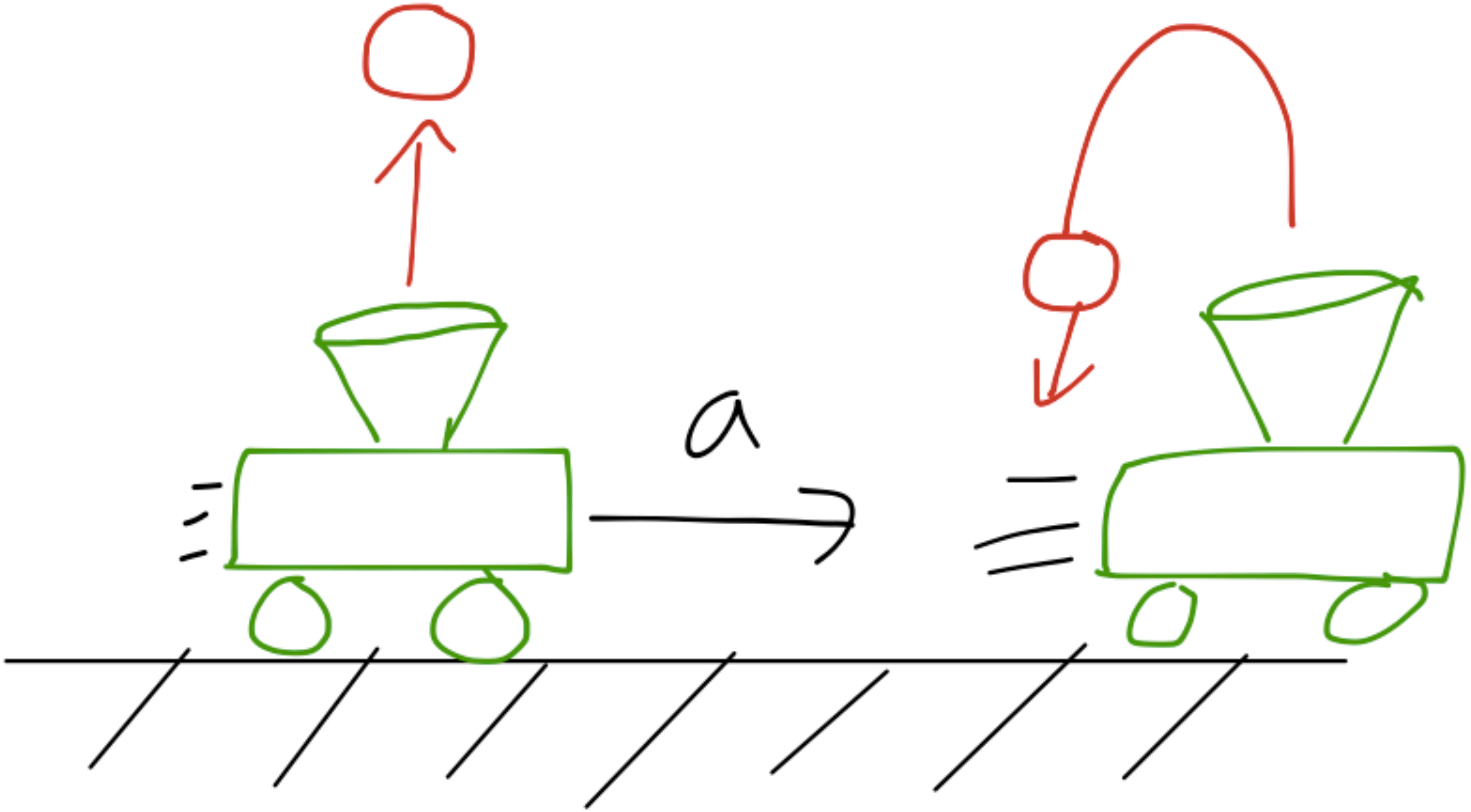
Starting from non-rotating non-inertial frames



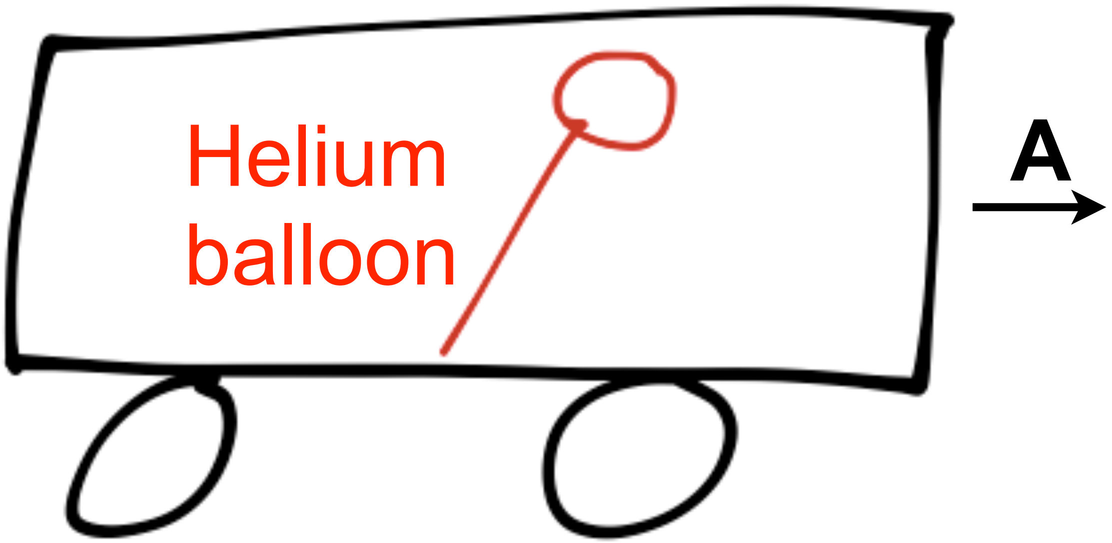
Starting from non-rotating non-inertial frames



Starting from non-rotating non-inertial frames



A fun side problem

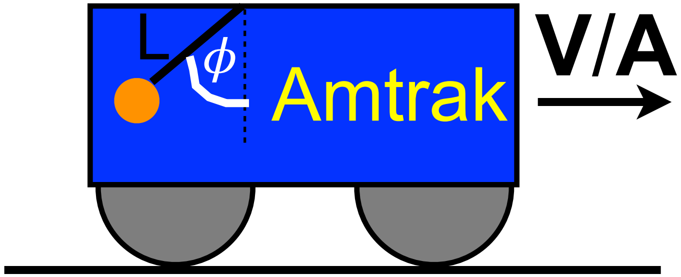


Air gets accelerated too... pressure gradient pushes balloon to the right

Let's work out Example 9.1

First let's follow the book...
and then what happens
in the Lagrangian formulation?

Let's work out Example 9.1 in an alternate way



$$x = -l\sin\theta + \frac{1}{2}at^2$$

$$y = l(1-\cos\theta)$$

$$\dot{x} = -l\cos\theta\dot{\theta} + at$$

$$\dot{x}^2 = l^2\cos^2\theta\dot{\theta}^2 + a^2t^2 - 2atl\cos\theta\dot{\theta}$$

$$\dot{y} = l\sin\theta\dot{\theta}$$

$$\dot{y}^2 = l^2\sin^2\theta\dot{\theta}^2$$

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy$$

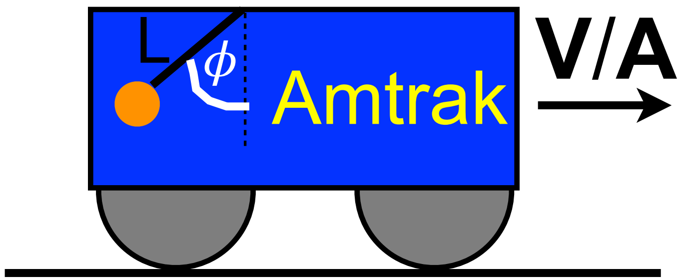
$$\mathcal{L} = \frac{m}{2}(l^2\dot{\theta}^2 + a^2t^2 - 2atl\cos\theta\dot{\theta}) - mgl(1-\cos\theta)$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = ml^2\dot{\theta} - matl\cos\theta$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = matl\sin\theta\dot{\theta} - mgl\sin\theta$$

$$\frac{d}{dt}\left(ml^2\dot{\theta} - matl\cos\theta\right) = matl\sin\theta\dot{\theta} - mgl\sin\theta$$

Let's work out Example 9.1 in an alternate way



$$\frac{d}{dt}\left(ml^2\dot{\theta} - matl\cos\theta\right) = matl\sin\theta\dot{\theta} - mgl\sin\theta$$

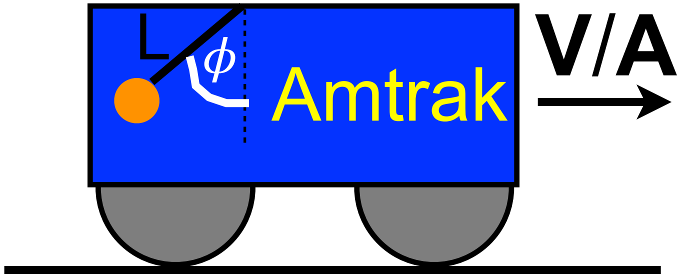
$$ml^2\ddot{\theta} + matl\sin\theta\dot{\theta} - mal\cos\theta = matl\sin\theta\dot{\theta} - mgl\sin\theta$$

$$ml^2\ddot{\theta} - mal\cos\theta = -mgl\sin\theta$$

$$l^2\ddot{\theta} = al\cos\theta - gl\sin\theta$$

$$\ddot{\theta} = (a/l)\cos\theta - (g/l)\sin\theta$$

What's the equilibrium?



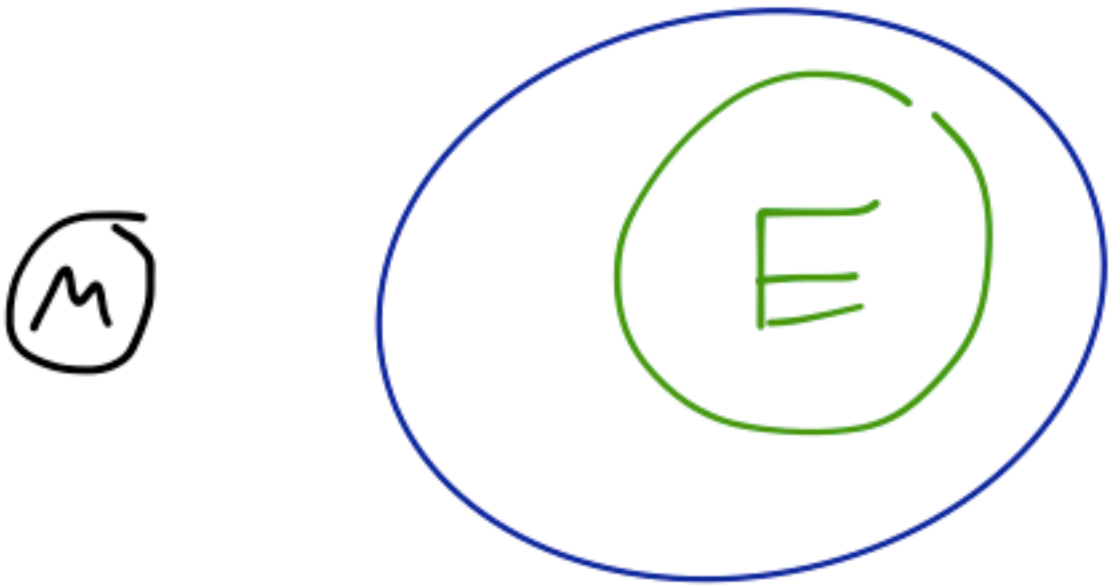
$$\ddot{\theta} = 0 = (a/l)\cos\theta - (g/l)\sin\theta \rightarrow$$

$$g\sin\theta = a\cos\theta$$

$$\tan\theta = a/g$$

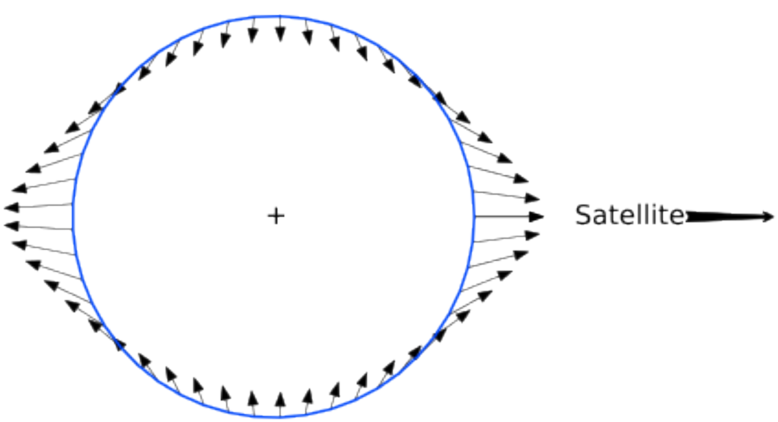
Let's move on to something more tricky - the tides

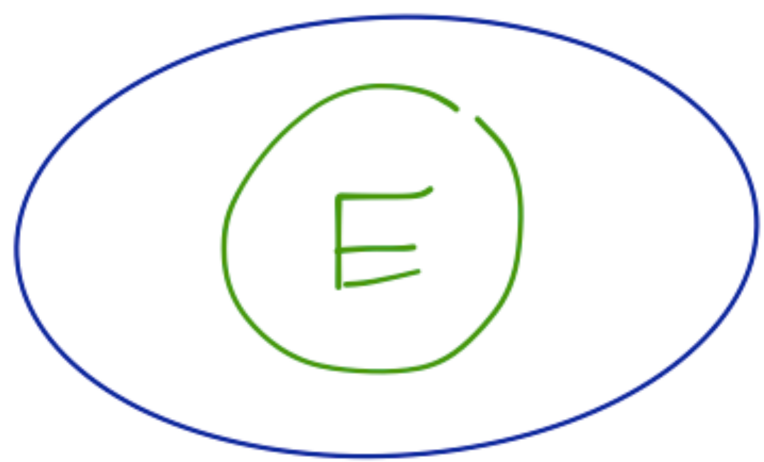
This picture is wrong! There are ~2 high tides per day. Tides are not due to the moon "pulling" at the water



