Armed with Newton and Lagrangian mechanics...

Let’s tackle a new problem (chapter 8 of Taylor)
Two-body central force motion

Center of Mass

Recall this from earlier in the course, now only with two particles
Two-body central force motion

Recall definition: If internal forces are along vector connecting particles, we call them **central forces**

\[ \mathbf{F}_{21} = -\mathbf{F}_{12} \]

\[ \mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \]

\[ M = m_1 + m_2 \]
Two-body central force motion

\[ \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \]

\[
\mathbf{F}_{12} = F(\mathbf{r}_1, \mathbf{r}_2)\hat{\mathbf{r}} \\
\mathbf{F}_1 = F(\mathbf{r}_1)\hat{\mathbf{r}} \\
\mathbf{F}_2 = -\mathbf{F}_{12} \\
U(\mathbf{r}_1, \mathbf{r}_2) = U(|\mathbf{r}_1 - \mathbf{r}_2|)\hat{\mathbf{r}} 	ext{ for conservative central force} \\
U = U(r)\]
What’s sorts of problems can we think of?
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What’s sorts of problems can we think of?
Writing down the Lagrangian

\[ L = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 - U(r) \]

Note that we are starting out with 6 degrees of freedom! Let’s hope that we can reduce this.
Writing down the Lagrangian

\[ \mathcal{L} = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 - U(r) \]

Good news when using the Lagrangian formalism is that we can pick 6 generalized coordinates. Which ones?
Writing down the Lagrangian

The center of mass of the system gives us 3 potentially useful coordinates.

Center of mass

\[ R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \]

\[ M = m_1 + m_2 \]
Writing down the Lagrangian

Recall that CoM moves as:

\[ \dot{P} = M\ddot{R} \]
\[ \dot{P} = M\ddot{R} = \sum F_{ext} \]

\[ F_{ext} = 0 \rightarrow \dot{P} = 0 \]
\[ \dot{P} = 0 \rightarrow \ddot{R} = \text{constant} \]

Free to choose inertial frame in which center of mass is at rest
Two-body central force motion

\[ \mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{M} \]

\[ M = m_1 + m_2 \]

\[ \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \]

\[ \mathbf{r}_1 = \mathbf{r} + \mathbf{r}_2 \]

\[ \mathbf{r}_2 = \frac{M \mathbf{R} - m_1 \mathbf{r}_1}{m_2} \]

\[ \mathbf{r}_1 = \mathbf{r} + \frac{M \mathbf{R} - m_1 \mathbf{r}_1}{m_2} \]

\[ \mathbf{r}_1 \left(1 + \frac{m_1}{m_2}\right) = \mathbf{r} + \frac{M \mathbf{R}}{m_2} \]

\[ \mathbf{r}_1 \frac{m_1 + m_2}{m_2} = \mathbf{r} + \frac{M \mathbf{R}}{m_2} \]

\[ \mathbf{r}_1 \frac{M}{m_2} = \mathbf{r} + \frac{M \mathbf{R}}{m_2} \]

\[ \mathbf{r}_1 = \frac{m_2}{M} \mathbf{r} + \mathbf{R} \]

Similarly,

\[ \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M} \mathbf{r} \]
Does this tell us what we might guess already?

\[ r_1 = R + \frac{m_2}{M} r \]
\[ r_2 = R - \frac{m_1}{M} r \]

For very large mass \( m_1 \gg m_2 \), \( M = (m_1 + m_2) \sim m_1 \) and \( r_1 \sim R, \ r_2 \sim R-r \)
Writing down the kinetic energy

\[ r_1 = \mathbf{R} + \frac{m_2}{M} \mathbf{r} \]

\[ r_2 = \mathbf{R} - \frac{m_1}{M} \mathbf{r} \]

\[ T = \frac{1}{2} (m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2) \]

\[ T = \frac{1}{2} \left( m_1 \left[ \dot{\mathbf{R}} + \frac{m_2}{M} \dot{\mathbf{r}} \right]^2 + m_2 \left[ \dot{\mathbf{R}} - \frac{m_1}{M} \dot{\mathbf{r}} \right]^2 \right) \]

\[ T = \frac{1}{2} \left( m_1 \left[ \dot{\mathbf{R}}^2 + \frac{m_2^2}{M^2} \dot{\mathbf{r}}^2 + 2 \dot{\mathbf{R}} \cdot \dot{\mathbf{r}} \frac{m_2}{M} \right] + m_2 \left[ \dot{\mathbf{R}}^2 + \frac{m_1^2}{M^2} \dot{\mathbf{r}}^2 - 2 \dot{\mathbf{R}} \cdot \dot{\mathbf{r}} \frac{m_1}{M} \right] \right) \]

