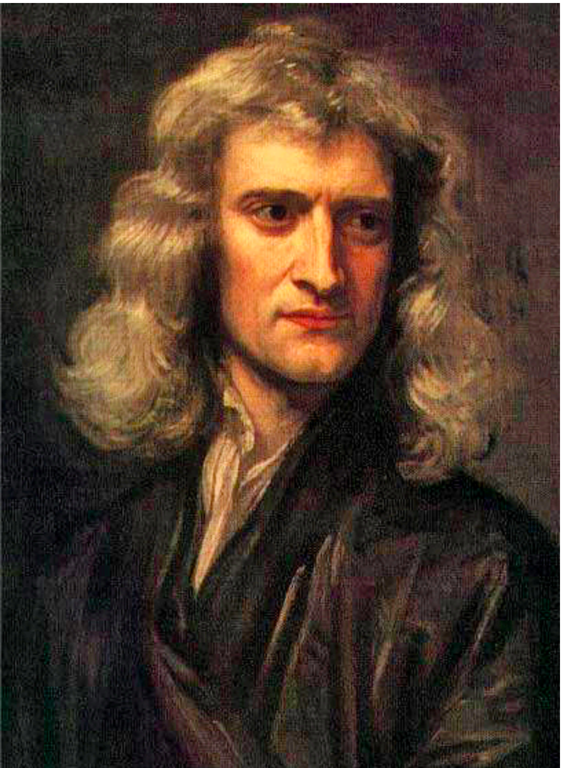
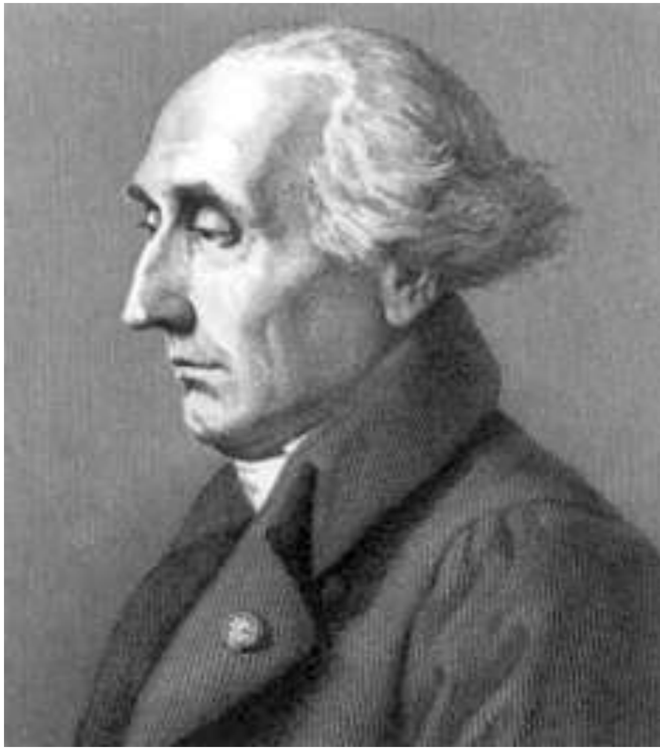
Armed with Newton and Lagrangian mechanics...

Let’s tackle a new problem (chapter 8 of Taylor)

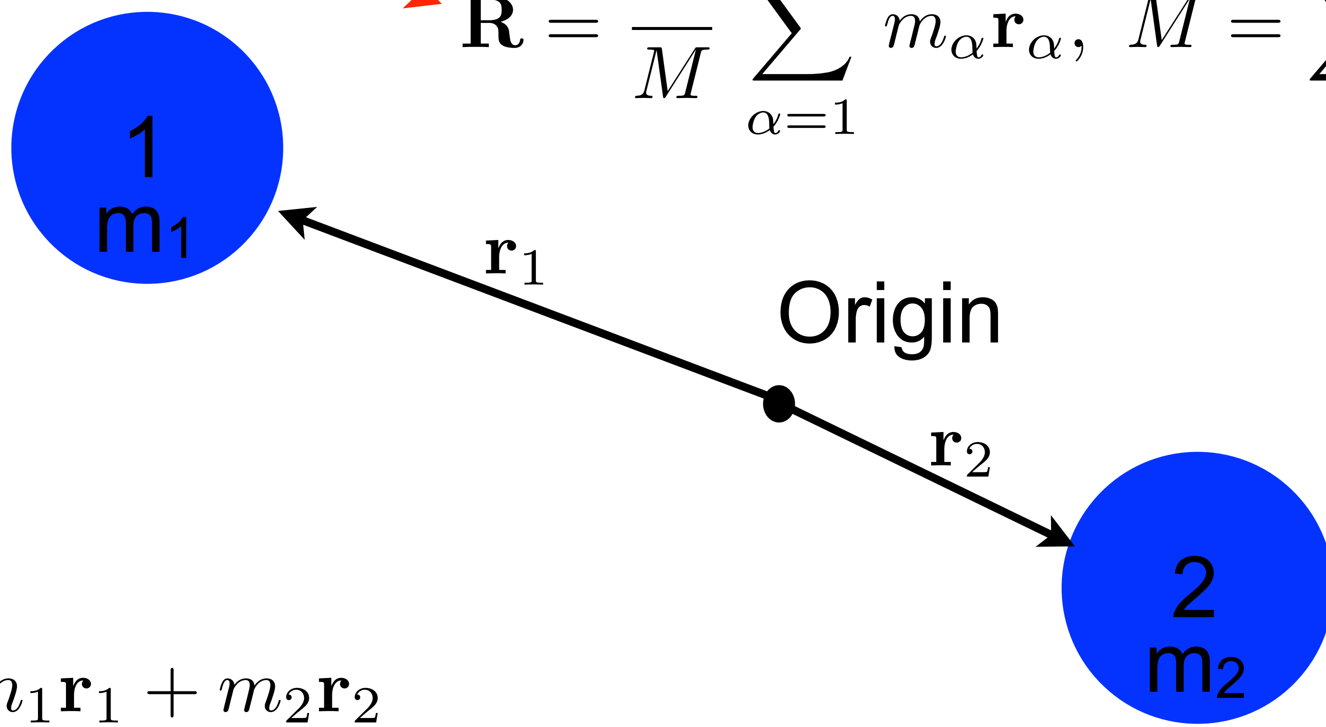




Two-body central force motion

Center of Mass

$$\mathbf{R} = \frac{1}{M} \sum_{\alpha=1}^{\alpha=N} m_\alpha \mathbf{r}_\alpha, \quad M = \sum m_\alpha$$