Tides (correct)

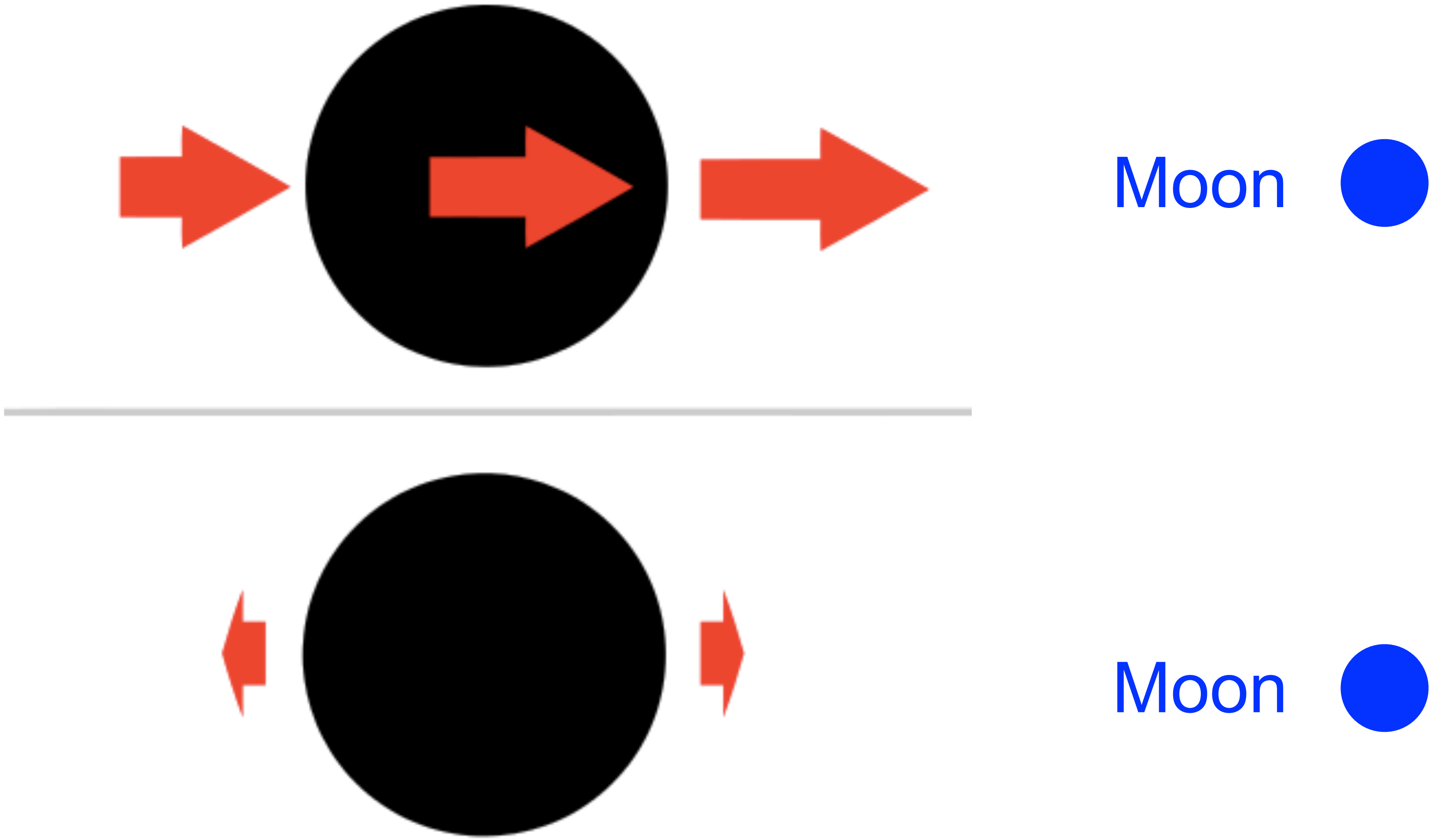
Tides are due to the differential force between the moon/sun and the earth's center of mass vs the moon/sun and the water, which is not at the center of mass. But CoM and the water are both accelerating! In other words, tides are due to the difference in inertial force vs position





In other words

Gravitational attraction towards moon



Difference between force at CoM

Let's calculate this using a non-inertial frame

$$m\dot{\mathbf{v}} = \mathbf{F} - m\mathbf{A}$$

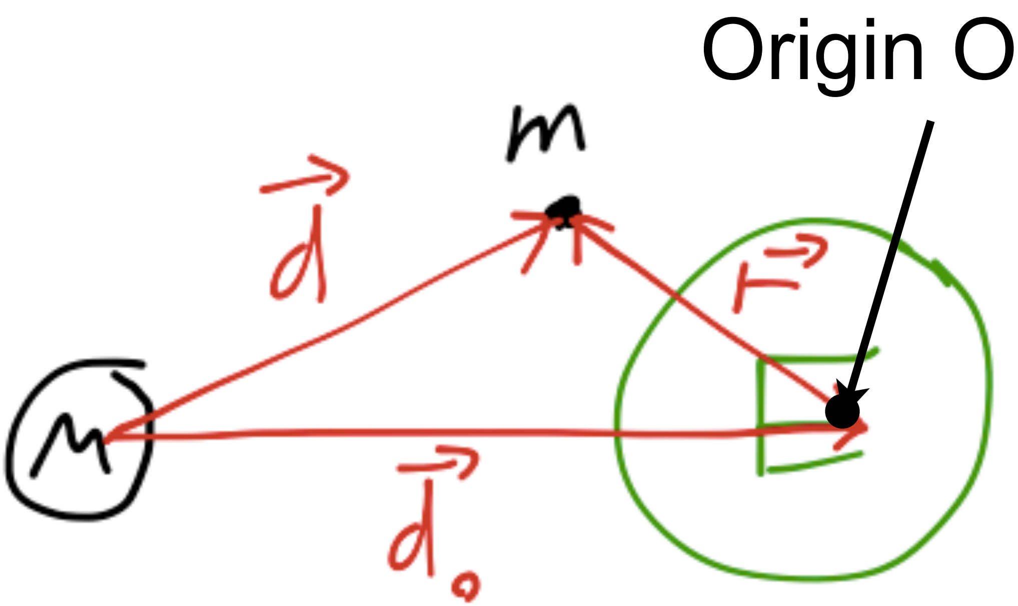
$$m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{A}$$

standard attraction to earth

$$mg\hat{\mathbf{r}}$$

$$\frac{-GM_m m}{d^2}\hat{\mathbf{d}}$$

standard grav. attraction to moon

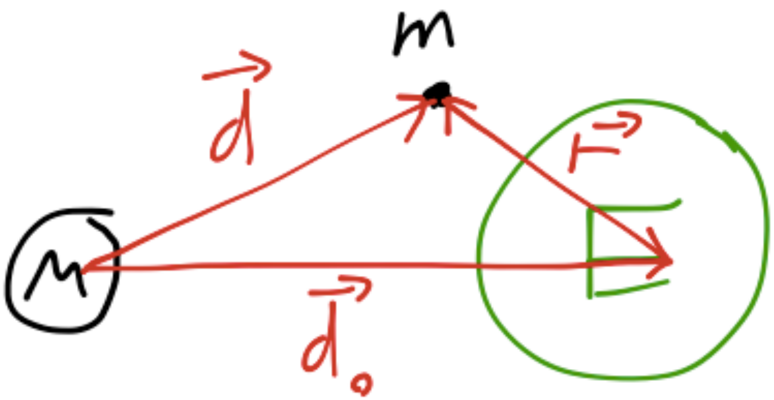


$$\mathbf{A} = -GM_m \frac{\hat{\mathbf{d}}_0}{d_0^2}$$

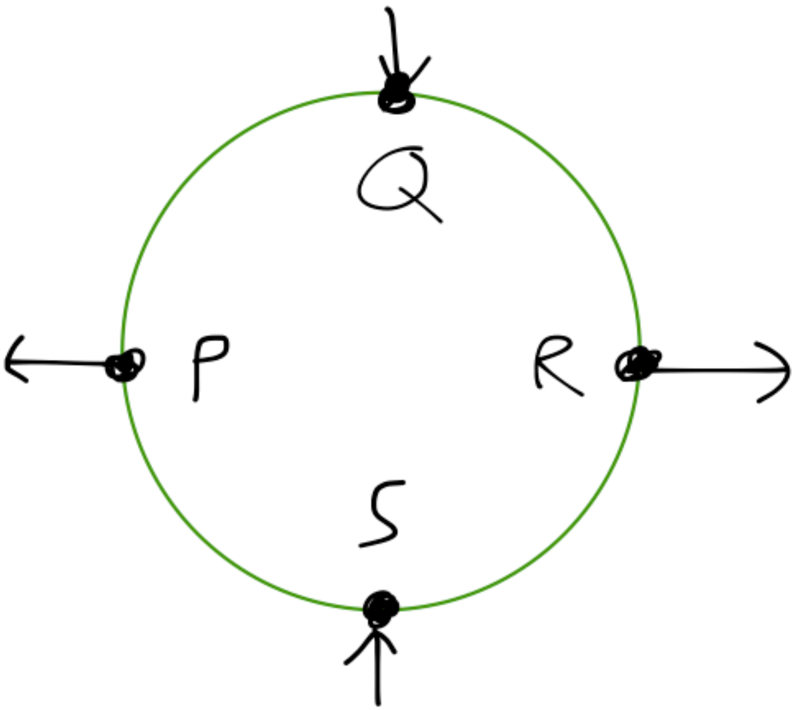
Centripetal acceleration of frame

$$m\ddot{\mathbf{r}} = mg\hat{\mathbf{r}} - GM_m m\frac{\hat{\mathbf{d}}}{d^2} + GM_m m\frac{\hat{\mathbf{d}}_0}{d_0^2}$$

Let's calculate this using a non-inertial frame

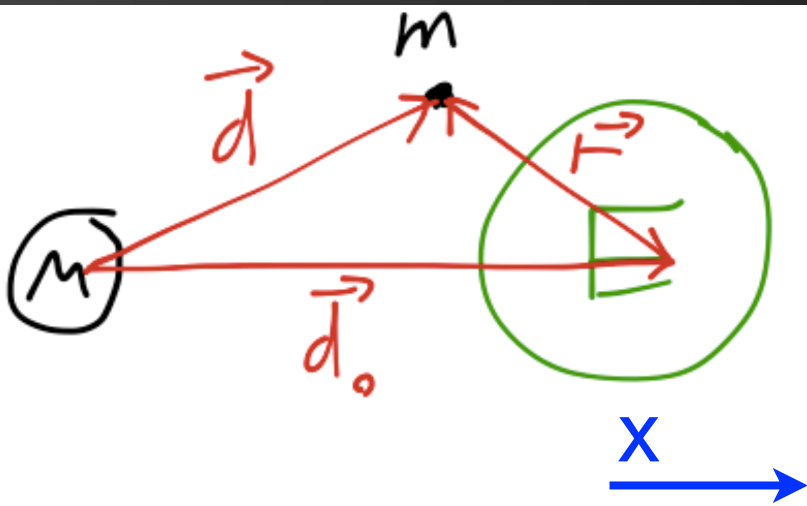


$$m\ddot{\mathbf{r}} = mg\hat{\mathbf{r}} - GM_m m\frac{\hat{\mathbf{d}}}{d^2} + GM_m m\frac{\hat{\mathbf{d}}_0}{d_0^2}$$



$$F_{tid} = GM_m m\left(\frac{\hat{\mathbf{d}}_0}{d_0^2} - \frac{\hat{\mathbf{d}}}{d^2}\right)$$

What is the potential energy associated with the tidal force?



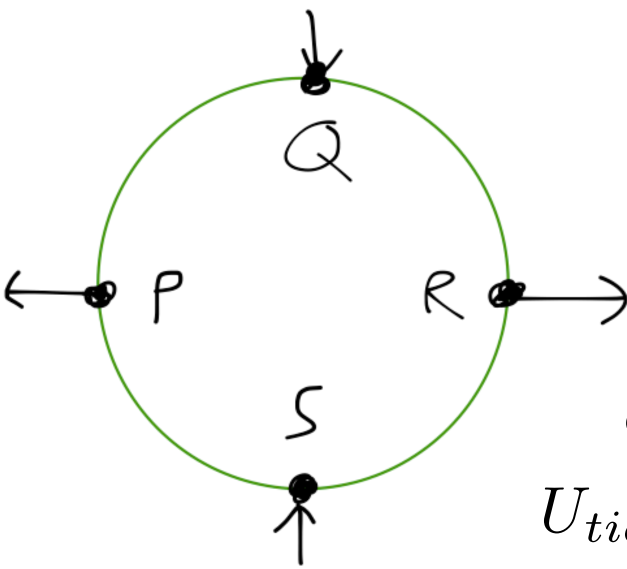
$$F_{tid} = GM_m m \left(\frac{\hat{\mathbf{d}}_0}{d_0^2} - \frac{\hat{\mathbf{d}}}{d^2} \right)$$

$$F_{tid} = -\nabla U_{tid}$$

$$U_{tid} = -GM_m m \left(\frac{x}{d_0^2} + \frac{1}{d} \right)$$

Is this clear?