\[ T = \frac{1}{2} \left( (m_1 + m_2) \dot{\mathbf{R}}^2 + \frac{m_1 m_2^2 + m_2 m_1^2}{M^2} \dot{\mathbf{r}}^2 + 2 \dot{\mathbf{R}} \cdot \dot{\mathbf{r}} \frac{m_1 m_2 - m_2 m_1}{M} \right) \]

\[ T = \frac{1}{2} \left( (m_1 + m_2) \dot{\mathbf{R}}^2 + \frac{(m_1 + m_2) (m_1 m_2)}{M^2} \dot{\mathbf{r}}^2 \right) \]

\[ T = \frac{1}{2} \left( M \dot{\mathbf{R}}^2 + \frac{m_1 m_2}{M} \dot{\mathbf{r}}^2 \right) \]
Simplifying the kinematic energy

Reduced mass
(always smaller than \(m_1\) and \(m_2\))

\[
\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2}
\]

\[
T = \frac{1}{2} \left( M \dot{R}^2 + \frac{m_1 m_2}{M} \dot{r}^2 \right)
\]

For very large mass \(m_1 \gg m_2\), \(\mu = \frac{(m_1 m_2)}{m_1} \sim m_2\)

For equal masses \(\mu = \frac{(m* m)}{(2m)} \sim m/2\)
Kinetic energy is the same as energy of two particles (not real!):
1) Particle with mass $M = m_1 + m_2$ moving with speed of the center of mass
2) Particle with mass $\mu$ moving with speed of relative position
Let’s write down the Lagrangian

\[ L = T - U = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu \dot{r}^2 - U(r) \]

\[ L = \frac{1}{2} M \dot{R}^2 + \left( \frac{1}{2} \mu \dot{r}^2 - U(r) \right) \]

\[ L = L_{cm} + L_{rel} \]

Only involves CoM velocity

Only involves relative coordinate and motion
With a Lagrangian, we can find the Equations of Motion

\[ \mathcal{L} = \frac{1}{2} M \ddot{R}^2 + \left( \frac{1}{2} \mu \dot{r}^2 - U(r) \right) \]

\[ \frac{\partial \mathcal{L}}{\partial \dot{R}} = 0 = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{R}} = M \frac{d}{dt} \dot{R} = \ddot{R} \]

\[ \frac{\partial \mathcal{L}}{\partial r} = - \frac{\partial U}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu \frac{d}{dt} \dot{r} = \mu \ddot{r} \]

\[ M \ddot{R} = 0 \rightarrow \dot{R} = \text{constant} \]

\[ \mu \ddot{r} = - \frac{\partial U}{\partial r} = F(r) \]

Center of mass moves with constant velocity

Only relevant equation/non-trivial motion involving the force
We can simplify further

\[ r_1 = \frac{m_2}{M} \mathbf{r} \]
\[ r_2 = -\frac{m_1}{M} \mathbf{r} \]

\[ M \ddot{\mathbf{R}} = 0 \rightarrow \ddot{\mathbf{R}} = \text{constant} \]
\[ \mu \ddot{\mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}} = \mathbf{F}(\mathbf{r}) \]

Choose inertial frame with \( \mathbf{R} \) at rest at the origin

Problem is now (for all \( m_1, m_2 \)) only three-dimensional. But consider the special case with very large mass \( m_1 \gg m_2, \mu \sim m_2 \)

and \( r_1 \sim 0, \) and \( r_2 \sim -\mathbf{r}. \) What does it look like?
What about angular momentum?

Total angular momentum of system in CM frame is just that of single particle at \( \mathbf{r} \) with mass \( \mu \)
What does conservation of angular momentum tell us?

$L = \mathbf{r} \times (\mu \dot{\mathbf{r}})$

$L$ is a constant (because there are no external forces), which means it always points in the same direction.

So our three-dimensional problem is now reduced to a two-dimensional problem (remember, we started with six dimensions!)