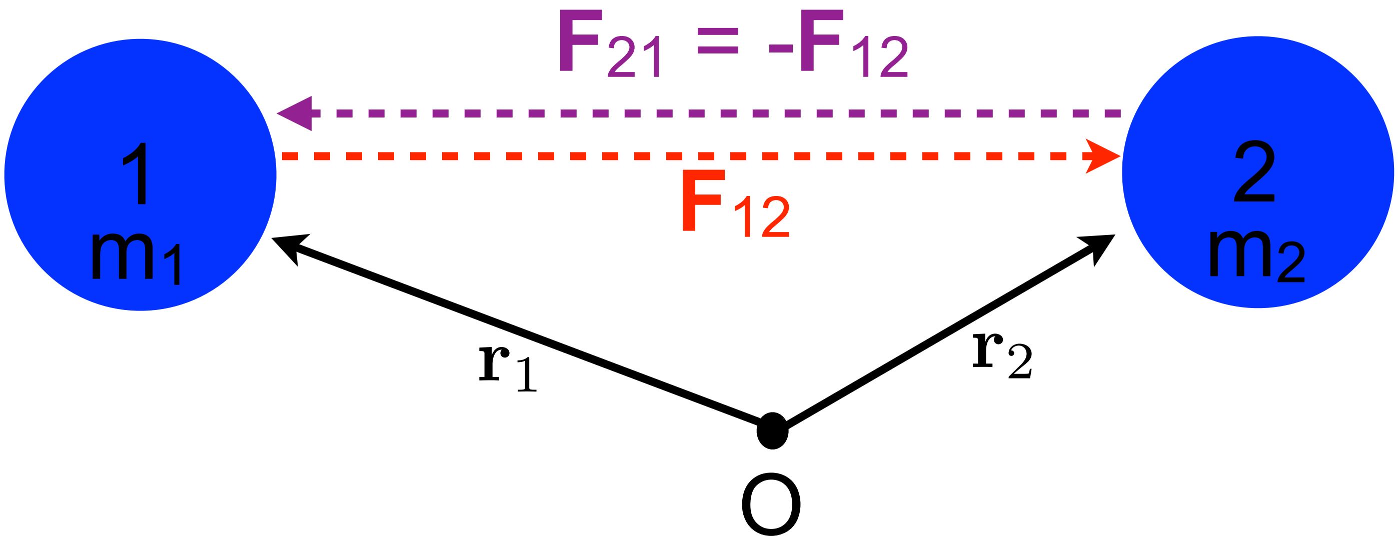


$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

$$M = m_1 + m_2$$

Recall this from earlier in the course, now only with two particles

Two-body central force motion

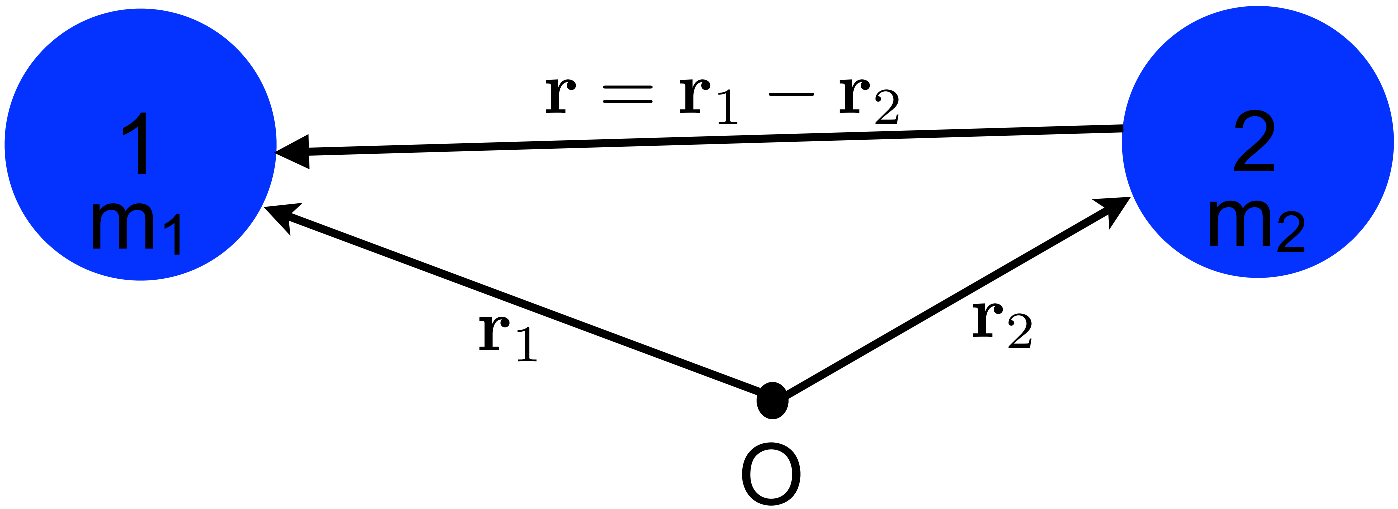


$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

$$M = m_1 + m_2$$

Recall definition: If internal forces are along vector connecting particles, we call them **central forces**

Two-body central force motion



$$\mathbf{F}_{12} = F(\mathbf{r}_1, \mathbf{r}_2)\hat{\mathbf{r}}$$

$$\mathbf{F}_{12} = F(\mathbf{r}_1 - \mathbf{r}_2)\hat{\mathbf{r}}$$

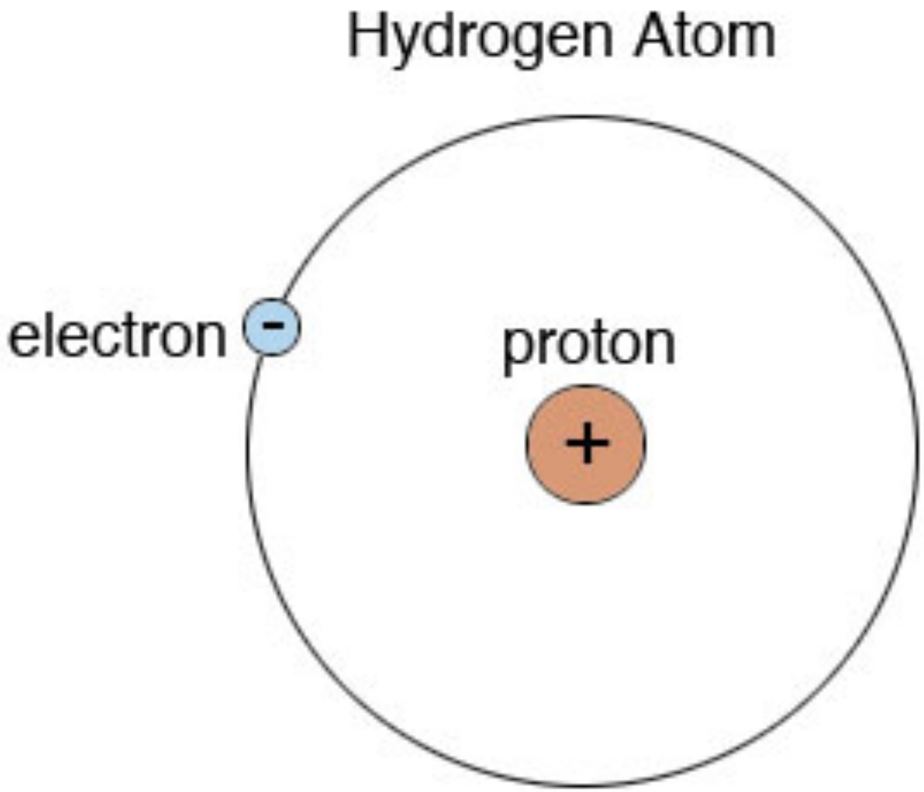
$$\mathbf{F}_{12} = F(\mathbf{r})\hat{\mathbf{r}}$$

$$\mathbf{F}_{21} = -\mathbf{F}_{12}$$

$$U(\mathbf{r}_1, \mathbf{r}_2) = U(|\mathbf{r}_1 - \mathbf{r}_2|)\hat{\mathbf{r}} \text{ for conservative central force}$$

$$U = U(r)$$

What's sorts of problems can we think of?