Ocean surface is equipotential

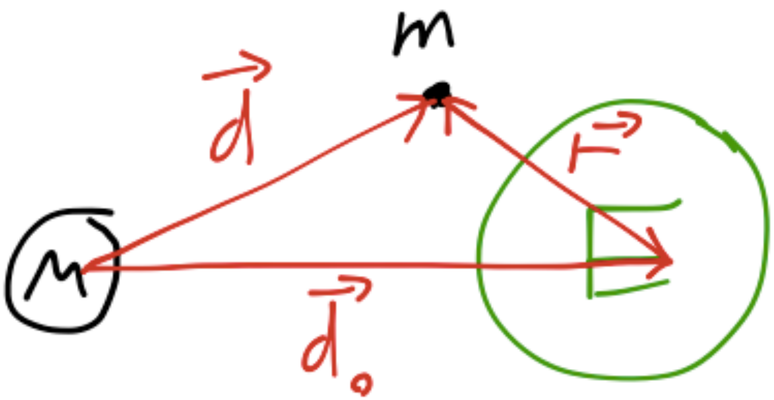


h is height difference between high and low tide

$$U_{tid}(P) + U_{grav}(P) = U_{tid}(Q) + U_{grav}(Q)$$

$$U_{tid}(P) - U_{tid}(Q) = U_{grav}(Q) - U_{grav}P = mgh$$

What is the potential energy associated with the tidal force?



$$U(P) - U(Q) = -mgh$$

$$U_{tid} = -GM_m m\left(\frac{x}{d_0^2} + \frac{1}{d}\right)$$

At point Q, $d = \sqrt{d_0^2 + r^2} \sim \sqrt{d_0^2 + R_e^2}$

$$U_{tid}(Q) = -GM_m m\left(\frac{x}{d_0^2} + \frac{1}{\sqrt{d_0^2 + R_e^2}}\right)$$

x(Q) = 0

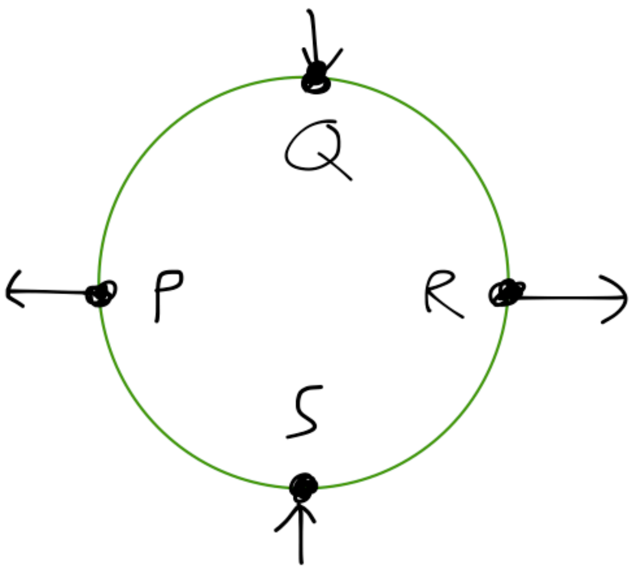
That clear?

$$U_{tid}(Q) = -GM_m m\left(\frac{1}{\sqrt{d_0^2 + R_e^2}}\right)$$

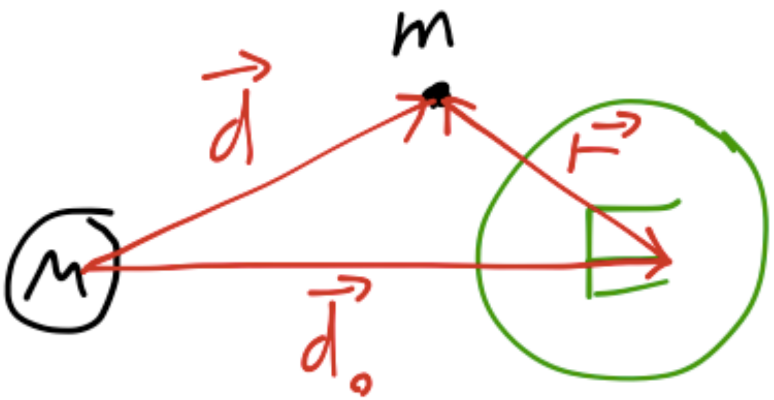
$$U_{tid}(Q) = -\frac{GM_m m}{d_0}\left(\frac{1}{\sqrt{1 + (R_e/d_0)^2}}\right)$$

$$U_{tid}(Q) = -\frac{GM_m m}{d_0}\left(1 + (R_e/d_0)^2\right)^{-1/2}$$

$$U_{tid}(Q) = -\frac{GM_m m}{d_0}\left(1 - \frac{R_e^2}{2d_0^2}\right)$$

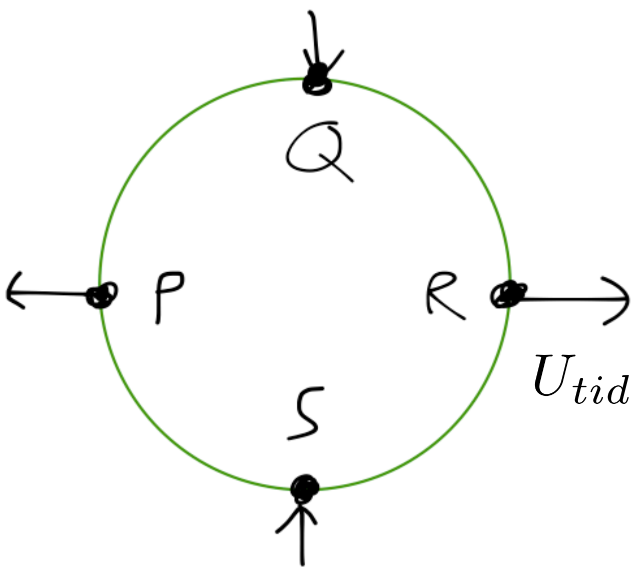


What is the potential energy associated with the tidal force?



$$U(P) - U(Q) = -mgh$$

$$U_{tid} = -GM_m m\left(\frac{x}{d_0^2} + \frac{1}{d}\right)$$



At point P, $d = d_0 - R_e, x = -R_e$

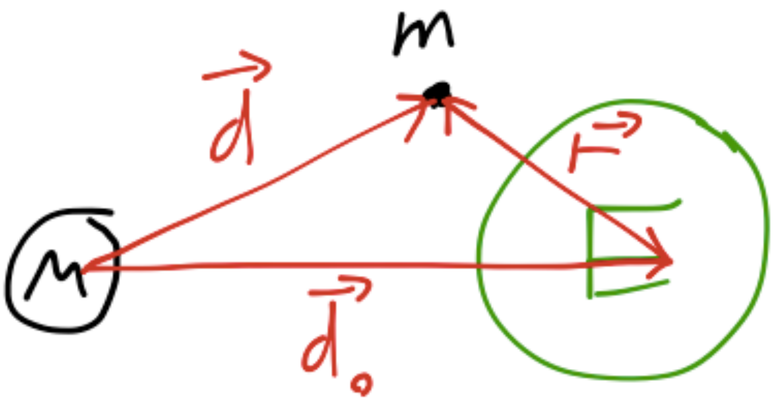
$$U_{tid}(P) = -GM_m m\left(\frac{-R_e}{d_0^2} + \frac{1}{d_0 - R_e}\right)$$

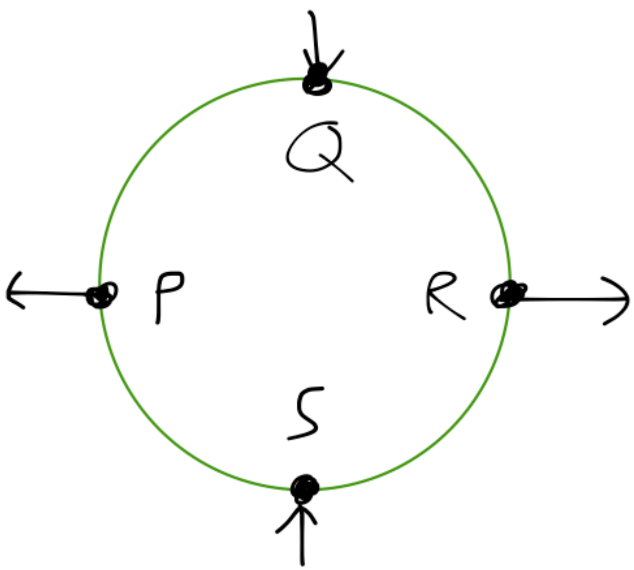
$$U_{tid}(P) = -\frac{GM_m m}{d_0}\left(\frac{-R_e}{d_0} + \frac{1}{1 - R_e/d_0}\right)$$

$$U_{tid}(P) = -\frac{GM_m m}{d_0}\left(\frac{-R_e}{d_0} + 1 + R_e/d_0 + (R_e/d_0)^2\right)$$

$$U_{tid}(P) = -\frac{GM_m m}{d_0}\left(1 + (R_e/d_0)^2\right)$$

Putting it together





$$U(P) - U(Q) = -mgh$$

$$U(P) - U(Q) = -mgh$$

$$U(P) = -\frac{GM_m m}{d_0}(1 + \frac{R_e^2}{d_0^2})$$

$$U(Q) = -\frac{GM_m m}{d_0}(1 - \frac{R_e^2}{2d_0^2})$$

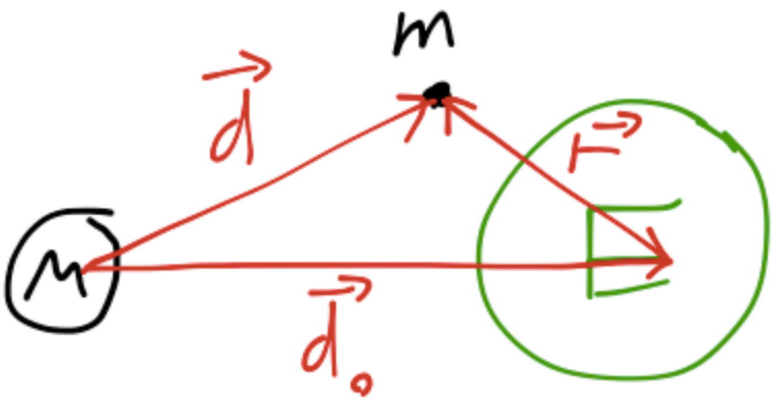
$$mgh = \frac{GM_m m}{d_0}\left(\frac{R_e^2}{d_0^2} + \frac{R_e^2}{2d_0^2}\right)$$

$$gh = \frac{GM_m R_e^2}{d_0^3}(1 + 1/2)$$

$$h = \frac{3GM_m R_e^2}{2gd_0^3}$$

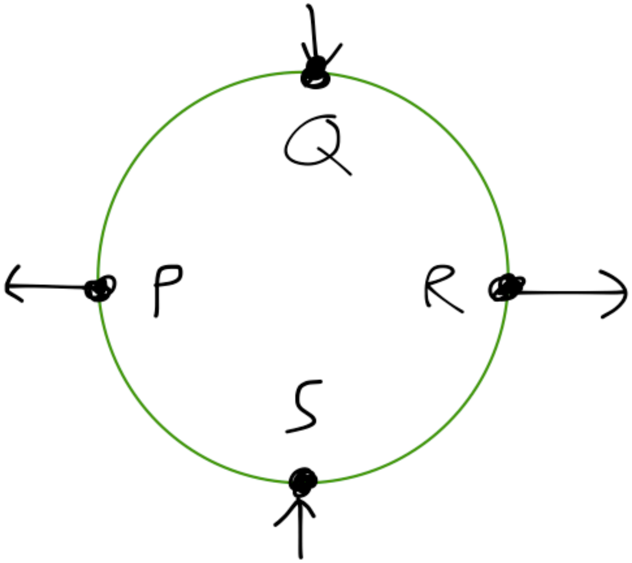
$$g = GM_e/R_e^2 \rightarrow \boxed{h = \frac{3}{2}\frac{M_m}{M_e}\frac{R_e^4}{d_0^3}}$$

Plugging in the numbers



$$h = \frac{3}{2}\frac{M_m}{M_e}\frac{R_e^4}{d_0^3}$$

h = 54 cm (moon)
h = 25 cm (sun)



Of course, this is a simplification, including land, seasonal and depth effects (not to mention wind)