Is this clear?
Let's finally write down a Lagrangian in detail

\[ \mathcal{L} = \frac{1}{2} \mu \dot{r}^2 - U(r) \]

\[ \mathcal{L} = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r) \]

\[ \frac{\partial \mathcal{L}}{\partial \phi} = 0 = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{d}{dt} (\mu r^2 \dot{\phi}) \]

\[ \mu r^2 \dot{\phi} = l = \text{constant} \]

\[ \frac{\partial \mathcal{L}}{\partial r} = \mu r \dot{\phi}^2 - \frac{\partial U}{\partial r} \]

\[ \frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu \dot{r} \]

\[ \frac{d}{dt} (\mu \dot{r}) = \mu r \dot{\phi}^2 - \frac{\partial U}{\partial r} \]

\[ \mu \ddot{r} = \mu r \dot{\phi}^2 - \frac{\partial U}{\partial r} \]

Conservation of angular momentum
Let’s simplify further

\[ \mu r^2 \dot{\phi} = l = \text{constant} \]

\[ \dot{\phi} = \frac{l}{\mu r^2} \]

\[ \mu \ddot{r} = \mu r \dot{\phi}^2 - \frac{\partial U}{\partial r} \]

\[ F_{cf} = \mu r \dot{\phi}^2 = \frac{l^2}{\mu r^3} \]

\[ F_{cf} = -\frac{d}{dr} \left( \frac{l^2}{2\mu r^2} \right) = -\frac{dU_{cf}}{dr} \]

\[ U_{cf}(r) = \frac{l^2}{2\mu r^2} \]
Let’s simplify further

\[ \mu \ddot{r} = \mu r \dot{\phi}^2 - \frac{\partial U}{\partial r} \]

\[ \mu \ddot{r} = -\frac{\partial U}{\partial r} + F_{cf} \]

\[ F_{cf} = \mu r \dot{\phi}^2 = \frac{l^2}{\mu r^3} \]

\[ F_{cf} = -\frac{d}{dr} \left( \frac{l^2}{2\mu r^2} \right) = -\frac{dU_{cf}}{dr} \]

\[ U_{cf}(r) = \frac{l^2}{2\mu r^2} \]

\[ \mu \ddot{r} = -\frac{d}{dr} \left[ U(r) + U_{cf}(r) \right] = -\frac{d}{dr} U_{eff}(r) \]

\[ U_{eff}(r) = U(r) + U_{cf}(r) = U(r) + \frac{l^2}{2\mu r^2} \]
Putting it together

Radial motion of this fictional particle with mass \( \mu \) behaves as if it was moving in a single dimensional effective potential given by the nominal one + the “fictitious” one

\[
\mu \ddot{r} = -\frac{d}{dr} \left[ U(r) + U_{cf}(r) \right] = -\frac{d}{dr} U_{eff}(r)
\]

\[
U_{eff}(r) = U(r) + U_{cf}(r) = U(r) + \frac{l^2}{2\mu r^2}
\]
Conservation of energy

\[ \mu \ddot{r} = -\frac{d}{dr} U_{eff}(r) \]

\[ \dot{r}(\mu \ddot{r}) = \dot{r} \left( -\frac{d}{dr} U_{eff}(r) \right) \]

\[ \frac{d}{dt} \left( \frac{1}{2} \mu \dot{r}^2 \right) = \frac{1}{2} \mu (2\dot{r}) \frac{d}{dt} \dot{r} = \mu \ddot{r} \dot{r} \]

\[ -\frac{d}{dt} U_{eff}(r) = -\frac{d}{dr} U_{eff}(r) \frac{dr}{dt} = \dot{r} \left( -\frac{d}{dr} U_{eff}(r) \right) \]

\[ \frac{d}{dt} \left( \frac{1}{2} \mu \dot{r}^2 \right) = -\frac{d}{dt} U_{eff}(r) \]

\[ \frac{d}{dt} \left( \frac{1}{2} \mu \dot{r}^2 + U_{eff}(r) \right) = 0 \]

\[ \frac{1}{2} \mu \dot{r}^2 + U_{eff}(r) = \text{const} \]
Conservation of energy

\[ \frac{1}{2} \mu \dot{r}^2 + U(r) + U_{cf}(r) = \text{const} \]

\[ \frac{1}{2} \mu \dot{r}^2 + U(r) + \frac{l^2}{2\mu r^2} = \text{const} \]

\[ \frac{1}{2} \mu \dot{r}^2 + U(r) + \frac{1}{2} \mu r^2 \dot{\phi}^2 = \text{const} = E \]

So everything we know about the 1d problem applies here, which simplifies things quite a bit!
Examples 8.1-8.2 together
Problem 8.7 in small groups or by yourself
Equations of the orbit

\[
\mu \ddot{r} = - \frac{d}{dr} [U(r) + U_{cf}(r)] = - \frac{d}{dr} U_{\text{eff}}(r)
\]

\[
U_{\text{eff}}(r) = U(r) + U_{cf}(r) = U(r) + \frac{l^2}{2\mu r^2}
\]

This gives us the equation \( r(t) \), but we might want to know \( r(\Phi) \) instead...