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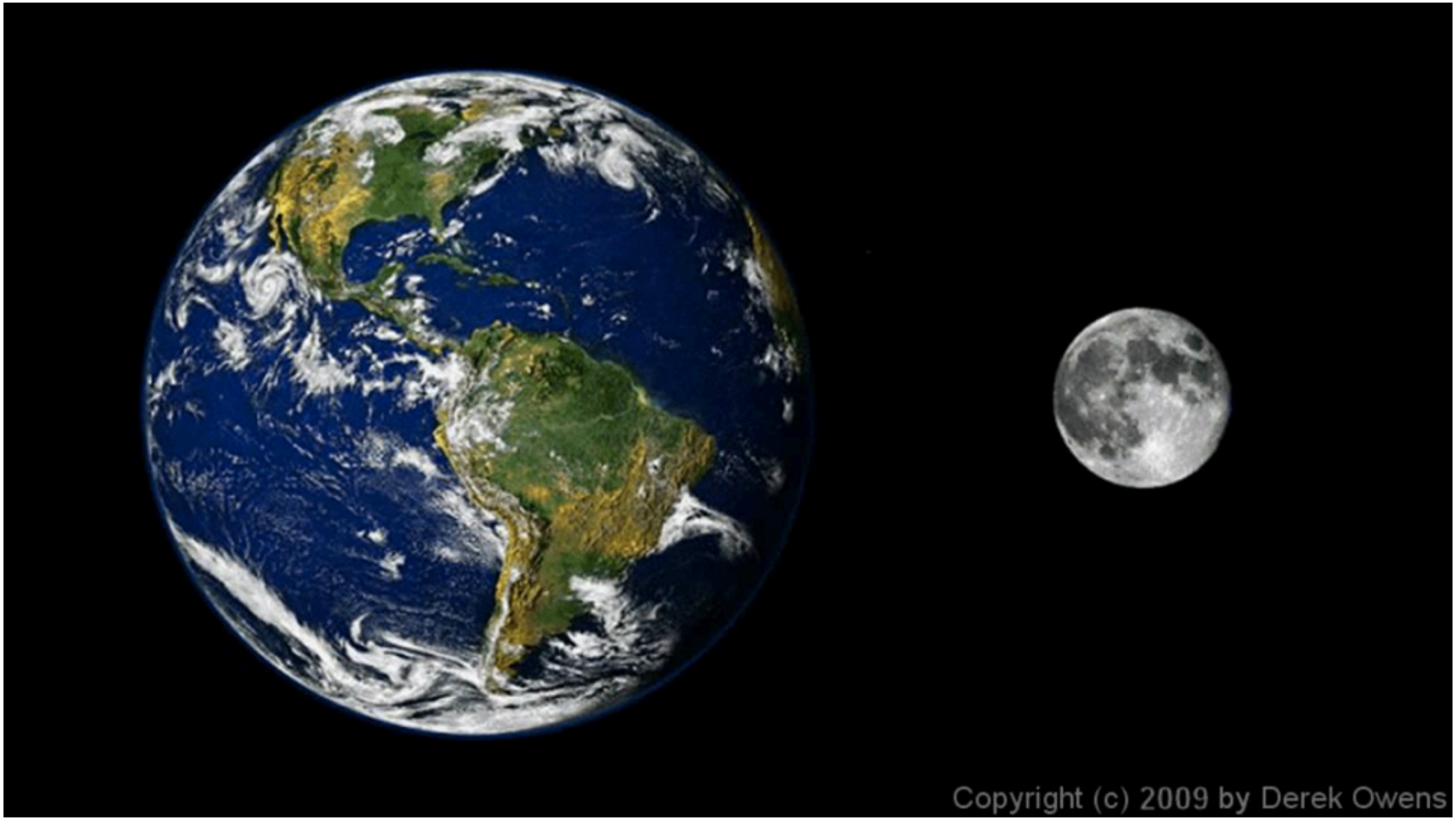
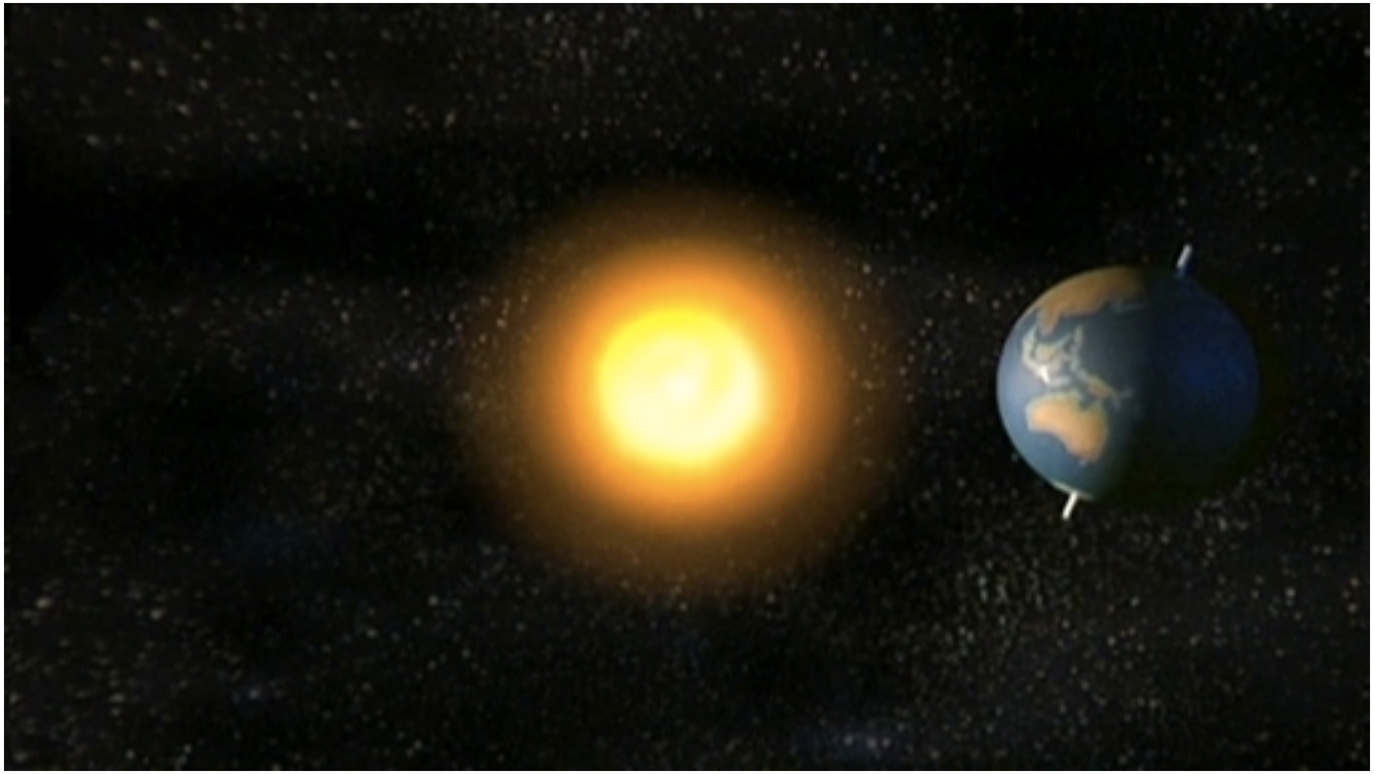
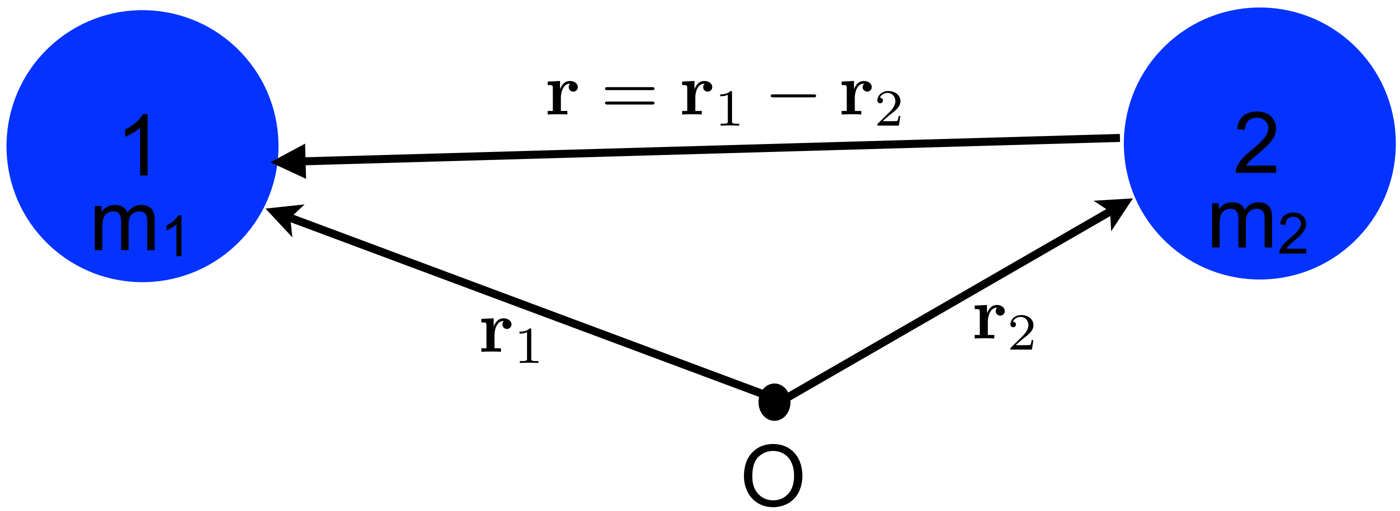


What's sorts of problems can we think of?



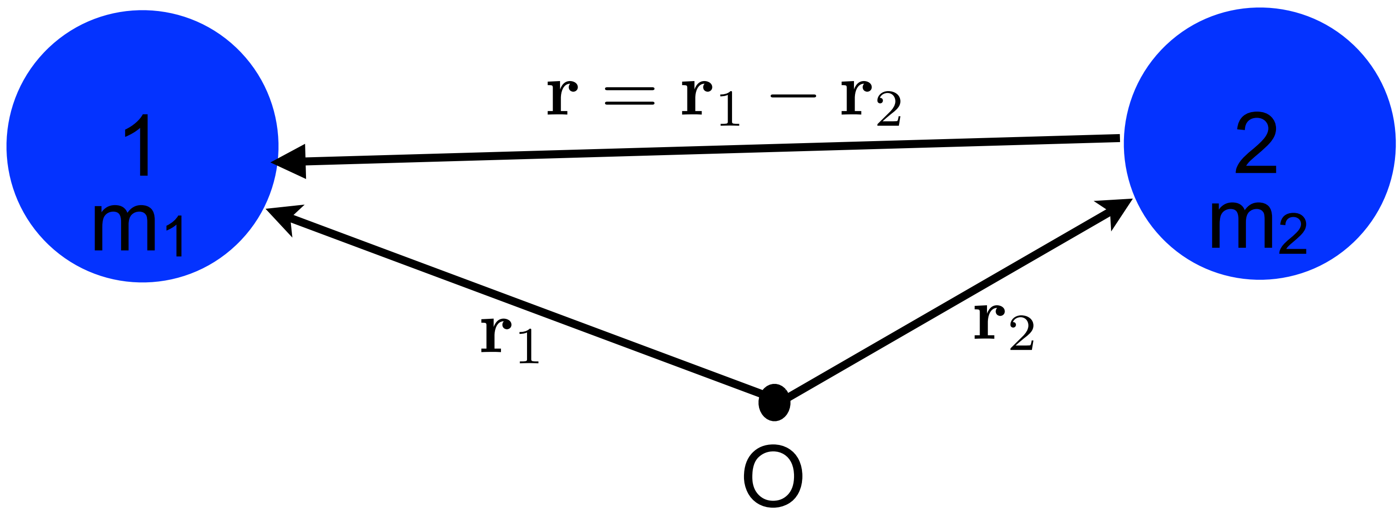
Writing down the Lagrangian



$$\mathcal{L} = \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2 - U(r)$$

Note that we are starting out with 6 degrees of freedom! Let's hope that we can reduce this