On to frames with rotation

On to frames with rotation

Don't forget the right-hand rule

$$\vec{\omega} = \omega \vec{u}$$

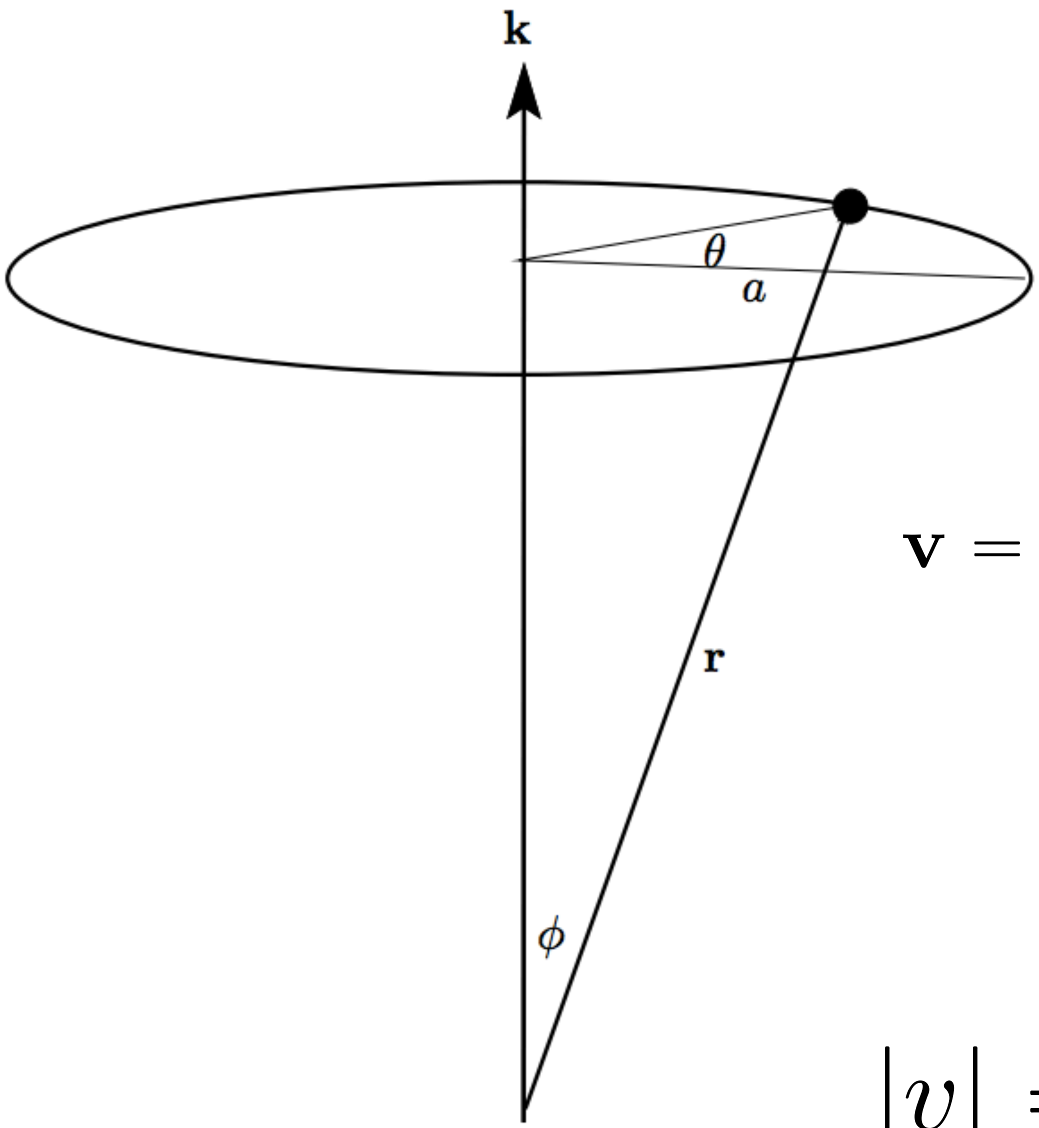
$\vec{\omega}$

$\vec{u}$

ω

Can imagine situations where any of these are constant, or functions of time

On to frames with rotation



$$\mathbf{r} = (a\cos\theta, a\sin\theta, h)$$

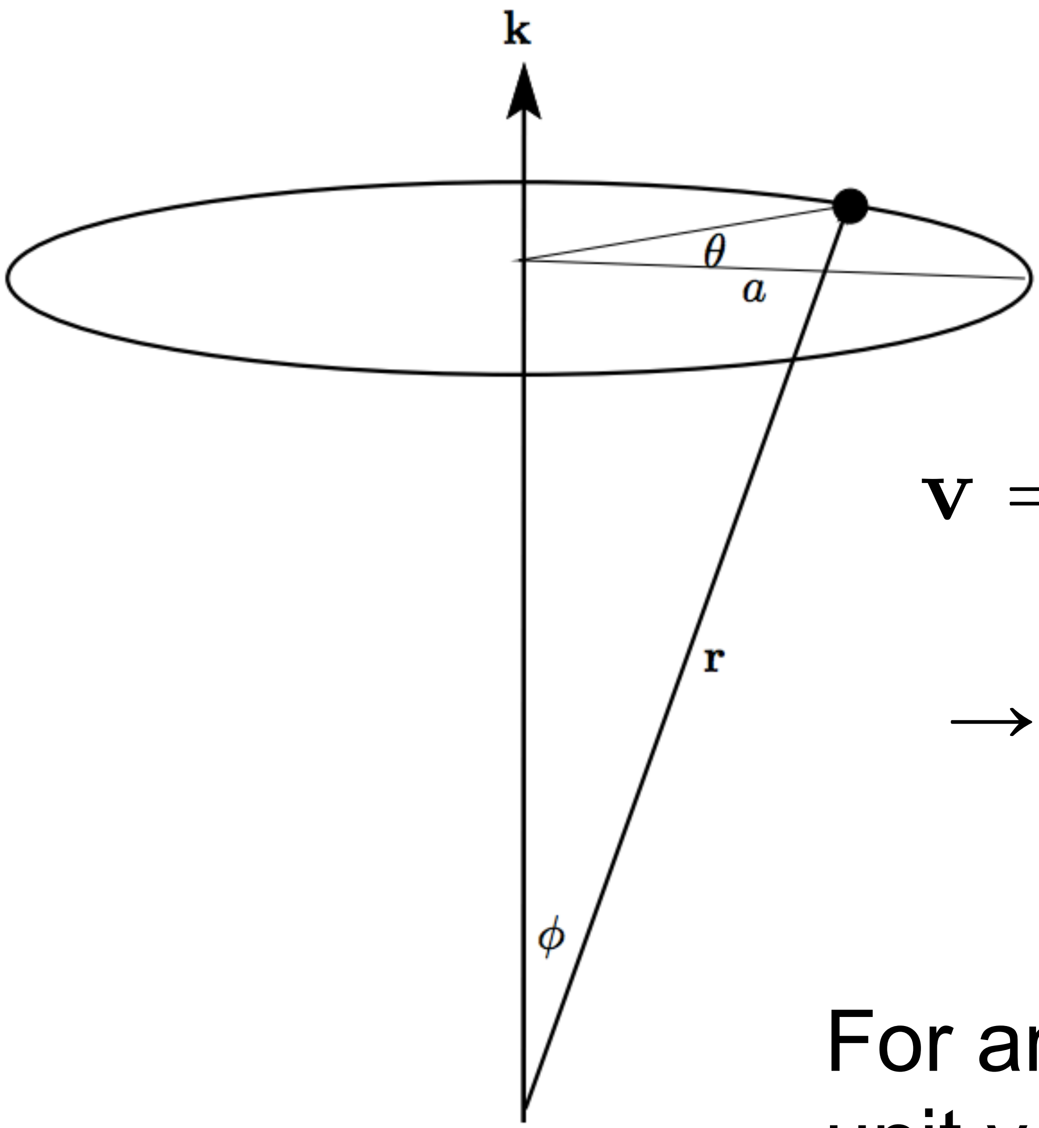
$$\mathbf{v} = \dot{\mathbf{r}} = (-a\dot{\theta}\sin\theta, a\dot{\theta}\cos\theta, 0)$$

$$\vec{\omega} = \dot{\theta}\hat{z}$$

$$\mathbf{v} = \vec{\omega} \times \mathbf{r}$$

$$|v| = \omega r \text{ if } \vec{\omega}\cdot\mathbf{r} = 0$$

We can use this to say more



$$\mathbf{v} = \vec{\omega} \times \mathbf{r} = \frac{d\mathbf{r}}{dt}$$

$$\rightarrow \frac{d\mathbf{e}}{dt} = \vec{\omega} \times \mathbf{e}$$

For any vector **e**, including unit vectors

Addition of angular velocities

$\mathbf{v}_{ij}$ = velocity of frame i relative to frame j

$$\mathbf{v}_{31} = \mathbf{v}_{32} + \mathbf{v}_{21}$$

$$\vec{\omega}_{31} \times \mathbf{r} = \vec{\omega}_{32} \times \mathbf{r} + \vec{\omega}_{21} \times \mathbf{r}$$

$$\vec{\omega}_{31} \times \mathbf{r} = (\vec{\omega}_{32} + \vec{\omega}_{21}) \times \mathbf{r}$$

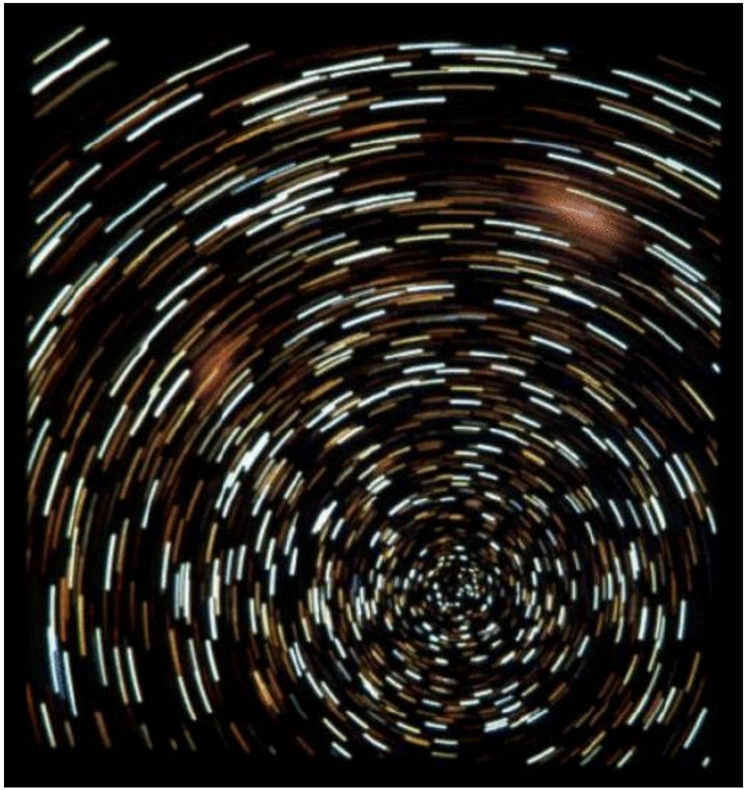
$$\rightarrow \vec{\omega}_{31} = \vec{\omega}_{32} + \vec{\omega}_{21}$$

$\boldsymbol{\omega}_{ij}$ = angular velocity of frame i relative to frame j

Vectors add just as translational vectors

Consider an inertial frame of references defined by S_0 and a second frame (of interest) S with shared origin, but rotating with respect to S_0 with angular velocity $\boldsymbol{\Omega}$

For example (Taylor) O has origin at enter of earth, S_0 is axes fixed to distant stars, S is non-inertial (earth rotates)



Earth's angular speed of rotation:

$$\Omega = 7.292\times10^{-5}\,s^{-1}$$

Time derivatives in a rotating frame

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{S_0} = \text{rate of change of } \mathbf{Q} \text{ relative to inertial frame } S_0$$

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{S} = \text{rate of change of } \mathbf{Q} \text{ relative to rotating frame } S$$

$$\mathbf{Q} = Q_1\mathbf{e}_1 + Q_2\mathbf{e}_2 + Q_3\mathbf{e}_3 = \sum_{i=1}^{i=3} Q_i\mathbf{e}_i$$

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{S} = \sum_i \frac{dQ_i}{dt}\mathbf{e}_i$$

valid in both frames, though $\mathbf{e}_i$ are constant in S but not in S_0

Since $\mathbf{e}_i$ are constant in S

Time derivatives in a rotating frame

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{S_0} = \text{rate of change of } \mathbf{Q} \text{ relative to inertial frame } S_0$$

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{S} = \text{rate of change of } \mathbf{Q} \text{ relative to rotating frame } S$$

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{S_0} = \sum_i \frac{dQ_i}{dt}\mathbf{e}_i + \sum_i Q_i \left(\frac{d\mathbf{e}_i}{dt}\right)_{S_0}$$

Recall: $\frac{d\mathbf{e}}{dt} = \vec{\Omega} \times \mathbf{e}$

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{S_0} = \sum_i \frac{dQ_i}{dt}\mathbf{e}_i + \sum_i Q_i(\vec{\Omega} \times \mathbf{e_i})$$

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{S_0} = \sum_i \frac{dQ_i}{dt}\mathbf{e}_i + \vec{\Omega} \times \sum_i Q_i\mathbf{e_i}$$

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{S_0} = \sum_i \frac{dQ_i}{dt}\mathbf{e}_i + \vec{\Omega} \times \mathbf{Q}$$

$$\boxed{\left(\frac{d\mathbf{Q}}{dt}\right)_{S_0} = \left(\frac{d\mathbf{Q}}{dt}\right)_{S} + \vec{\Omega} \times \mathbf{Q}}$$

A quick exercise

Let's do
Problem 9.7
together

Now we can move back to Newton's Laws

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{S_0} = \left(\frac{d\mathbf{Q}}{dt}\right)_{S} + \vec{\Omega} \times \mathbf{Q}$$

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S_0} = \mathbf{F}$$

$$\left(\frac{d\mathbf{r}}{dt}\right)_{S_0} = \left(\frac{d\mathbf{r}}{dt}\right)_{S} + \vec{\Omega} \times \mathbf{r}$$

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S_0}\left(\frac{d\mathbf{r}}{dt}\right)_{S_0}$$

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S_0}\left[\left(\frac{d\mathbf{r}}{dt}\right)_{S} + \vec{\Omega} \times \mathbf{r}\right]$$

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S}\left[\left(\frac{d\mathbf{r}}{dt}\right)_{S} + \vec{\Omega} \times \mathbf{r}\right] + \vec{\Omega} \times \left[\left(\frac{d\mathbf{r}}{dt}\right)_{S} + \vec{\Omega} \times \mathbf{r}\right]$$

Need some simplification

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_S \left[\left(\frac{d\mathbf{r}}{dt}\right)_S + \vec{\Omega} \times \mathbf{r}\right] + \vec{\Omega} \times \left[\left(\frac{d\mathbf{r}}{dt}\right)_S + \vec{\Omega} \times \mathbf{r}\right]$$

Evaluate in frame where **Ω** const, so d/dt(**Ω**) = 0

Dots are with respect to rotating frame S

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S_0} = \ddot{\mathbf{r}} + \vec{\Omega} \times \dot{\mathbf{r}} + \vec{\Omega} \times \left[\dot{\mathbf{r}} + \vec{\Omega} \times \mathbf{r}\right]$$

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S_0} = \ddot{\mathbf{r}} + 2\vec{\Omega} \times \dot{\mathbf{r}} + \vec{\Omega} \times (\vec{\Omega} \times \mathbf{r})$$

Putting it together

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S_0} = \mathbf{F} = m\ddot{\mathbf{r}} + 2m\vec{\Omega} \times \dot{\mathbf{r}} + m\vec{\Omega} \times (\vec{\Omega} \times \mathbf{r})$$