Time to play some tricks
Rewriting the radial equation

\[ \mu \ddot{r} = F(r) + \frac{l^2}{\mu r^3} \]

\[ l = \mu r^2 \dot{\phi} \]

\[ u = \frac{1}{r}, \quad r = \frac{1}{u} \]

\[ \frac{d}{dt} = \frac{d\phi}{dt} \frac{d}{d\phi} = \dot{\phi} \frac{d}{d\phi} = \frac{l}{\mu r^2} \frac{d}{d\phi} = \frac{lu^2}{\mu} \frac{d}{d\phi} \]

1/r^2 = u^2
Rewriting the radial equation

Careful!

\[ \mu \neq u \]

\[ \frac{dr}{dt} = \frac{du}{\mu} \frac{d\phi}{d\phi} \]

\[ \dot{r} = \frac{d}{dt} r = \frac{lu^2}{\mu} \frac{d}{d\phi} r = \frac{lu^2}{\mu} \frac{d}{d\phi} \left( \frac{1}{u} \right) = \frac{lu^2}{\mu} \frac{-1}{u^2} \frac{du}{d\phi} \]

\[ \ddot{r} = \frac{d}{dt} \dot{r} = \frac{lu^2}{\mu} \frac{d}{d\phi} \left( \frac{-lu}{\mu} \frac{du}{d\phi} \right) \]

\[ \ddot{r} = -\frac{l^2 u^2}{\mu^2} \frac{d^2 u}{d\phi^2} \]
Plugging back in

Careful!

\[ \mu \ddot{r} = F(r) + \frac{l^2}{\mu r^3} \]

\[ \mu \ddot{r} = F(1/u) + \frac{u^3 l^2}{\mu} \]

\[ \ddot{r} = - \frac{l^2 u^2}{\mu^2} \frac{d^2 u}{d\phi^2} \]

\[ -\mu \left( \frac{l^2 u^2}{\mu^2} \frac{d^2 u}{d\phi^2} \right) = F(1/u) + \frac{u^3 l^2}{\mu} \]

\[ \frac{d^2 u}{d\phi^2} = -\mu \frac{l^2 u^2}{\mu} F(1/u) - u \]

More compact:

\[ u''(\phi) = - \frac{\mu}{l^2 u^2} F - u(\phi) \]
The two-body gravitational problem

\[ u''(\phi) = -\frac{\mu}{l^2 u^2} F - u(\phi) \]

\[ F(r) = \frac{-G m_1 m_2}{r^2} \hat{r} \]

\[ F = -\frac{\gamma}{r^2} = -\gamma u^2 \]
The two-body gravitational problem

\[ u''(\phi) = -\frac{\mu}{l^2u^2}F - u(\phi) \]

\[ u'' = -\frac{\mu}{l^2u^2}(-\gamma u^2) - u \]

\[ u'' = \frac{\mu\gamma}{l^2} - u \]

\[ w = u - \frac{\mu\gamma}{l^2} \]

\[ w' = u', \quad w'' = u'' \]

\[ w'' = \frac{\mu\gamma}{l^2} - u = \frac{\mu\gamma}{l^2} - (w + \frac{\mu\gamma}{l^2}) \]

\[ w'' = \frac{\mu\gamma}{l^2} - w - \frac{\mu\gamma}{l^2} \]

\[ w'' = -w \rightarrow w(\phi) = A \cos(\phi - \delta) \]

\[ u(\phi) = A \cos(\phi - \delta) + \frac{\mu\gamma}{l^2} \]

\[ u(\phi) = \frac{\mu\gamma}{l^2} (1 + \epsilon \cos \phi) \]

Freedom to define coordinates so that \( \delta = 0 \)

\[ \epsilon = \frac{A l^2}{\mu \gamma} \]
Solution to the two-body gravitational problem

\[ u(\phi) = \frac{\mu \gamma}{l^2} (1 + \epsilon \cos \phi) = \frac{1}{r} \]

\[ r(\phi) = \frac{c}{1 + \epsilon \cos \phi} \]

\[ c = \frac{l^2}{\mu \gamma} = \frac{l^2}{G m_1 m_2 \mu} \]

\[ \epsilon < 1 \rightarrow r(\phi) \text{ bounded} \]

\[ \epsilon > 1 \rightarrow r(\phi) \text{ can grow to } \infty \]
Studying the bounded orbits