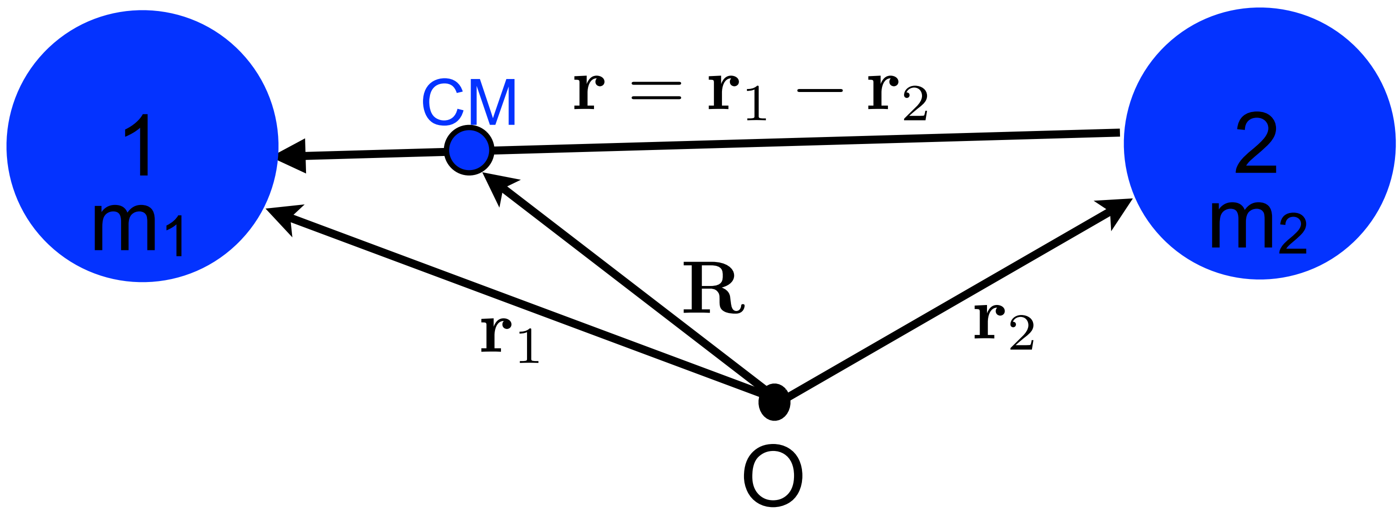
Writing down the Lagrangian



$$\mathcal{L} = \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2 - U(r)$$

Good news when using the Lagrangian formalism is that we can pick 6 generalized coordinates. Which ones?

Writing down the Lagrangian

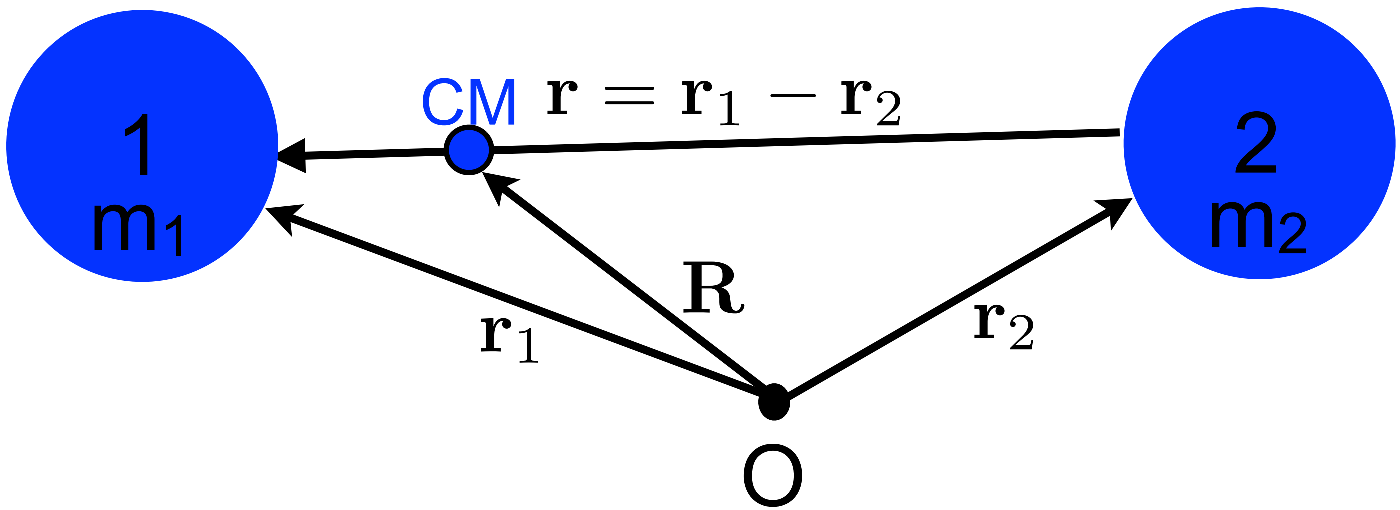


$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

$$M = m_1 + m_2$$

Center of mass of system gives us 3 potentially useful coordinates

Writing down the Lagrangian



Recall that CoM moves as:

$$\mathbf{P} = M\dot{\mathbf{R}}$$

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \sum \mathbf{F}^{ext}$$

$$\mathbf{F}^{\mathbf{ext}} = \mathbf{0} \rightarrow \dot{\mathbf{P}} = \mathbf{0}$$

$$\dot{\mathbf{P}} = 0 \rightarrow \dot{R} = \text{constant}$$

Free to choose inertial frame in which center of mass is at rest

Two-body central force motion

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}$$

$$M = m_1 + m_2$$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

$$\mathbf{r}_1 = \mathbf{r} + \mathbf{r}_2$$

$$\mathbf{r}_2 = \frac{M\mathbf{R} - m_1\mathbf{r}_1}{m_2}$$

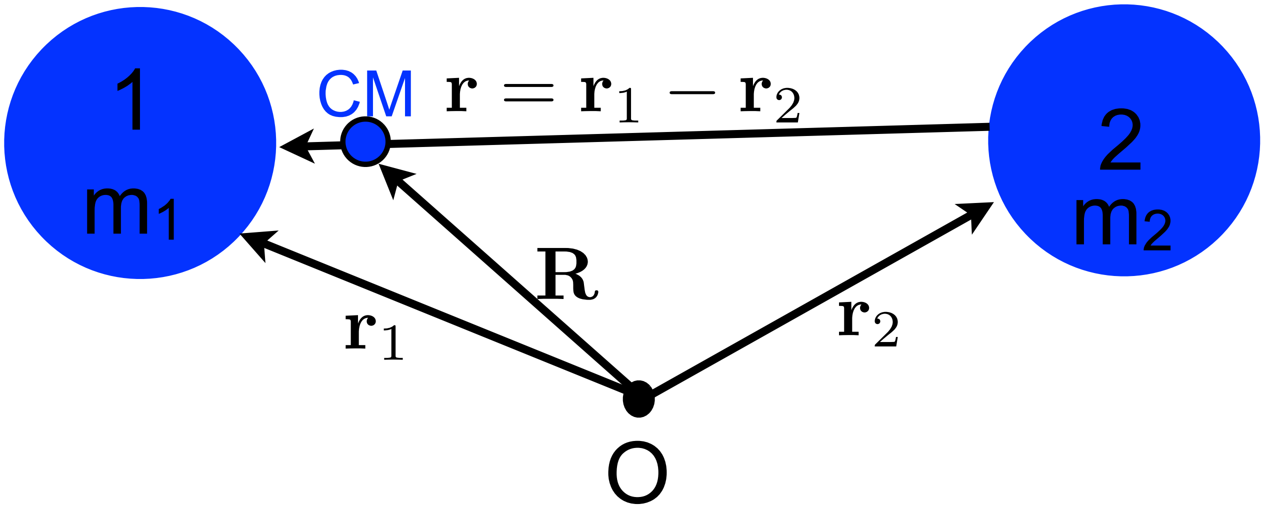
$$\mathbf{r}_1 = \mathbf{r} + \frac{M\mathbf{R} - m_1\mathbf{r}_1}{m_2}$$

$$\mathbf{r}_1(1 + \frac{m_1}{m_2}) = \mathbf{r} + \frac{M\mathbf{R}}{m_2}$$

$$\mathbf{r}_1\frac{m_1 + m_2}{m_2} = \mathbf{r} + \frac{M\mathbf{R}}{m_2}$$

$$\mathbf{r}_1\frac{M}{m_2} = \mathbf{r} + \frac{M\mathbf{R}}{m_2}$$

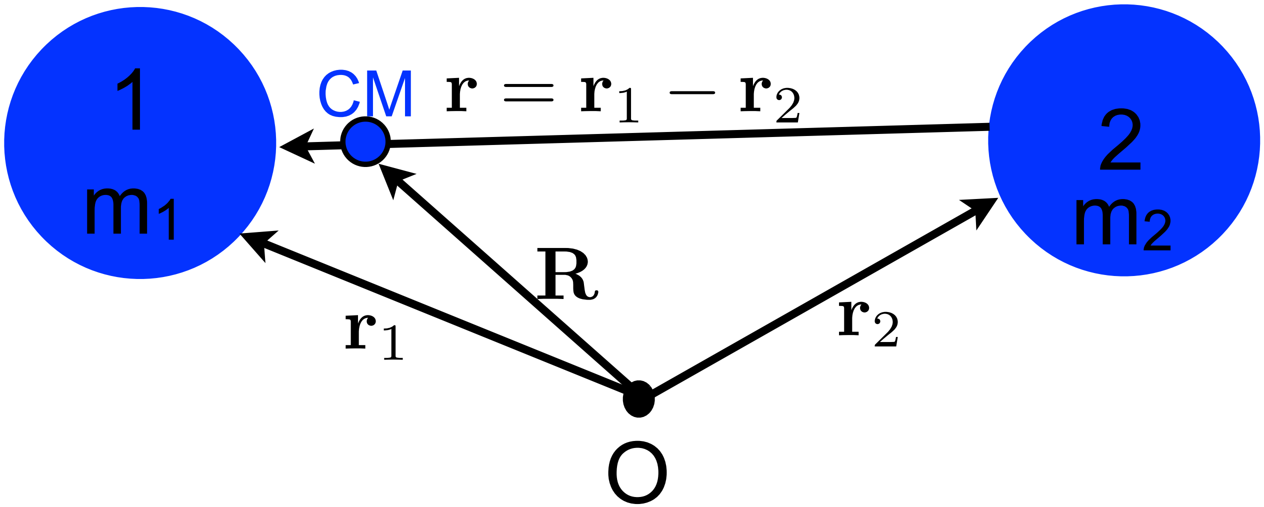
$$\mathbf{r}_1 = \frac{m_2}{M}\mathbf{r} + \mathbf{R}$$



Similarly,

$$\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}$$

Does this tell us what we might guess already?