Changing cross product order cancels minus signs...

$$m\ddot{\mathbf{r}} = F + 2m\dot{\mathbf{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \mathbf{r}) \times \vec{\Omega}$$

Coriolis Force

Centrifugal Force

Coriolis force and centrifugal force

$$m\ddot{\mathbf{r}} = F + 2m\dot{\mathbf{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \mathbf{r}) \times \vec{\Omega}$$

Coriolis Force = 0
if v = 0

Centrifugal Force

$$m\ddot{\mathbf{r}} = F + 2m\dot{\mathbf{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \mathbf{r}) \times \vec{\Omega}$$

V ~ rΩ ~ speed of rotation on earth ~1000 mi/h

$$F_{cor} \sim mv\Omega$$

$$F_{cf} \sim mr\Omega^2$$

$$\frac{F_{cor}}{F_{cf}} \sim \frac{v}{r\Omega} \sim \frac{v}{V}$$

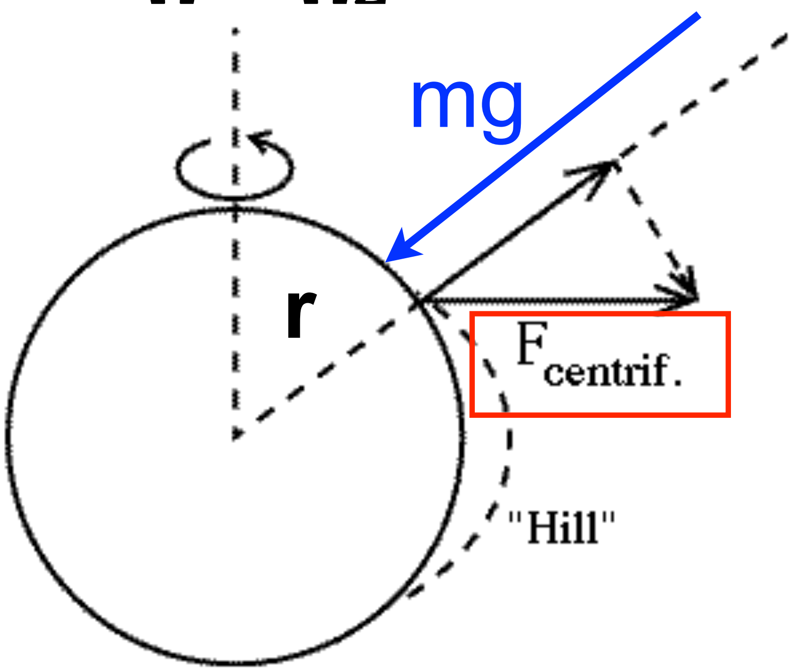
v ~ v as seen on rotating earth

A quick exercise

Let's do problem 9.8 together.
Tricky! Let's define a coordinate system. x = easterly, y = northerly, z = up (ie radially out).
What is Ω at equator? y direction
What is Ω near (but not at) the north pole? Mostly z direction, but a little y direction
What is r? Always z direction

Centrifugal force modifies gravity (tangential component towards equator and reduces overall magnitude)

$$\vec{\Omega} = \Omega\hat{\mathbf{z}}$$



On earth, **r** points from center (origin) to where we stand

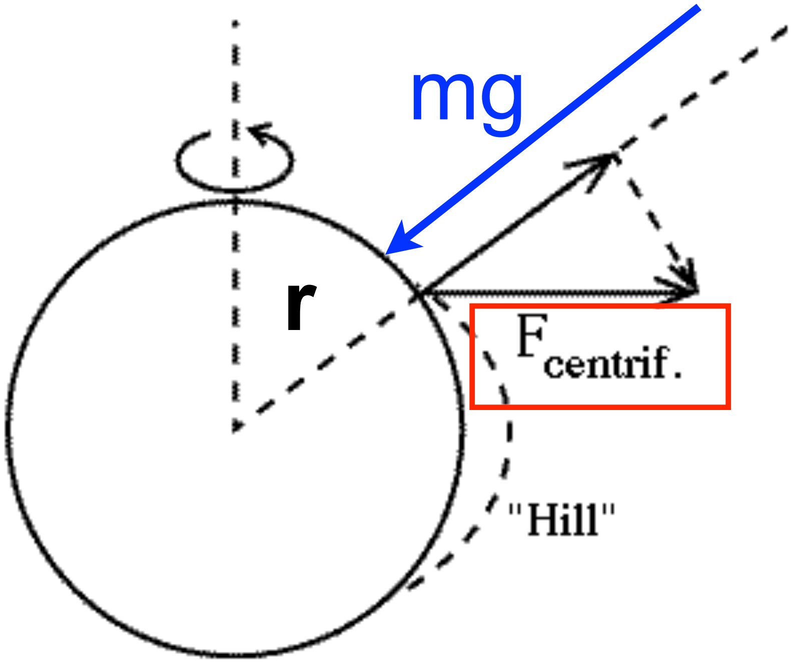
Centrifugal force

$$\mathbf{r} = r\hat{\mathbf{z}} \rightarrow$$

$$(\vec{\Omega} \times \mathbf{r}) \times \vec{\Omega} = r\Omega^2(\hat{\mathbf{z}} \times \hat{\mathbf{z}}) \times \hat{\mathbf{z}} = 0$$

At poles, no effect at all

$$\vec{\Omega} = \Omega\hat{\mathbf{z}}$$

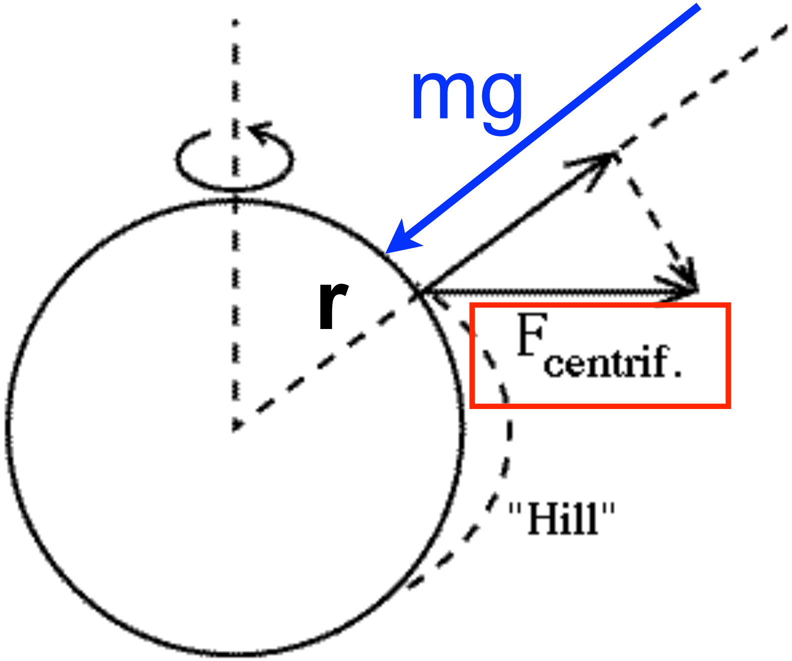


Centrifugal force

$$\mathbf{r} = r\hat{\mathbf{x}} \rightarrow$$

$$(\vec{\Omega} \times \mathbf{r}) \times \vec{\Omega} = r\Omega^2(\hat{\mathbf{z}} \times \hat{\mathbf{x}}) \times \hat{\mathbf{z}} = r\Omega^2\hat{\mathbf{y}} \times \hat{\mathbf{z}} = r\Omega^2\hat{\mathbf{x}} = \Omega^2\mathbf{r}$$

$$\vec{\Omega} = \Omega\hat{\mathbf{z}}$$

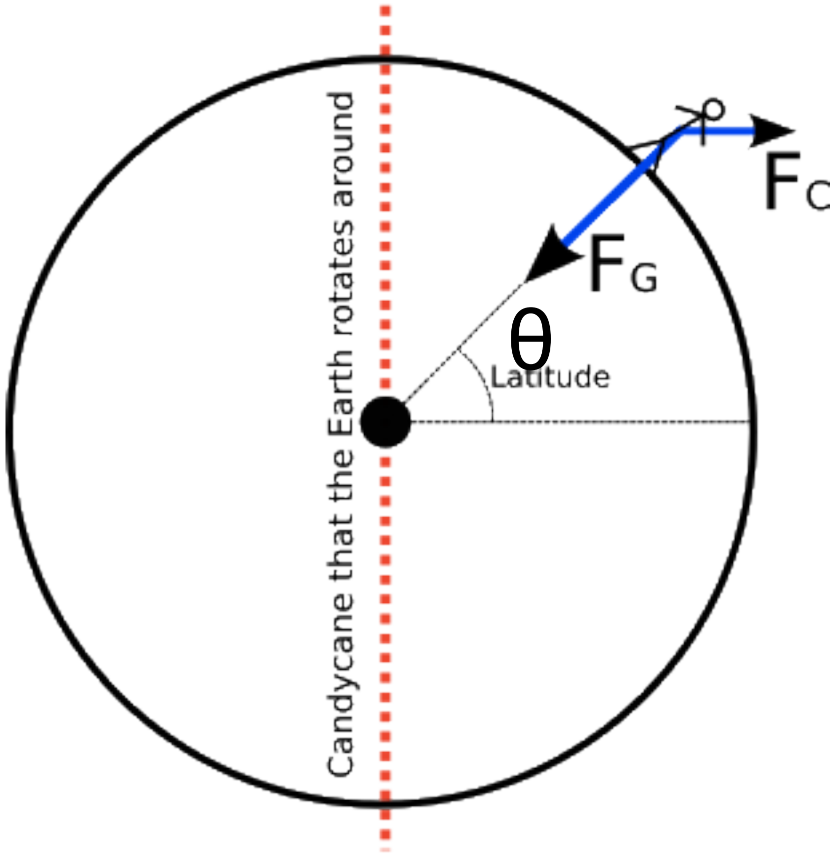


At equator, centrifugal force is in exact opposition to gravity, with magnitude $R\Omega^2$ = 0.3% of gravity

Centrifugal force

$$\mathbf{r} = r\hat{\mathbf{x}} \rightarrow$$

$$(\vec{\Omega} \times \mathbf{r}) \times \vec{\Omega} = r\Omega^2(\hat{\mathbf{z}} \times \hat{\mathbf{x}}) \times \hat{\mathbf{z}} = r\Omega^2\hat{\mathbf{y}} \times \hat{\mathbf{z}} = r\Omega^2\hat{\mathbf{x}} = \Omega^2\mathbf{r}$$

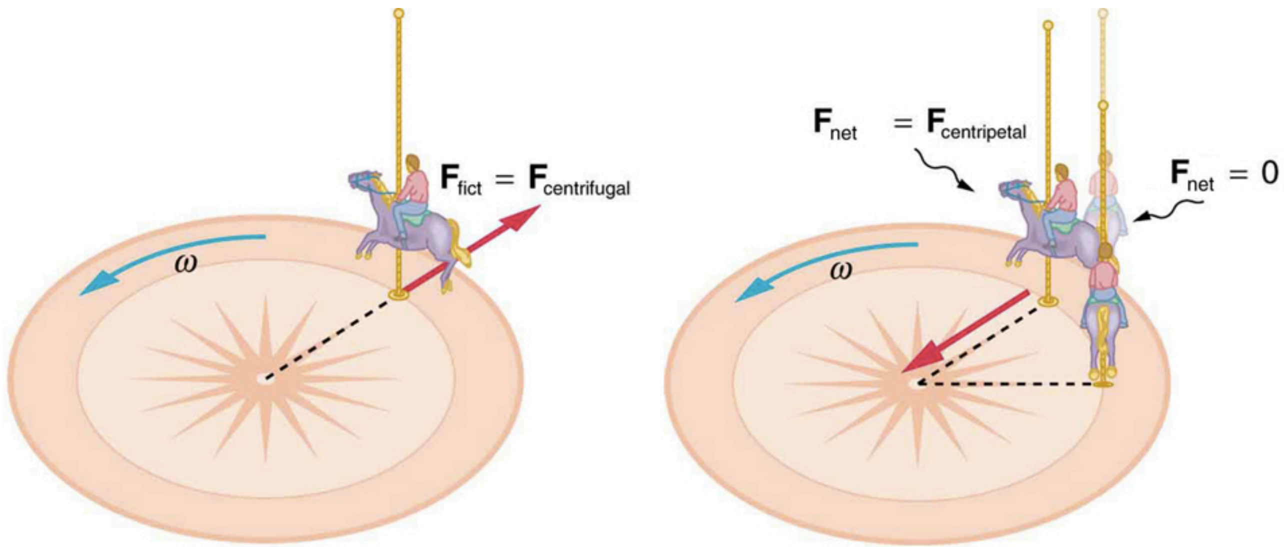


At equator, centrifugal force is in exact opposition to gravity, with magnitude $R\Omega^2$ = 0.3% of gravity

But off the equator, the direction changes, too ...

What is the centrifugal force?