\[ r(\phi) = \frac{c}{1 + \epsilon \cos \phi} \]

\[ r_{\text{min}} = \frac{c}{1 + \epsilon} \quad \text{perihelion/perigee (}\phi=0\text{)} \]

\[ r_{\text{max}} = \frac{c}{1 - \epsilon} \quad \text{aphelion/apogee (}\phi=\pi\text{)} \]
Rewriting the solution (Problem 8.16)

\[ r = \sqrt{x^2 + y^2}, \cos \phi = x/r \]

\[ r = \frac{c}{1 + \epsilon \cos \phi} \]

\[ r(1 + \epsilon x/r) = c \]

\[ r + \epsilon x = c \]

\[ r = c - \epsilon x \]

\[ r^2 = (c - \epsilon x)^2 = x^2 + y^2 \]

\[ c^2 + \epsilon^2 x^2 - 2c\epsilon x = x^2 + y^2 \]

\[ (1 - \epsilon^2)x^2 + 2c\epsilon x + y^2 = c^2 \]

\[ x^2 + \frac{2c\epsilon}{1 - \epsilon^2}x + \frac{y^2}{1 - \epsilon^2} = c^2/(1 - \epsilon^2) \]

\[ (x + d)^2 + \frac{y^2}{1 - \epsilon^2} = c^2/(1 - \epsilon^2) + d^2 \]

Where completing the square means

\[ d = \frac{c\epsilon}{1 - \epsilon^2} \]
Rewriting the solution (Problem 8.16)

\[
(x + \frac{c\epsilon}{1 - \epsilon^2})^2 + \frac{y^2}{1 - \epsilon^2} = \frac{c^2}{1 - \epsilon^2} + \frac{c^2\epsilon^2}{(1 - \epsilon^2)^2}
\]

\[
(x + \frac{c\epsilon}{1 - \epsilon^2})^2 + \frac{y^2}{1 - \epsilon^2} = \frac{c^2 - c^2\epsilon^2 + c^2\epsilon^2}{(1 - \epsilon^2)^2}
\]

\[
(x + d)^2 \frac{1}{a^2} + \frac{y^2}{b^2} = 1
\]

With constants:

\[
a = \frac{c}{1 - \epsilon^2}, \quad b = \frac{c}{\sqrt{1 - \epsilon^2}}, \quad d = \frac{c\epsilon}{1 - \epsilon^2}
\]

\[
a = \frac{c}{1 - \epsilon^2}, \quad b = \frac{c}{\sqrt{1 - \epsilon^2}}, \quad d = a\epsilon
\]
Looking at the solution

\[
\frac{(x + d)^2}{a^2} + \frac{y^2}{b^2} = 1
\]

\[d = \frac{ce}{1 - \epsilon^2}\]

Distance from origin to sun

This is an ellipse, with the center of the ellipse offset by d along the x axis from the origin.

Ellipse

\[f + g\] always adds to the same value
Definition of eccentricity $e$ of an ellipse

$$e = \sqrt{\frac{a^2 - b^2}{a^2}}$$

$$e = \sqrt{\frac{a^2}{1-e^2} - \frac{b^2}{1-e^2}}$$

Quick check:
What do we get for a circle?

So $e$ is the eccentricity

$$\frac{(x + d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a = \frac{c}{1-e^2}, \quad b = \frac{c}{\sqrt{1-e^2}}, \quad d = a\epsilon$$

$$e^2 = \frac{c^2 - c^2 - \epsilon^2 c^2}{(1-e^2)^2}$$

$$e^2 = \frac{c^2}{(1-e^2)^2}$$

$$e^2 = \epsilon^2$$

$$e = \epsilon$$
A reminder about ellipses

Kepler’s First Law

As expected, $\varepsilon=0 \rightarrow d=0$

$\frac{c\varepsilon}{1-\varepsilon^2} = a\varepsilon = \text{ellipse focus}$

Drawing an ellipse: loop string around thumb tacks at each focus and stretch string tight with a pencil while moving the pencil around the tacks. The Sun is at one focus.
A brief step back to Section 3.4 (Kepler’s 2nd law)

As a planet moves around the sun, what can we say about the area swept out by the orbit?

In a small time $\Delta t$, the area swept out is the area of the triangle $OAB$.

\[
x = r\cos \theta
\]
\[
y = r\sin \theta
\]
A brief step back to Section 3.4 (Kepler’s 2nd law)

As a planet moves around the sun, what can we say about the area swept out by the orbit?