$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}$$

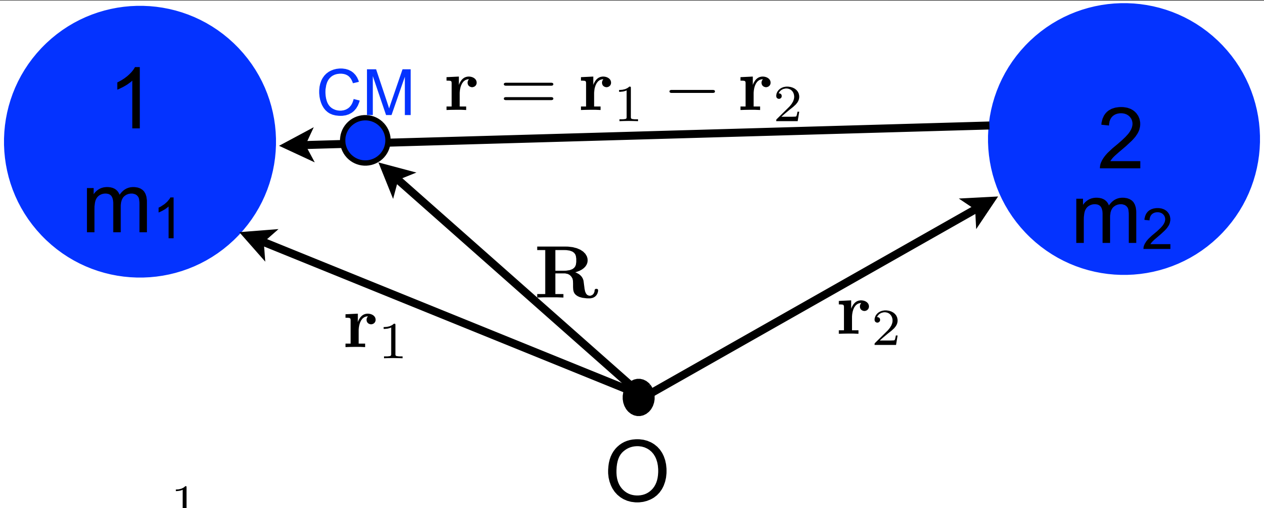
$$\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}$$

For very large mass $m_1 >> m_2$, $M=(m_1+m_2)\sim m_1$ and $\mathbf{r}_1 \sim \mathbf{R}$, $\mathbf{r}_2 \sim \mathbf{R-r}$

Writing down the kinetic energy

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}$$

$$\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}$$



$$T = \frac{1}{2}\left(m_1\dot{\mathbf{r}}_1^2 + m_2\dot{\mathbf{r}}_2^2\right)$$

$$T = \frac{1}{2}\left(m_1\left[\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right]^2 + m_2\left[\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right]^2\right)$$

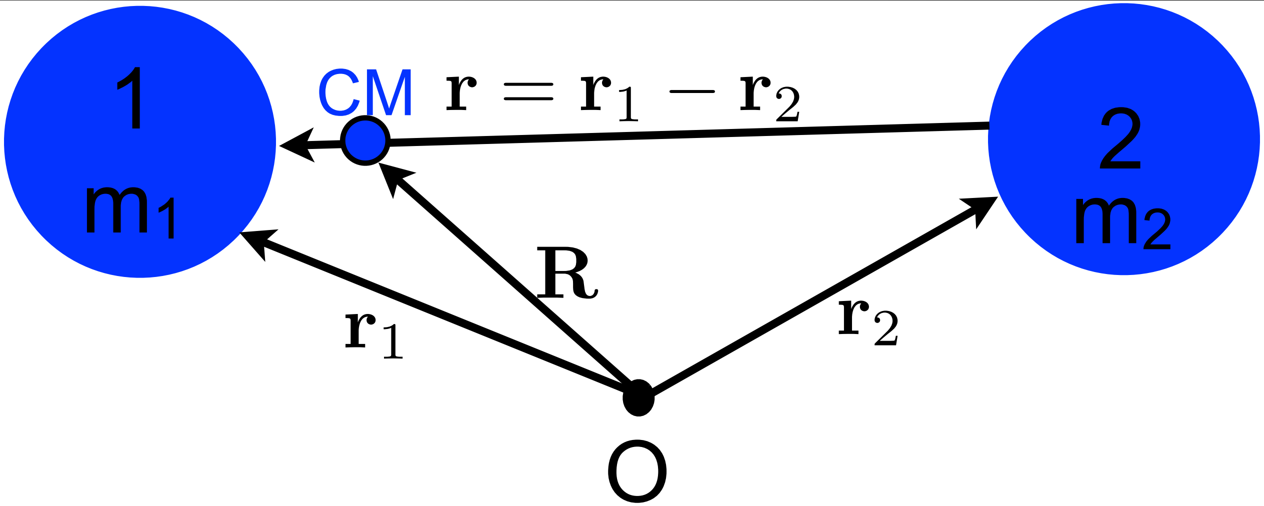
$$T = \frac{1}{2}\left(m_1\left[\dot{\mathbf{R}}^2 + \frac{m_2^2}{M^2}\dot{\mathbf{r}}^2 + 2\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}\frac{m_2}{M}\right] + m_2\left[\dot{\mathbf{R}}^2 + \frac{m_1^2}{M^2}\dot{\mathbf{r}}^2 - 2\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}\frac{m_1}{M}\right]\right)$$

$$T = \frac{1}{2}\left((m_1+m_2)\dot{\mathbf{R}}^2 + \frac{m_1m_2^2 + m_2m_1^2}{M^2}\dot{\mathbf{r}}^2 + 2\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}\frac{m_1m_2 - m_2m_1}{M}\right)$$

$$T = \frac{1}{2}\left((m_1+m_2)\dot{\mathbf{R}}^2 + \frac{(m_1+m_2)(m_1m_2)}{M^2}\dot{\mathbf{r}}^2\right)$$

$$T = \frac{1}{2}\left(M\dot{\mathbf{R}}^2 + \frac{m_1m_2}{M}\dot{\mathbf{r}}^2\right)$$

Simplifying the kinematic energy



Reduced mass (always smaller than m_1 and m_2)

$$\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2}$$

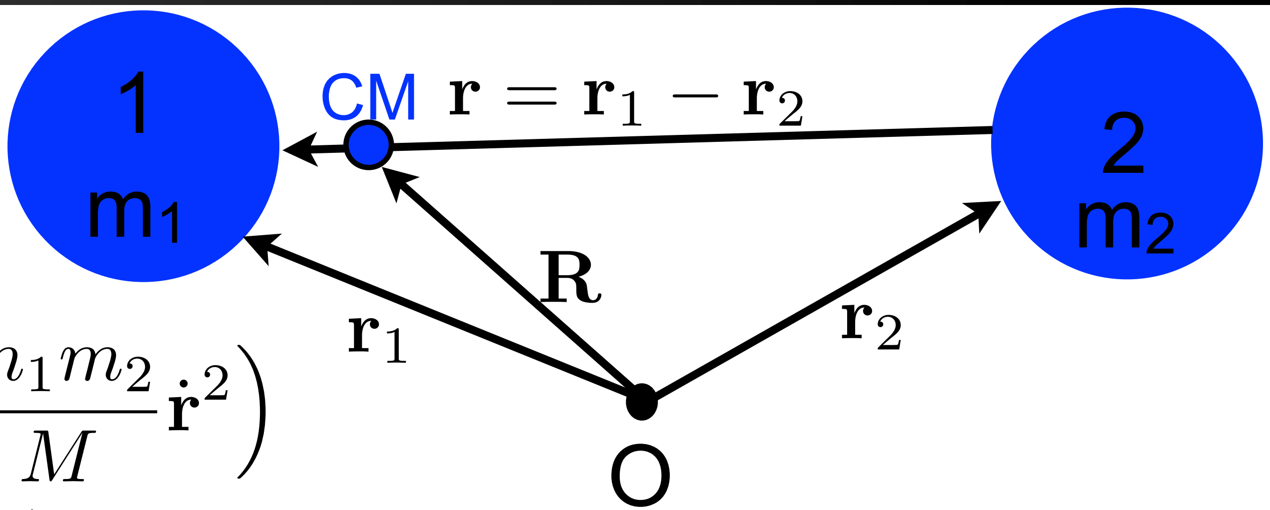
$$T = \frac{1}{2}\left(M\dot{\mathbf{R}}^2 + \frac{m_1 m_2}{M}\dot{\mathbf{r}}^2\right)$$

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2$$

For very large mass $m_1 >> m_2$, $\mu=(m_1 m_2)/m_1 \sim m_2$

For equal masses $\mu=(m*m)/(2m) \sim m/2$

What's the meaning/signifiance?