Some might refer to it as “fictitious” but that is a bit unfair to it! It’s a result of the inertia of the system as it is continually accelerated in a rotating system. It draws the system away from the center of rotation (ie away from Earth, for example)



Merry-go-round’s rotating frame of reference
(a)

Inertial frame of reference
(b)

More familiar formulas:

$$\mathbf{v} = \vec{\Omega} \times \mathbf{r}$$

$$v = \Omega r$$

$$\mathbf{F}_{cf} = mv^2/r$$

On earth, **r** points from center (origin) to where we stand

$$\vec{\Omega} = \Omega \hat{\mathbf{z}}$$

$$\mathbf{r} = r\hat{\mathbf{x}} \rightarrow$$

$$(\vec{\Omega} \times \mathbf{r}) \times \vec{\Omega} = r\Omega^2(\hat{\mathbf{z}} \times \hat{\mathbf{x}}) \times \hat{\mathbf{z}} = r\Omega^2\hat{\mathbf{y}} \times \hat{\mathbf{z}} = r\Omega^2\hat{\mathbf{x}} = \Omega^2\mathbf{r}$$

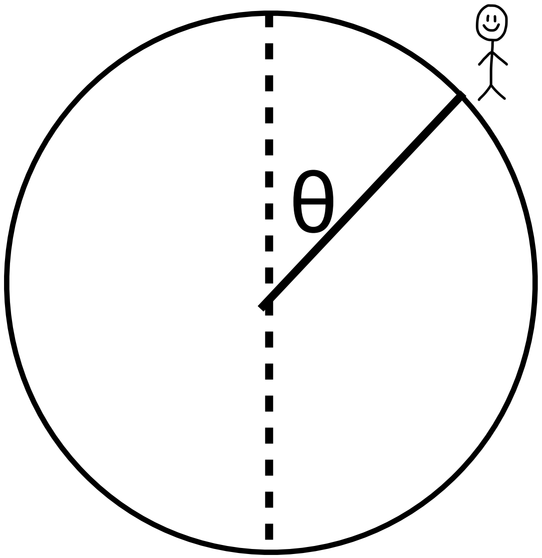
$$\mathbf{r} = r\hat{\mathbf{y}} \rightarrow$$

$$(\vec{\Omega} \times \mathbf{r}) \times \vec{\Omega} = r\Omega^2(\hat{\mathbf{z}} \times \hat{\mathbf{y}}) \times \hat{\mathbf{z}} = -r\Omega^2\hat{\mathbf{x}} \times \hat{\mathbf{y}} = r\Omega^2\hat{\mathbf{y}} = \Omega^2\mathbf{r}$$

$$\mathbf{r} = r\hat{\mathbf{z}} \rightarrow$$

$$(\vec{\Omega} \times \mathbf{r}) \times \vec{\Omega} = r\Omega^2(\hat{\mathbf{z}} \times \hat{\mathbf{z}}) \times \hat{\mathbf{z}} = 0$$

Angle between “apparent” and “true” gravity



α=Angle between combined force and pure gravity

$$\mathbf{F}_{grav} = (-g\sin\theta, 0, -g\cos\theta)$$

$$\mathbf{F}_{cen} = (\omega^2 R\sin\theta, 0, 0)$$

$$\mathbf{F} = (-g\sin\theta + \omega^2 R\sin\theta, 0, -g\cos\theta)$$

$$\mathbf{F}\cdot\mathbf{F}_{grav} = |\mathbf{F}_{grav}||\mathbf{F}|\cos\alpha$$

$$\mathbf{F}\cdot\mathbf{F}_{grav} = g^2\sin^2\theta - gR\omega^2\sin^2\theta + g^2\cos^2\theta$$

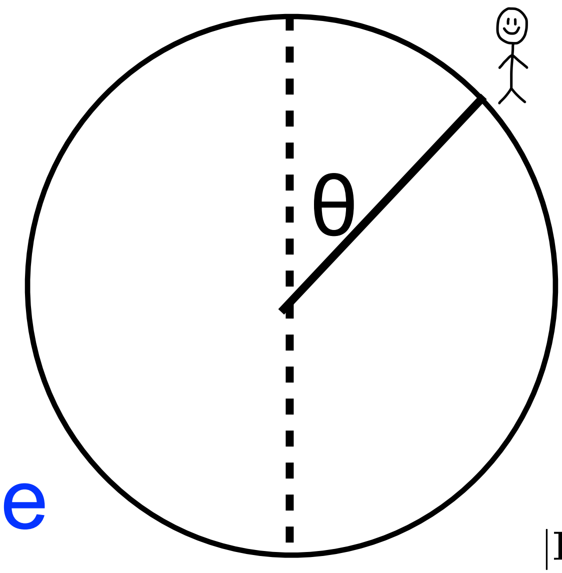
$$\mathbf{F}\cdot\mathbf{F}_{grav} = g^2 - gR\omega^2\sin^2\theta$$

$$|\mathbf{F}_{grav}| = g$$

$$|\mathbf{F}| = \sqrt{g^2\sin^2\theta + \omega^4 R^2\sin^2\theta - 2gR\omega^2\sin^2\theta + g^2\cos^2\theta}$$

$$|\mathbf{F}| = \sqrt{g^2 + \omega^4 R^2\sin^2\theta - 2gR\omega^2\sin^2\theta}$$

Angle between "apparent" and "true" gravity

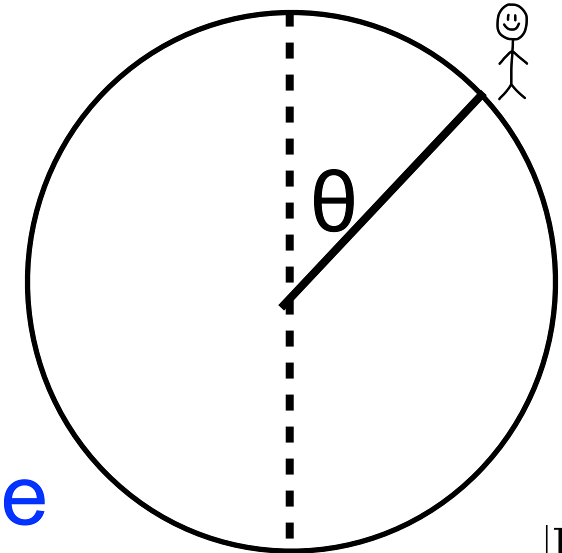


$$\mathbf{F}_{grav} = (-g\sin\theta, 0, -g\cos\theta)$$
$$\mathbf{F}_{cen} = (\omega^2 R\sin\theta, 0, 0)$$
$$\mathbf{F} = (-g\sin\theta + \omega^2 R\sin\theta, 0, -g\cos\theta)$$
$$\mathbf{F}\cdot\mathbf{F}_{grav} = |\mathbf{F}_{grav}||\mathbf{F}|\cos\alpha$$
$$\mathbf{F}\cdot\mathbf{F}_{grav} = g^2\sin^2\theta - gR\omega^2\sin^2\theta + g^2\cos^2\theta$$
$$\mathbf{F}\cdot\mathbf{F}_{grav} = g^2 - gR\omega^2\sin^2\theta$$
$$|\mathbf{F}_{grav}| = g$$
$$|\mathbf{F}| = \sqrt{g^2\sin^2\theta + \omega^4 R^2\sin^2\theta - 2gR\omega^2\sin^2\theta + g^2\cos^2\theta}$$
$$|\mathbf{F}| = \sqrt{g^2 + \omega^4 R^2\sin^2\theta - 2gR\omega^2\sin^2\theta}$$
$$\cos\alpha = \frac{g^2 - gR\omega^2\sin^2\theta}{g\sqrt{g^2 + \omega^4 R^2\sin^2\theta - 2gR\omega^2\sin^2\theta}}$$
$$\cos\alpha = \frac{g^2 - gR\omega^2\sin^2\theta}{g^2}\left(1 + \frac{\omega^4 R^2\sin^2\theta}{g^2} - \frac{2R\omega^2\sin^2\theta}{g}\right)^{-1/2}$$
$$\cos\alpha \sim \frac{g^2 - gR\omega^2\sin^2\theta}{g^2}\left(1 - \frac{2R\omega^2\sin^2\theta}{g}\right)^{-1/2}$$
$$\cos\alpha \sim \left(1 - \frac{R\omega^2\sin^2\theta}{g}\right)\left(1 + \frac{R\omega^2\sin^2\theta}{g}\right)$$
$$\cos\alpha \sim \left(1 - \frac{R^2\omega^4\sin^4\theta}{g^2}\right)$$

α=Angle between combined force and pure gravity

$$R\omega^2 \sim 0.003g$$

Angle between "apparent" and "true" gravity



$$\mathbf{F}_{grav} = (-g\sin\theta, 0, -g\cos\theta)$$
$$\mathbf{F}_{cen} = (\omega^2 R\sin\theta, 0, 0)$$
$$\mathbf{F} = (-g\sin\theta + \omega^2 R\sin\theta, 0, -g\cos\theta)$$
$$\mathbf{F}\cdot\mathbf{F}_{grav} = |\mathbf{F}_{grav}||\mathbf{F}|\cos\alpha$$
$$\mathbf{F}\cdot\mathbf{F}_{grav} = g^2\sin^2\theta - gR\omega^2\sin^2\theta + g^2\cos^2\theta$$
$$\mathbf{F}\cdot\mathbf{F}_{grav} = g^2 - gR\omega^2\sin^2\theta$$
$$|\mathbf{F}_{grav}| = g$$
$$|\mathbf{F}| = \sqrt{g^2\sin^2\theta + \omega^4 R^2\sin^2\theta - 2gR\omega^2\sin^2\theta + g^2\cos^2\theta}$$
$$|\mathbf{F}| = \sqrt{g^2 + \omega^4 R^2\sin^2\theta - 2gR\omega^2\sin^2\theta}$$
$$\cos\alpha = \frac{g^2 - gR\omega^2\sin^2\theta}{g\sqrt{g^2 + \omega^4 R^2\sin^2\theta - 2gR\omega^2\sin^2\theta}}$$

α=Angle between combined force and pure gravity

$$\cos\alpha = \frac{g^2 - gR\omega^2\sin^2\theta}{g^2}\left(1 + \frac{\omega^4 R^2\sin^2\theta}{g^2} - \frac{2R\omega^2\sin^2\theta}{g}\right)^{-1/2}$$

$$R\omega^2 \sim 0.003g$$

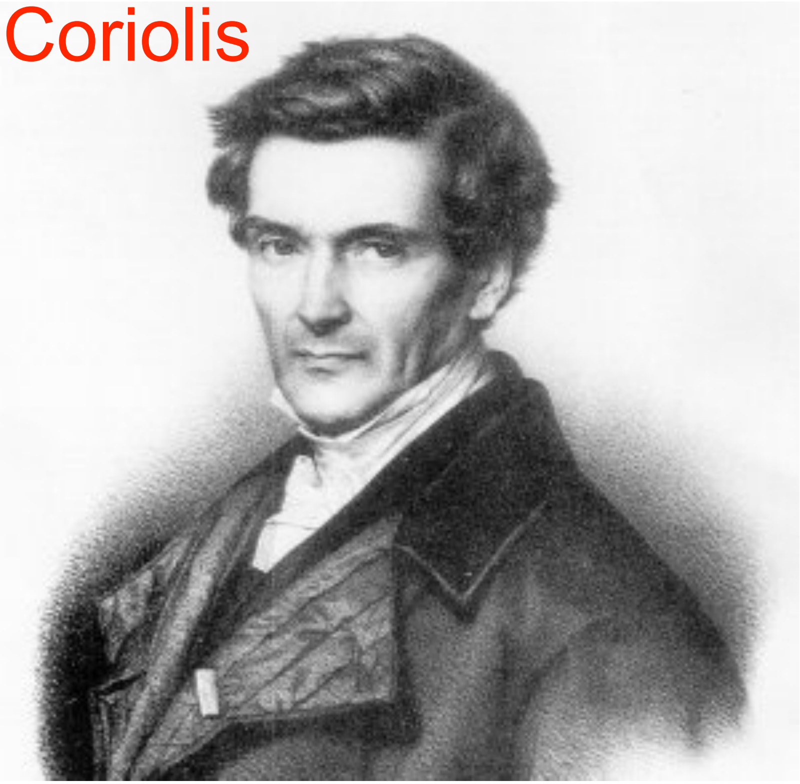
$$\cos\alpha \sim \frac{g^2 - gR\omega^2\sin^2\theta}{g^2}\left(1 - \frac{2R\omega^2\sin^2\theta}{g}\right)^{-1/2}$$
$$\cos\alpha \sim \left(1 - \frac{R\omega^2\sin^2\theta}{g}\right)\left(1 + \frac{R\omega^2\sin^2\theta}{g}\right)$$
$$\cos\alpha \sim \left(1 - \frac{R^2\omega^4\sin^4\theta}{g^2}\right)$$

$$\theta = 45^o \rightarrow \cos\alpha = \left(1 - \frac{R^2\omega^4}{4g^2}\right)$$
$$\cos x \sim 1 - \frac{x^2}{2} \rightarrow \alpha = \frac{R\omega^2}{2g} \sim 0.1^o$$

Coriolis Force

This one is a bit more difficult to picture, but is again due to the fact that we are in a rotating frame, and our intuition about inertia only holds in a non-rotating inertial frame

Gaspard-Gustave Coriolis



Note: if v=0, no Coriolis force!