In a small time $\Delta t$, the area swept out is the area of the triangle $\text{OAB}...$ but $\Delta t$ is small, so $\Delta \theta$ is small. $\cos(\Delta \theta) \sim 1$, $\sin(\Delta \theta) \sim \Delta \theta$

$$dA = 0.5xy = 0.5r^2\Delta \theta$$
A brief step back to Section 3.4 (Kepler’s 2nd law)

As a planet moves around the sun, what can we say about the area swept out by the orbit?

![Diagram of a planet orbiting the sun with area A, angular displacement Δθ, and radial displacement Δr.]

\[ A = \frac{1}{2} \int_{\theta_0}^{\theta_1} r^2 d\theta \]

Recall \( l = mr^2 \dot{\theta} = mr^2 \frac{d\theta}{dt} \)

\[ d\theta = \frac{ldt}{mr^2} \]

\[ A = \frac{1}{2} \int_{t(\theta_0)}^{t(\theta_1)} r^2 \frac{ldt}{mr^2} \]

\[ A = \frac{l}{2m} \int_{t(\theta_0)}^{t(\theta_1)} dt \]

\[ \frac{dA}{dt} = \frac{l}{2m} = \frac{mr^2 \dot{\theta}}{2m} = \frac{r^2 \omega}{2} \]

Constant for this system!
On to Kepler's third law

\[ \frac{dA}{dt} = \frac{r^2 \omega}{2} \]

\[ A = \pi ab \]

\[ \tau = \frac{A}{dA/dt} = \frac{2\pi ab}{r^2 \omega} \]

\[ \tau^2 = \frac{4\pi^2 a^2 b^2}{r^4 \omega^2} \]

\[ \tau^2 = \frac{4\pi^2 a^4 (1 - \epsilon^2)}{r^4 \omega^2} \]

\[ \tau^2 = \frac{4\pi^2 a^4 (1 - \epsilon^2)}{r^4 (c \gamma)/(\mu r^4)} \]

\[ \tau^2 = \frac{4\mu \pi^2 a^3}{\gamma} \]

Recall:

\[ a = \frac{c}{1 - \epsilon^2}, \quad b = \frac{c}{\sqrt{1 - \epsilon^2}} \]

\[ \frac{b}{a} = \sqrt{1 - \epsilon^2} \]

\[ b^2 = a^2 (1 - \epsilon^2) \]

\[ c = \frac{l^2}{\mu \gamma} = \frac{\mu^2 r^4 \omega^2}{\mu \gamma} = \frac{\mu r^4 \omega^2}{\gamma} \]

\[ \omega^2 = \frac{c \gamma}{\mu r^4} \]
Continuing with Kepler’s third law

\[ \tau^2 = \frac{4\mu\pi^2 a^3}{\gamma} \]

\[ \gamma = Gm_em_s, \mu = \frac{m_em_s}{m_e + m_s} \]

\[ \frac{\mu}{\gamma} = \frac{m_em_s}{(m_e + m_s)(Gm_em_s)} = \frac{1}{G(m_e + m_s)} \]

\[ \frac{\mu}{\gamma} \sim \frac{1}{Gm_s} \]

\[ \tau^2 = \frac{4\pi^2}{Gm_s} a^3 \]

For example, for earth and sun, where \( m_s >> m_e \)

Square of period proportional to cube of semimajor axis
Energy of the orbit

\[ \frac{1}{2} \mu \dot{r}^2 + U_{\text{eff}}(r) = E \]

\[ \dot{r}(r_{\text{min}}) = 0 \rightarrow U_{\text{eff}}(r_{\text{min}}) = E \]

\[ U(r_{\text{min}}) + \frac{l^2}{2\mu r_{\text{min}}^2} = E \]

\[ \frac{\gamma}{r_{\text{min}}} + \frac{l^2}{2\mu r_{\text{min}}^2} = E \]

\[ r(\phi) = \frac{c}{1 + \epsilon \cos \phi} \]

\[ r_{\text{min}} = \frac{c}{1 + \epsilon} \]

\[ r_{\text{max}} = \frac{c}{1 - \epsilon} \]

\[ c = \frac{l^2}{\mu \gamma} \]
Energy of the orbit

\[-\frac{\gamma}{r_{\text{min}}} + \frac{l^2}{2\mu r_{\text{min}}^2} = E\]

\[r_{\text{min}} = \frac{l^2}{\mu \gamma (1 + \epsilon)}\]

\[-\frac{\gamma^2 \mu (1 + \epsilon)}{l^2} + \frac{\mu \gamma^2 (1 + \epsilon)^2}{2l^2} = E\]

\[E = \frac{\mu \gamma^2 ((1 + \epsilon)^2 - 2(1 + \epsilon))}{2l^2}\]

\[E = \frac{\mu \gamma^2 (\epsilon^2 - 1)}{2l^2}\]

\[E > 0 \text{ when } \epsilon > 1\]
\[E < 0 \text{ when } \epsilon < 1\]
\[E = 0 \text{ when } \epsilon = 1\]
What is the orbit if $\varepsilon=0$?