$$T = \frac{1}{2}\left(M\dot{\mathbf{R}}^2 + \frac{m_1 m_2}{M}\dot{\mathbf{r}}^2\right)$$

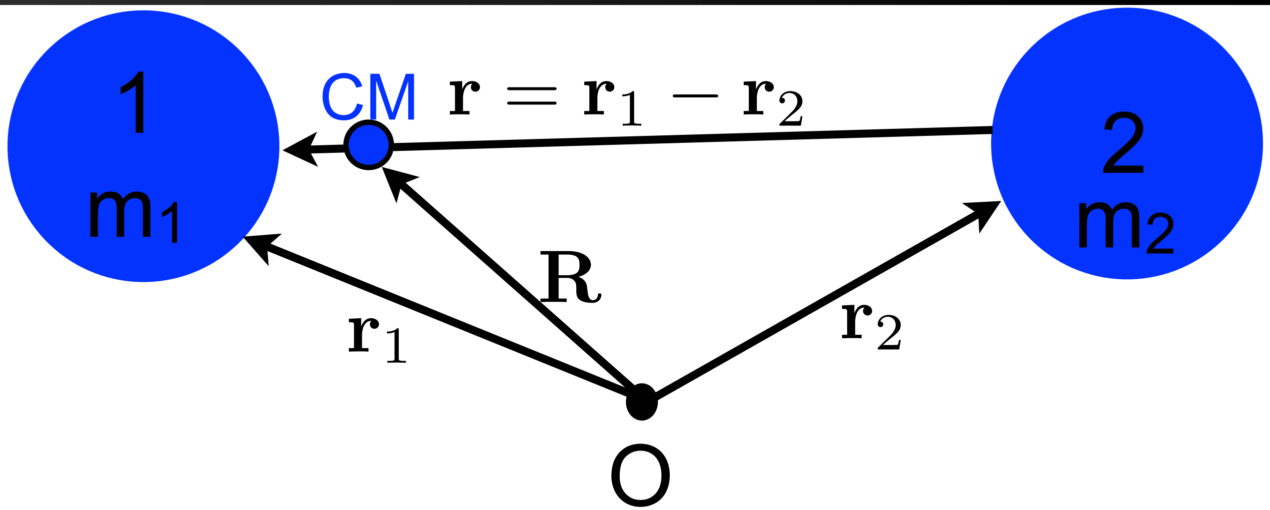
$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2$$

Kinetic energy is the same as energy of two particles (not real!):

1) Particle with mass $M = m_1 + m_2$ moving with speed of the center of mass

2) Particle with mass μ moving with speed of relative position

Let's write down the Lagrangian



$$\mathcal{L} = T - U = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \mu \dot{\mathbf{r}}^2 - U(r)$$

$$\mathcal{L} = \frac{1}{2} M \dot{\mathbf{R}}^2 + \left(\frac{1}{2} \mu \dot{\mathbf{r}}^2 - U(r) \right)$$

$$\mathcal{L} = \mathcal{L}_{cm} + \mathcal{L}_{rel}$$

Only involves CoM velocity

Only involves relative coordinate and motion

With a Lagrangian, we can find the Equations of Motion

$$\mathcal{L} = \frac{1}{2} M \dot{\mathbf{R}}^2 + \left(\frac{1}{2} \mu \dot{\mathbf{r}}^2 - U(r) \right)$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{R}} = 0 = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{R}}} = M \frac{d}{dt} \dot{\mathbf{R}} = \ddot{\mathbf{R}}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{r}} = -\frac{\partial U}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \mu \frac{d}{dt} \dot{\mathbf{r}} = \mu \ddot{\mathbf{r}}$$

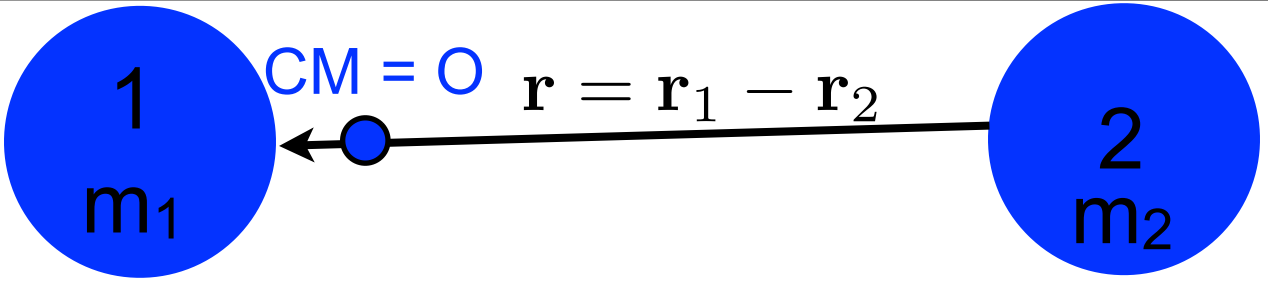
$$M \ddot{\mathbf{R}} = 0 \rightarrow \dot{\mathbf{R}} = \text{constant}$$

$$\mu \ddot{\mathbf{r}} = -\frac{\partial U}{\partial r} = \mathbf{F}(\mathbf{r})$$

Center of mass moves with constant velocity

Only relevant equation/non-trivial motion involving the force

We can simplify further



$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

$$\mathbf{r}_1 = \frac{m_2}{M}\mathbf{r}$$

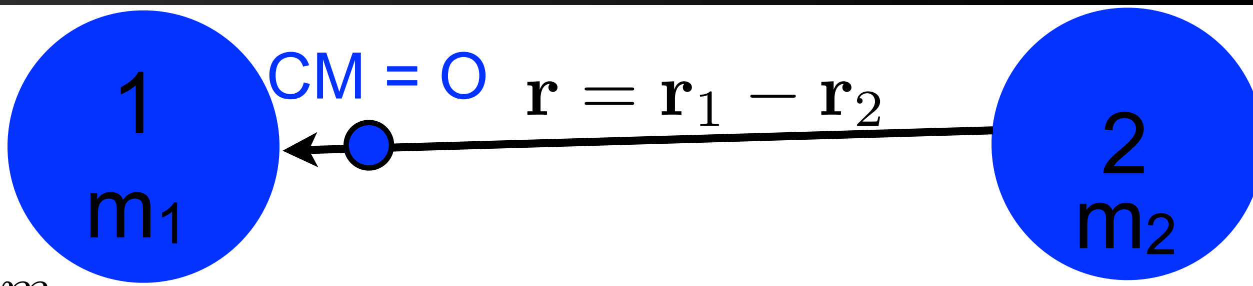
$$\mathbf{r}_2 = -\frac{m_1}{M}\mathbf{r}$$

$$M\ddot{\mathbf{R}} = 0 \rightarrow \dot{\mathbf{R}} = \text{constant} \qquad \mu\ddot{\mathbf{r}} = -\frac{\partial U}{\partial r} = \mathbf{F}(\mathbf{r})$$

Choose inertial frame with **R** at rest at the origin

Problem is now (for all m_1, m_2) only three-dimensional. But consider the special case with very large mass $m_1 >> m_2$, $\mu \sim m_2$ and $\mathbf{r}_1 \sim O$, and $r_2 \sim -\mathbf{r}$. What does it look like?

What about angular momentum?



$$\mathbf{r}_1 = \frac{m_2}{M}\mathbf{r}$$

$$\mathbf{r}_2 = -\frac{m_1}{M}\mathbf{r}$$

$$\mathbf{L} = \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2$$

$$\mathbf{L} = \frac{m_2}{M}\mathbf{r} \times (m_1\dot{\mathbf{r}}_1) - \frac{m_1}{M}\mathbf{r} \times (m_2\dot{\mathbf{r}}_2)$$

$$\mathbf{L} = \frac{m_2}{M}\mathbf{r} \times (m_1\frac{m_2}{M}\dot{\mathbf{r}}) - \frac{m_1}{M}\mathbf{r} \times (m_2\frac{-m_1}{M}\dot{\mathbf{r}})$$

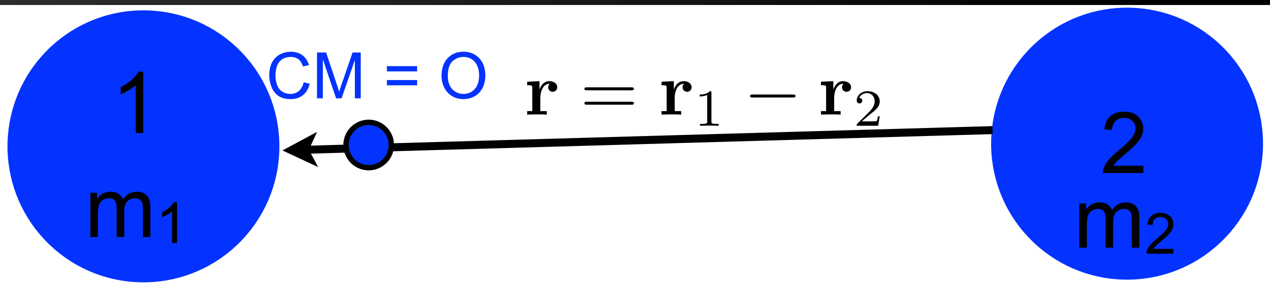
$$\mathbf{L} = \frac{m_1 m_2}{M^2}\Big(\mathbf{r} \times (m_2\dot{\mathbf{r}}) + \mathbf{r} \times (m_1\dot{\mathbf{r}})\Big)$$

$$\mathbf{L} = \frac{\mu}{m_1 + m_2}(m_1 + m_2)(\mathbf{r} \times \dot{\mathbf{r}})$$

$$\mathbf{L} = \mathbf{r} \times (\mu\dot{\mathbf{r}})$$

Total angular momentum of system in CM frame is just that of single particle at **r** with mass μ

What does conservation of angular momentum tell us?