And if v is parallel to Ω, also no Coriolis force

$$\mathbf{F}_{cor} = 2m\dot{\mathbf{r}} \times \Omega = 2m\mathbf{v} \times \Omega$$

Coriolis Force

$$\mathbf{F}_{cor} = 2m\dot{\mathbf{r}} \times \Omega = 2m\mathbf{v} \times \Omega$$

$$\Omega = 7.3\text{x}10^{-5}s^{-1}$$

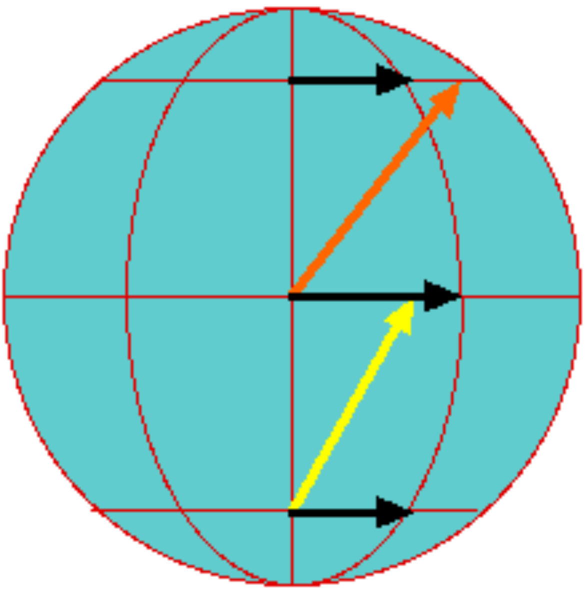
$$F_{cor}^{max}/F_{grav} = (1.5\text{x}10^{-5})(v)$$

$$v = 67 \text{ km/s for Coriolis force to equal grav force!}$$

A simple but useful problem

Let's look at example 9.2 together

Coriolis Force in a picture

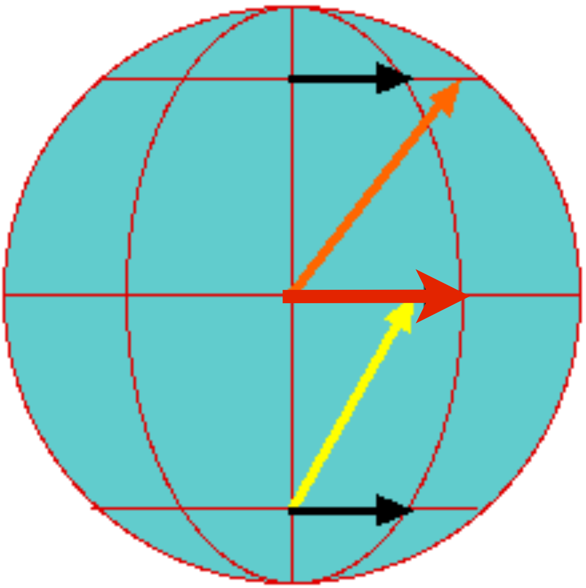


As it travels north, orange projectile travels further eastward than the earth beneath it

As it travels north, yellow projective travels less eastward than the ground beneath it

Curve to the right in northern hemisphere, to left in southern hemisphere

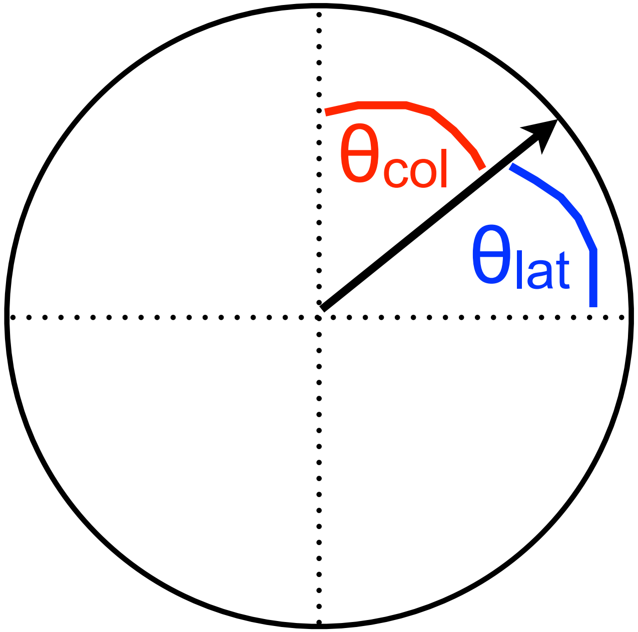
Coriolis Force in another picture



What about red projectile? It is traveling faster now than the ground beneath it and thus it will want to fly out/off the earth (inertial Coriolis force is up)

On coordinate systems

In these discussion, we sometimes need to be careful about our choice of latitude vs colatitude. Subtle but important. When the book uses θ, it's typically using colatitude, ie the angle from the z axis, which runs from 0 (north pole) to 180 (south pole). Geographers typically use the latitutde, which runs from +90 (north pole) to -90 (south pole)



Particle in free-fall

Include centrifugal force in **g**

Coriolis term

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\vec{\omega} \times \dot{\mathbf{r}}$$

$$\int_0^t \ddot{\mathbf{r}} dt = \int_0^t \left[\mathbf{g} - 2\vec{\omega} \times \dot{\mathbf{r}}\right] dt$$

$$\dot{\mathbf{r}} - \dot{\mathbf{r}}(0) = \mathbf{g}t - 2\vec{\omega} \times \mathbf{r} + 2\vec{\omega} \times \mathbf{r}(0)$$

$$\dot{\mathbf{r}} = \mathbf{g}t - 2\vec{\omega} \times \mathbf{r} + 2\vec{\omega} \times \mathbf{r}(0)$$

Let's start particle from rest (ie dropped from rest), so v(0) = 0

Let's plug the last line into the first line ...

Particle in free-fall

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\vec{\omega} \times \dot{\mathbf{r}}$$

$$\dot{\mathbf{r}} = \mathbf{g}t - 2\vec{\omega} \times \mathbf{r} + 2\vec{\omega} \times \mathbf{r}(0)$$

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\vec{\omega} \times \left(\mathbf{g}t - 2\vec{\omega} \times \mathbf{r} + 2\vec{\omega} \times \mathbf{r}(0)\right)$$

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\vec{\omega} \times \left(\mathbf{g}t + 2\vec{\omega} \times (\mathbf{r}(0) - \mathbf{r})\right)$$

small compared to **g**t

$$\ddot{\mathbf{r}} = \mathbf{g} - 2t\vec{\omega} \times \mathbf{g}$$

This we can solve

$$\ddot{\mathbf{r}} = \mathbf{g} - 2t\vec{\omega} \times \mathbf{g}$$

$$\int_0^t \ddot{\mathbf{r}} dt = \int_0^t \left[\mathbf{g} - 2t\vec{\omega} \times \mathbf{g}\right] dt$$

v(0) = 0

$$\dot{\mathbf{r}} dt + \dot{\mathbf{r}}(0) = \mathbf{g}t - t^2\vec{\omega} \times \mathbf{g}$$

$$\int_0^t \dot{\mathbf{r}} dt = \int_0^t \left[\mathbf{g}t - t^2\vec{\omega} \times \mathbf{g}\right] dt$$

$$\mathbf{r}(t) - \mathbf{r}(0) = \frac{1}{2}\mathbf{g}t^2 - \frac{1}{3}t^3\vec{\omega} \times \mathbf{g}$$

$$\mathbf{r}(t) = \frac{1}{2}\mathbf{g}t^2 - \frac{1}{3}t^3\vec{\omega} \times \mathbf{g} + \mathbf{r}(0)$$

How to interpret

$$\mathbf{r}(t) = \frac{1}{2}\mathbf{g}t^2 - \frac{1}{3}t^3\vec{\omega} \times \mathbf{g} + \mathbf{r}(0)$$

Define coordinates with x = easterly, y = northerly, z = up (ie radially out)

Assume we're on the equator

r = (x,y,z), **g** = (0,0,-g), **ω** = (0,ω,0)

If we drop a particle down a well over depth h
r(0) = (0,0,R+h) where R is radius of earth

$$-\frac{1}{3}t^3\vec{\omega} \times \mathbf{g} = \frac{1}{3}\omega g t^3 \hat{\mathbf{x}}$$

Deflection in easterly direction!

Deflection in the well

$$\mathbf{r}(t) = \frac{1}{2}\mathbf{g}t^2 - \frac{1}{3}t^3\vec{\omega} \times \mathbf{g} + \mathbf{r}(0)$$

Deflection still quite small, so t for descent down the well is given by standard formula of

$$t = \sqrt{2d/g} \sim \sqrt{2h/g}$$

$$-\frac{1}{3}t^3\vec{\omega} \times \mathbf{g} = \frac{1}{3}\omega g t^3\hat{\mathbf{x}}$$

$$\text{Deflection} = \frac{1}{3}\omega g(2h/g)^{3/2}\hat{\mathbf{x}} = \sqrt{8/(9g)}\omega h^{3/2}\hat{\mathbf{x}}$$

h = 100m (well) $\rightarrow$ 2.2 cm

h = 2 km (skydiving) $\rightarrow$ 2 m

Another quick exercise

Let's do
problem 9.9
together

Foucault pendulum

Foucault pendulum

Pendulum with length L

$$m\ddot{\mathbf{r}} = m\mathbf{g} - 2\vec{\omega} \times \dot{\mathbf{r}} + \mathbf{T}$$

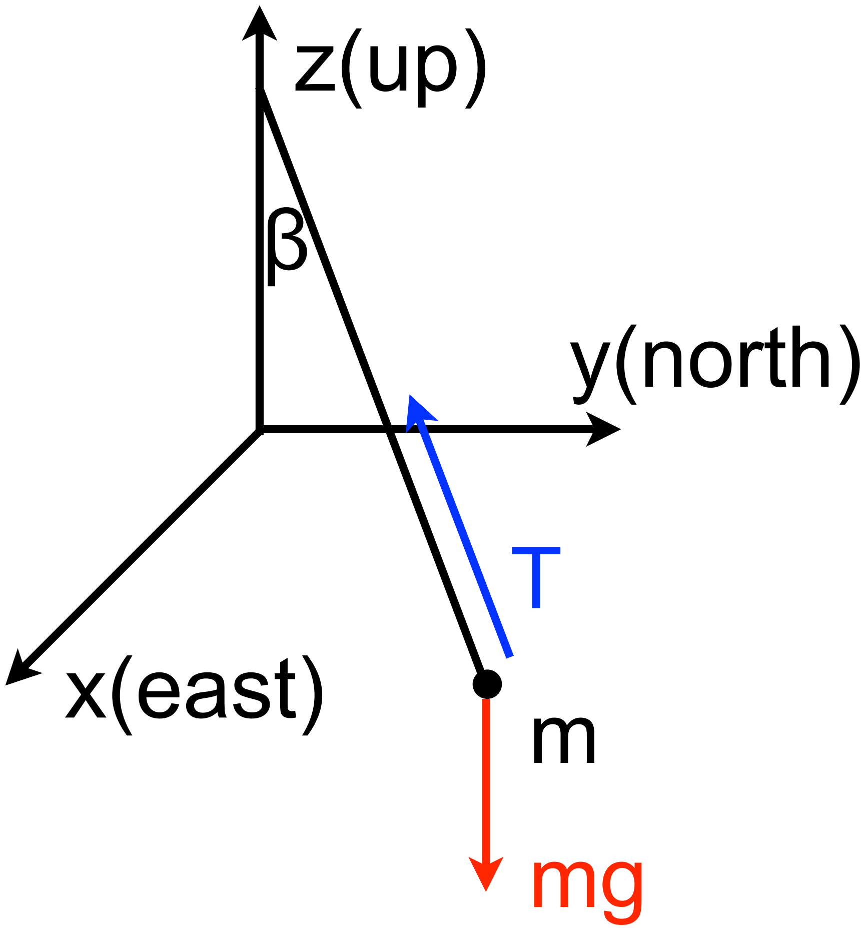
$$x^2 + y^2 + z^2 = L^2$$

$$T_x \sim T(x/L)$$

$$T_y \sim T(y/L)$$

$$T_z \sim T(z/L)$$

$$T_z \sim mg$$



Foucault pendulum

Pendulum with length L

$$\vec{\omega} = (0, \omega \sin\theta, \omega \cos\theta)$$

$$\ddot{x} = -gx/L + 2\dot{y}\omega\cos\theta - 2\dot{z}\omega\sin\theta$$

$$\ddot{y} = -gy/L - 2\omega\dot{x}\cos\theta$$

For small oscillations, expect v_z small

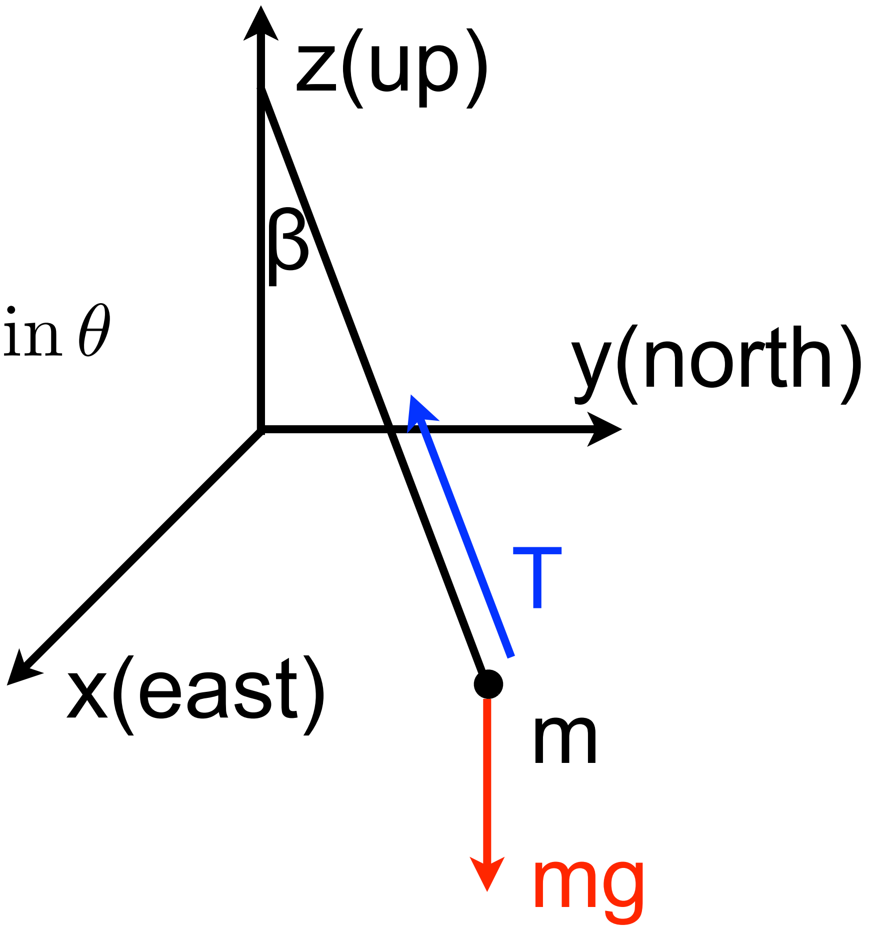
$$\ddot{x} = -gx/L + 2\dot{y}\omega\cos\theta$$