\[ r = \sqrt{x^2 + y^2}, \cos \phi = \frac{x}{r} \]

\[ r = \frac{c}{1 + \cos \phi} \]

\[ r(1 + x/r) = c \]

\[ r + x = c \]

\[ r = c - x \]

\[ r^2 = (c - x)^2 = x^2 + y^2 \]

\[ c^2 + x^2 - 2xc = x^2 + y^2 \]

\[ c^2 - 2cx = y^2 \]

Get a parabola when energy $= 0$.
What is the orbit if $\varepsilon > 1$?

\[ r = \sqrt{x^2 + y^2}, \cos \phi = \frac{x}{r} \]

\[ r = \frac{c}{1 + \varepsilon \cos \phi} \]

\[ r \to \infty \text{ when } \varepsilon \cos \phi_{\text{max}} = -1 \]

There is a maximum angle that the satellite can reach!
Similar math to before for $\varepsilon > 1$

$$r = \sqrt{x^2 + y^2}, \cos \phi = x/r$$

$$r = \frac{c}{1 + \varepsilon \cos \phi}$$

$$r(1 + \varepsilon x/r) = c$$

$$r + \varepsilon x = c$$

$$r = c - \varepsilon x$$

$$r^2 = (c - \varepsilon x)^2 = x^2 + y^2$$

$$c^2 + \varepsilon^2 x^2 - 2ce \varepsilon x = x^2 + y^2$$

$$(1 - \varepsilon^2) x^2 + 2ce \varepsilon x + y^2 = c^2$$

Note sign swap compared to before

$$x^2 + \frac{2ce}{1 - \varepsilon^2} x + \frac{y^2}{1 - \varepsilon^2} = \frac{c^2}{(1 - \varepsilon^2)}$$

$$x^2 - \frac{2ce}{\varepsilon^2 - 1} x - \frac{y^2}{\varepsilon^2 - 1} = -\frac{c^2}{(\varepsilon^2 - 1)}$$

$$(x - d)^2 - \frac{y^2}{\varepsilon^2 - 1} = -\frac{c^2}{(1 - \varepsilon^2)} + d^2$$

Now $$(1-\varepsilon^2) < 0$$
Hyperbola:

\[
\frac{(x - \delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1
\]
Putting it all together

e=1  e=2

e=0.5  e=0

F
Let's work on some problems

8.28 and 8.29 in small groups or by yourself
Launching a satellite (not in Taylor)

Given a satellite launch with burnout (when the rocket shuts down) at a certain angle, how can we calculate the eccentricity and apogee/perigee of the orbit?
Consider burnout a distance $r_b$ from the center of the earth with velocity $v_b$

Call the angle between $v_b$ and $r_b$ to be $\gamma$

Angular momentum about center of earth is a constant $= mr_b v_b \sin \gamma = L$. At perigee/apogee, $mrv = L$ (since $r$ and $v$ are perpendicular)
Launching a satellite

Energy at burnout = \( \frac{m}{2}v_b^2 - \frac{GM_em}{r_b} = \text{Constant} \)

Energy later = \( \frac{m}{2}v^2 - \frac{GM_em}{r} = \frac{m}{2}v_b^2 - \frac{GM_em}{r_b} \)

\( v^2 - v_b^2 = 2GM_e \left( \frac{1}{r} - \frac{1}{r_b} \right) \)
Launching a satellite

**E conservation**

\[ v^2 - v_b^2 = 2GM_e \left( \frac{1}{r} - \frac{1}{r_b} \right) \]

\[ rv = r_b v_b \sin \gamma \]