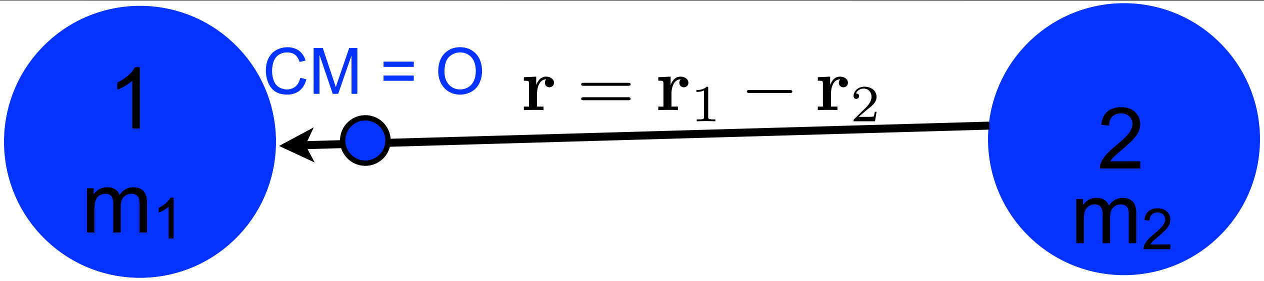
$$\mathbf{L} = \mathbf{r} \times (\mu \dot{\mathbf{r}})$$

L is a constant (because there are no external forces), which means it always points in same direction.

So our three-dimensional problem is now reduced to a two-dimensional problem (remember, we started with six dimensions!)

Is this clear? →

Let's finally write down a Lagrangian in detail



$$\mathcal{L} = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)$$

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{d}{dt}(\mu r^2\dot{\phi})$$

$$\mu r^2\dot{\phi} = l = \text{constant}$$

Conservation of angular momentum

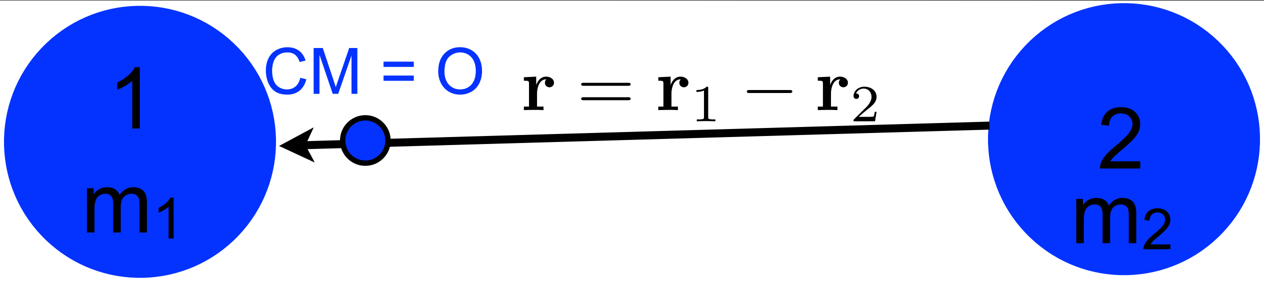
$$\frac{\partial \mathcal{L}}{\partial r} = \mu r\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu\dot{r}$$

$$\frac{d}{dt}(\mu\dot{r}) = \mu r\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

$$\mu\ddot{r} = \mu r\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Let's simplify further



$$\mu r^2 \dot{\phi} = l = \text{constant}$$

$$\dot{\phi} = \frac{l}{\mu r^2}$$

$$\mu \ddot{r} = \mu r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

$$\mu \ddot{r} = -\frac{\partial U}{\partial r} + F_{cf}$$

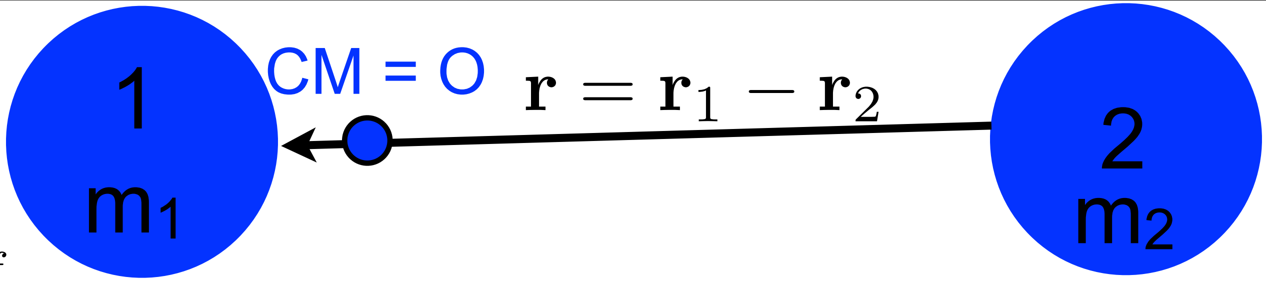
$$F_{cf} = \mu r \dot{\phi}^2 = \frac{l^2}{\mu r^3}$$

Centrifugal Force

$$F_{cf} = -\frac{d}{dr}\left(\frac{l^2}{2\mu r^2}\right) = -\frac{dU_{cf}}{dr}$$

$$U_{cf}(r) = \frac{l^2}{2\mu r^2}$$

Let's simplify further



$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

$$\mu\ddot{r} = \mu r\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

$$\mu\ddot{r} = -\frac{\partial U}{\partial r} + F_{cf}$$

$$F_{cf} = \mu r\dot{\phi}^2 = \frac{l^2}{\mu r^3}$$

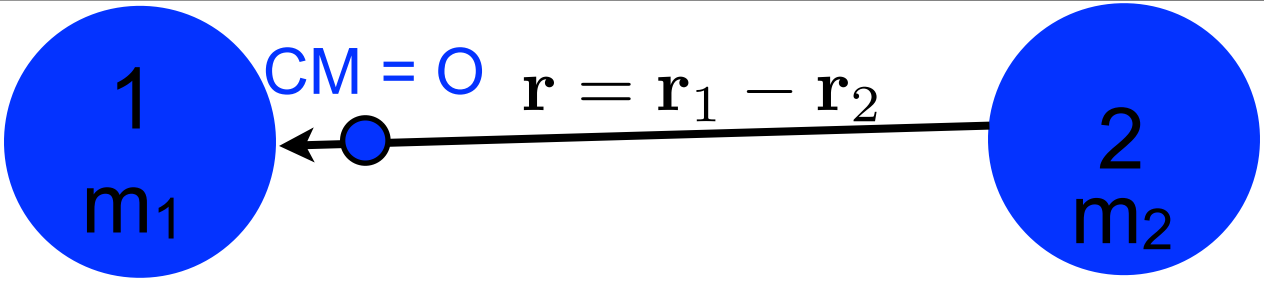
$$F_{cf} = -\frac{d}{dr}\left(\frac{l^2}{2\mu r^2}\right) = -\frac{dU_{cf}}{dr}$$

$$U_{cf}(r) = \frac{l^2}{2\mu r^2}$$

$$\mu\ddot{r} = -\frac{d}{dr}\left[U(r) + U_{cf}(r)\right] = -\frac{d}{dr}U_{eff}(r)$$

$$U_{eff}(r) = U(r) + U_{cf}(r) = U(r) + \frac{l^2}{2\mu r^2}$$

Putting it together



$$\mu\ddot{r} = -\frac{d}{dr}\left[U(r) + U_{cf}(r)\right] = -\frac{d}{dr}U_{eff}(r)$$

$$U_{eff}(r) = U(r) + U_{cf}(r) = U(r) + \frac{l^2}{2\mu r^2}$$

Radial motion of this fictional particle with mass μ behaves as if it was moving in a single dimensional effective potential given by the nominal one + the “fictitious” one

Conservation of energy

$$\mu\ddot{r} = -\frac{d}{dr}U_{eff}(r)$$

$$\dot{r}(\mu\ddot{r}) = \dot{r}\left(-\frac{d}{dr}U_{eff}(r)\right)$$

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = \frac{1}{2}\mu(2\dot{r})\frac{d}{dt}\dot{r} = \mu\dot{r}\ddot{r}$$

$$-\frac{d}{dt}U_{eff}(r) = -\frac{d}{dr}U_{eff}(r)\frac{dr}{dt} = \dot{r}\left(-\frac{d}{dr}U_{eff}(r)\right)$$

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{d}{dt}U_{eff}(r)$$

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)\right) = 0$$

$$\frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) = \text{const}$$

Conservation of energy

$$\frac{1}{2}\mu\dot{r}^2 + U(r) + U_{cf}(r) = \text{const}$$

$$\frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{l^2}{2\mu r^2} = \text{const}$$

$$\frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{1}{2}\mu r^2\dot{\phi}^2 = \text{const} = \text{E}$$

So everything we know about the 1d problem applies here, which simplifies things quite a bit!