$$\ddot{y} = -gy/L - 2\dot{x}\omega\cos\theta$$

$$\omega_0^2 = g/L, \omega\cos\theta = \omega_z$$

$$\ddot{x} - 2\omega_z\dot{y} + \omega_0^2 x = 0$$

$$\ddot{y} + 2\omega_z\dot{x} + \omega_0^2 y = 0$$



Solution using earlier trick with complex numbers

$$\ddot{x} - 2\omega_z \dot{y} + \omega_0^2 x = 0$$

$$\ddot{y} + 2\omega_z \dot{x} + \omega_0^2 y = 0$$

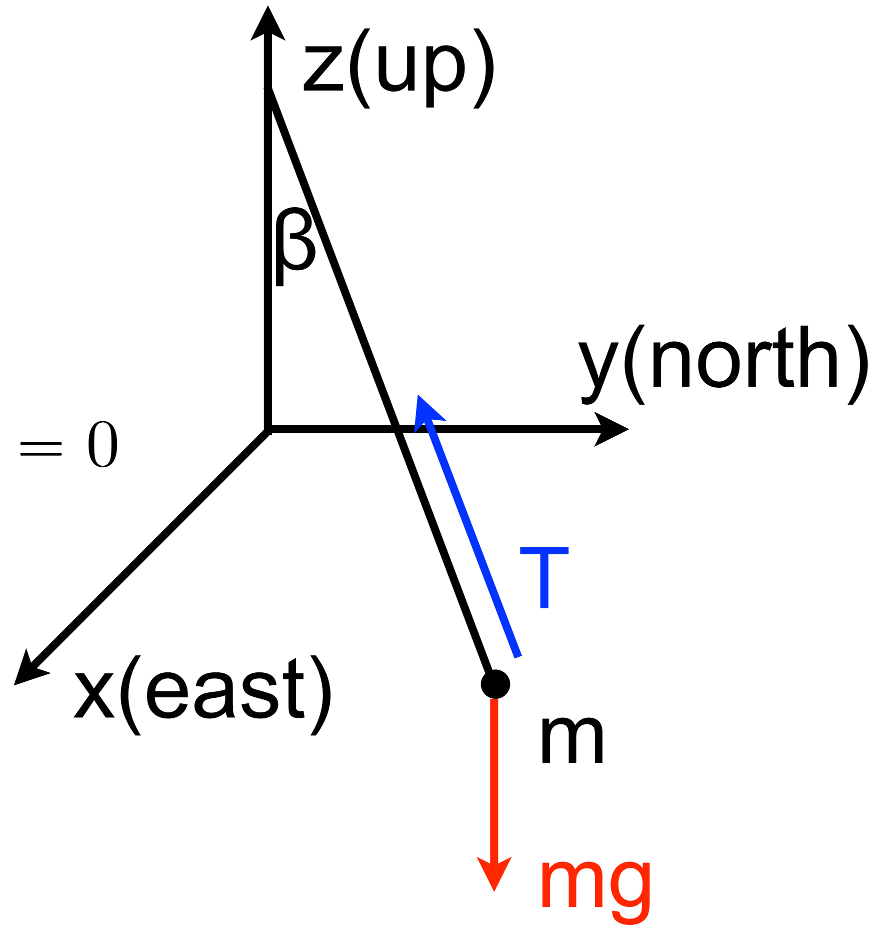
$$\eta = x + iy$$

Sum

$$i\ddot{y} + 2i\omega_z \dot{x} + i\omega_0^2 y = 0$$

$$\ddot{x} + i\ddot{y} + 2\omega_z(i\dot{x} - \dot{y}) + \omega_0^2(x + iy) = 0$$

$$\ddot{\eta} + 2i\omega_z \dot{\eta} + \omega_0^2 \eta = 0$$



$$\eta(t) = e^{-i\alpha t}$$

$$-\alpha^2 + 2i\omega_z(-i\alpha) + \omega_0^2 = 0$$

$$-\alpha^2 + 2\alpha\omega_z + \omega_0^2 = 0$$

$$\alpha^2 - 2\alpha\omega_z - \omega_0^2 = 0$$

$$\alpha = \left(2\omega_z \pm \sqrt{4\omega_z^2 + 4\omega_0^2}\right)/2$$

$$\alpha \sim \omega_z \pm \sqrt{\omega_z^2 + \omega_0^2}$$

$$\omega_0 >> \omega_z \rightarrow$$

$$\alpha \sim \omega_z \pm \omega_0$$

Solution for pendulum

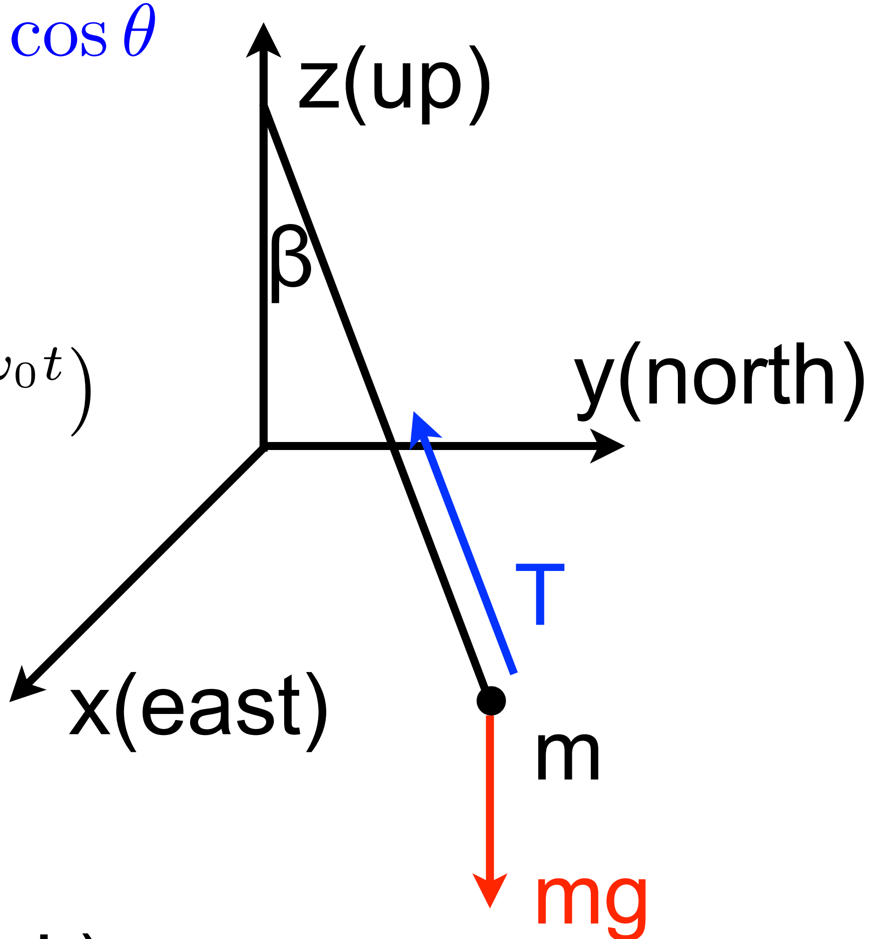
$$\omega_z = \omega\cos\theta$$

$$\eta(t) = e^{-i\alpha t}$$

$$\alpha \sim \omega_z \pm \omega_0$$

$$\eta = e^{-i\omega_z t}\left(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}\right)$$

What does this term do?



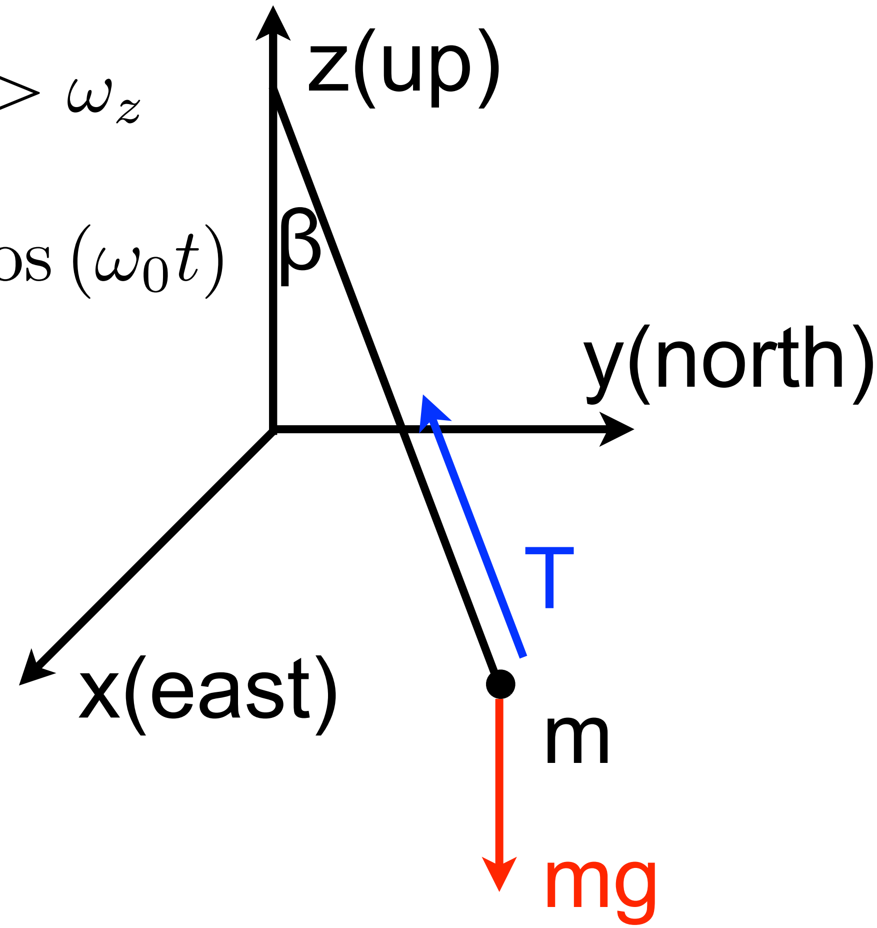
Let's check (as in the book) what happens if at t=0, v=0, x=A, y=0 (so that $C_1 = C_2 = A/2$ - let's see why). Take a close look at Fig 9.17

Solution for pendulum

$$\omega_z = \omega \cos\theta \qquad \omega_0 >> \omega_z$$

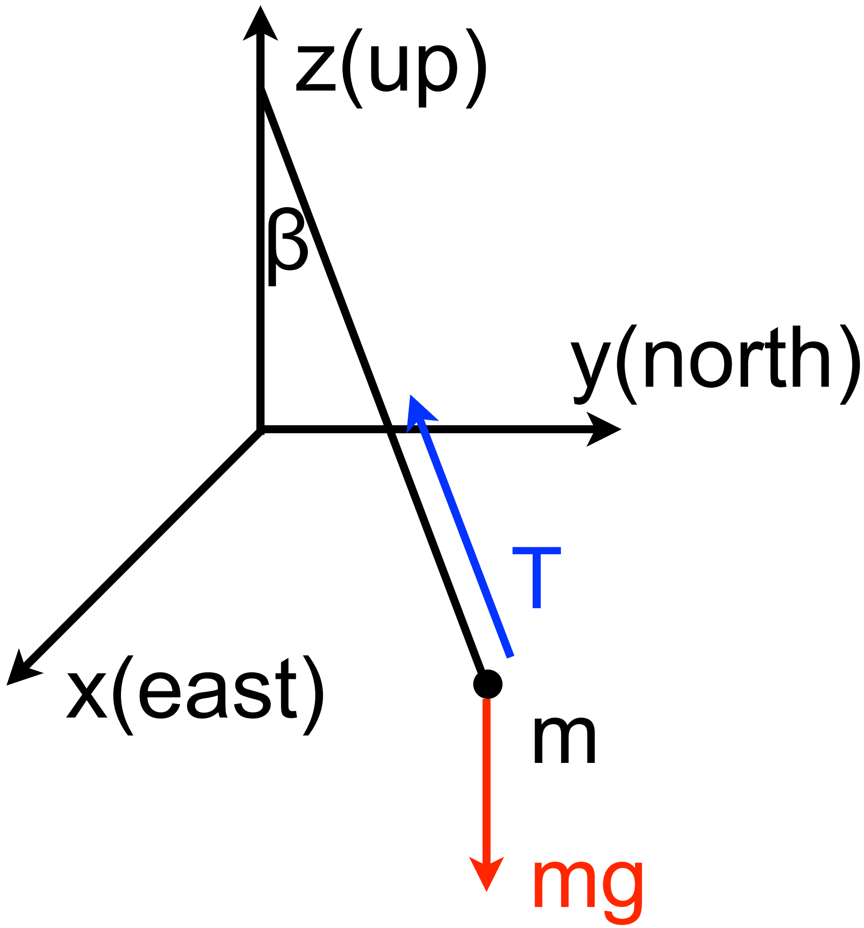
$$\eta(t) = x(t) + iy(t) = Ae^{-i\omega_z t} \cos(\omega_0 t)$$

At first, natural frequency much larger than the extra term in front, but eventually, that rotates motion in x direction to y direction and back



Rotation of the plane of the pendulum

At first, natural frequency much larger than the extra term in front, but eventually, that rotates motion in x direction to y direction and back. Chicago co-latitude = 48.2 degrees, so ω_z = ωcos(48.2 deg)=2/3 (360 degrees/day) = 240 degrees / 24 hours



90 degree rotation (complete shift to y direction) in just 9 hours!

One last word on these subjects

Recall from earlier in the course:

$$F_r = m(\ddot{r} - r\dot{\phi}^2)$$

$$F_\phi = m(2\dot{r}\dot{\phi} + r\ddot{\phi})$$

Choice of frame determines whether accelerations or forces are complicated (alternatively, which one is simple)

Homework (due as usual in 1 week)

9.2, 9.14, 9.25,9.26, 9.28, 9.29