**L conservation**

\[ v^2 = (r_b/r)^2 v_b^2 \sin^2 \gamma \]

\[ (r_b/r)^2 v_b^2 \sin^2 \gamma - v_b^2 = 2GM_e \left( \frac{1}{r} - \frac{1}{r_b} \right) \]

\[ \sin^2 \gamma - \left( \frac{r^2}{r_b^2} \right) = \frac{2GM_e}{v_b^2} \left( \frac{r}{r_b^2} - \frac{r^2}{r_b^3} \right) \]

\[ \sin^2 \gamma - \left( \frac{r^2}{r_b^2} \right) = \frac{2GM_e}{v_b^2 r_b} \left( \frac{r}{r_b} - \frac{r^2}{r_b^2} \right) \]

\[ \left( \frac{r}{r_b} \right)^2 \left( \frac{2GM_e}{r_b v_b^2} - 1 \right) - \frac{2GM_e}{r_b v_b^2} \left( \frac{r}{r_b} \right) + \sin^2 \gamma = 0 \]
Solving the quadratic equation

\[ \left(\frac{r}{r_b}\right)^2 \left(\frac{2GM_e}{r_b v_b^2} - 1\right) - \frac{2GM_e}{r_b v_b^2} \left(\frac{r}{r_b}\right) + \sin^2 \gamma = 0 \]

\[ k = \frac{2GM_e}{r_b v_b^2}, \quad x = \frac{r}{r_b} \]

\[ x^2 (k - 1) - kx + \sin^2 \gamma = 0 \]

\[ x = \frac{k \pm \sqrt{k^2 - 4(k - 1) \sin^2 \gamma}}{2(k - 1)} \]

Two solutions - smaller for perigee, larger for apogee
A satellite launched from earth burns out at a height of 300 km at 8,500 m/s and a zenith angle = 85 degrees. What are the orbit apogee, perigee and eccentricity?

\[
\left( \frac{r}{r_b} \right)^2 \left( \frac{2GM_e}{r_b v_b^2} - 1 \right) - \frac{2GM_e}{r_b v_b^2} \left( \frac{r}{r_b} \right) + \sin^2 \gamma = 0
\]

\[
k = \frac{2GM_e}{r_b v_b^2}, \ x = r/r_b
\]

\[
x^2(k-1) - kx + \sin^2 \gamma = 0
\]

\[
x = \frac{k \pm \sqrt{k^2 - 4(k-1)\sin^2 \gamma}}{2(k-1)}
\]

Careful! \( r \) is height from center of earth (need 6.38e6 meters extra)
x = 0.979, 1.56

Altitude at perigee = 
0.979(6.38e6 + 300e3) meters 
= 160 km above the earth

Altitude at apogee = 
1.56(6.38e6 + 300e3) meters 
=4000 km above the earth
Altitude at perigee = 
0.979(6.38e6 + 300e3) = 6540 km above earth center

Altitude at apogee = 
1.56(6.38e6 + 300e3) = 10420 km above earth center

\[
r(\phi) = \frac{c}{1 + \epsilon \cos \phi}
\]

\[
\frac{r_{\text{min}}}{r_{\text{max}}} = \frac{1 - \epsilon}{1 + \epsilon} = 0.63
\]

so eccentricity = 0.23
Hi, MOM!

Know that $r$ of first orbit at given $\Phi = r$ of second orbit (after a thrust push)
A special case of thrust

Consider thrust at perigee/apogee in tangential direction (so direction of velocity doesn’t change)
A special case of thrust

\[ r(\phi) = \frac{c}{1 + \epsilon \cos \phi} \]

\[ r_1(0) = \frac{c_1}{1 + \epsilon_1} \]

\[ r_2(0) = \frac{c_2}{1 + \epsilon_2} \]

\[ r_1(0) = r_2(0) \rightarrow \frac{c_1}{1 + \epsilon_1} = \frac{c_2}{1 + \epsilon_2} \]

Thrust factor

\[ c = \frac{l^2}{\mu \gamma} \rightarrow c_2 = \lambda^2 c_1 \]

\[ v_2 = \lambda v_1 \]

\[ l = \mu r v \text{ constant} \rightarrow l_2 = \lambda v_1 \]
A special case of thrust

\[
\frac{c_1}{1 + \epsilon_1} = \frac{c_2}{1 + \epsilon_2}
\]

\[c_2 = \lambda^2 c_1\]

\[
\frac{c_1}{1 + \epsilon_1} = \frac{c_1 \lambda^2}{1 + \epsilon_2}
\]

\[1 + \epsilon_2 = \lambda^2 (1 + \epsilon_1)\]

\[\epsilon_2 = \lambda^2 \epsilon_1 + (\lambda^2 - 1)\]

If \(\lambda\) positive, what does that mean for eccentricity? Similarly, what happens if it is negative?
Let’s go over

First part of example
8.6 together
Homework, as usual due in 1 week

8.3, 8.12, 8.15, 8.17, 8.18, 8.33