Let's work out

Examples 8.1-8.2 together
Problem 8.7 in small groups or by yourself

Equations of the orbit

$$\mu\ddot{r} = -\frac{d}{dr}\left[U(r) + U_{cf}(r)\right] = -\frac{d}{dr}U_{eff}(r)$$

$$U_{eff}(r) = U(r) + U_{cf}(r) = U(r) + \frac{l^2}{2\mu r^2}$$

This gives us the equation r(t), but we might want to know r(Φ) instead...

Time to play some tricks

Rewriting the radial equation

$$\mu\ddot{r} = F(r) + \frac{l^2}{\mu r^3}$$

$$l = \mu r^2 \dot{\phi}$$

Careful!

$\mu \neq u$

$$u = \frac{1}{r}, r = \frac{1}{u}$$

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{l}{\mu r^2}\frac{d}{d\phi} = \frac{lu^2}{\mu}\frac{d}{d\phi}$$

$1/r^2 = u^2$

Rewriting the radial equation

Careful!

$\mu \neq u$

$$\frac{d}{dt} = \frac{lu^2}{\mu}\frac{d}{d\phi}$$

$$\dot{r} = \frac{d}{dt}r = \frac{lu^2}{\mu}\frac{d}{d\phi}r = \frac{lu^2}{\mu}\frac{d}{d\phi}\left(\frac{1}{u}\right) = \frac{lu^2}{\mu}\frac{-1}{u^2}\frac{du}{d\phi}$$

$$\dot{r} = \frac{-l}{\mu}\frac{du}{d\phi}$$

$$\ddot{r} = \frac{d}{dt}\dot{r} = \frac{lu^2}{\mu}\frac{d}{d\phi}\left(\frac{-l}{\mu}\frac{du}{d\phi}\right)$$

$$\ddot{r} = -\frac{l^2u^2}{\mu^2}\frac{d^2u}{d\phi^2}$$

Plugging back in

Careful!

$\mu \neq u$

$$\mu\ddot{r} = F(r) + \frac{l^2}{\mu r^3}$$

$$\mu\ddot{r} = F(1/u) + \frac{u^3 l^2}{\mu}$$

$$\ddot{r} = -\frac{l^2 u^2}{\mu^2}\frac{d^2 u}{d\phi^2}$$

$$-\mu\left(\frac{l^2 u^2}{\mu^2}\frac{d^2 u}{d\phi^2}\right) = F(1/u) + \frac{u^3 l^2}{\mu}$$

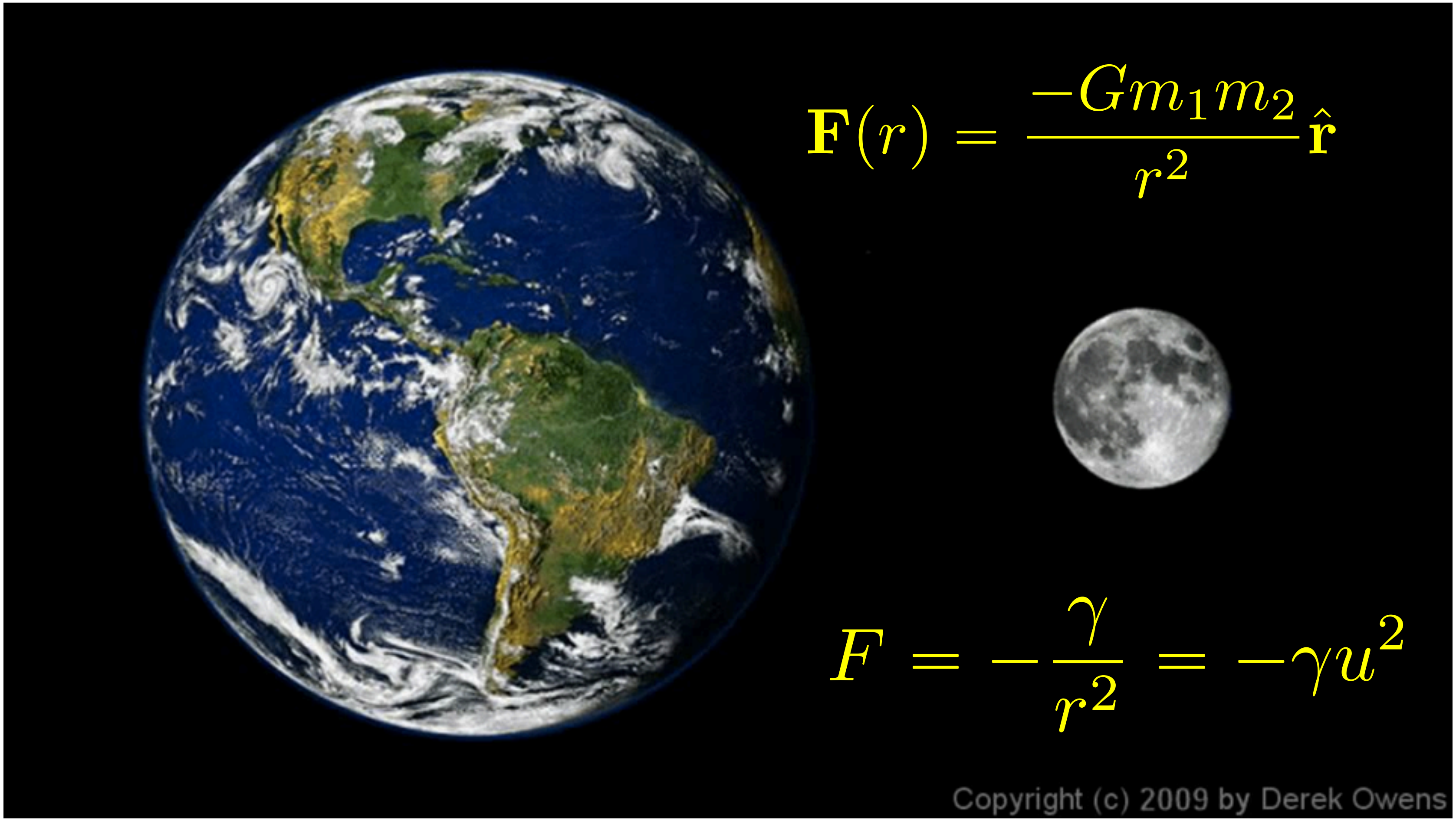
$$\frac{d^2 u}{d\phi^2} = \frac{-\mu}{l^2 u^2}F(1/u) - u$$

More compact:

$$u''(\phi) = -\frac{\mu}{l^2 u^2}F - u(\phi)$$

The two-body gravitational problem

$$u''(\phi) = -\frac{\mu}{l^2 u^2} F - u(\phi)$$



The two-body gravitational problem

$$u''(\phi) = -\frac{\mu}{l^2u^2}F - u(\phi)$$

$$u'' = -\frac{\mu}{l^2u^2}(-\gamma u^2) - u$$

$$u'' = \frac{\mu\gamma}{l^2} - u$$

$$w = u - \frac{\mu\gamma}{l^2}$$

$$w' = u', w'' = u''$$

$$w'' = \frac{\mu\gamma}{l^2} - u = \frac{\mu\gamma}{l^2} - (w + \frac{\mu\gamma}{l^2})$$

$$w'' = \frac{\mu\gamma}{l^2} - w - \frac{\mu\gamma}{l^2}$$

$$w'' = -w \rightarrow w(\phi) = A\cos(\phi - \delta)$$

$$u(\phi) = A\cos(\phi - \delta) + \frac{\mu\gamma}{l^2}$$

$$u(\phi) = \frac{\mu\gamma}{l^2}(1 + \epsilon\cos\phi)$$

Freedom to define coordinates so that δ=0

$$\epsilon = \frac{Al^2}{\mu\gamma}$$

Solution to the two-body gravitational problem

$$u(\phi) = \frac{\mu\gamma}{l^2}(1 + \epsilon\cos\phi) = \frac{1}{r}$$

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi}$$

$$c = \frac{l^2}{\mu\gamma} = \frac{l^2}{G m_1 m_2 \mu}$$

$$\epsilon < 1 \rightarrow r(\phi) \text{ bounded}$$

$$\epsilon > 1 \rightarrow r(\phi) \text{ can grow to } \infty$$

Studying the bounded orbits

$$r(\phi) = \frac{c}{1 + \epsilon \cos \phi}$$

$$r_{min} = \frac{c}{1 + \epsilon}$$ perihelion/perigee (ϕ=0)

$$r_{max} = \frac{c}{1 - \epsilon}$$ aphelion/apogee (ϕ=π)

Rewriting the solution (Problem 8.16)

$$r = \sqrt{x^2 + y^2}, \cos\phi = x/r$$

$$r = \frac{c}{1 + \epsilon\cos\phi}$$

$$r(1 + \epsilon x/r) = c$$

$$r + \epsilon x = c$$

$$r = c - \epsilon x$$

$$r^2 = (c - \epsilon x)^2 = x^2 + y^2$$

$$c^2 + \epsilon^2 x^2 - 2c\epsilon x = x^2 + y^2$$

$$(1 - \epsilon^2)x^2 + 2c\epsilon x + y^2 = c^2$$

$$x^2 + \frac{2c\epsilon}{1 - \epsilon^2}x + \frac{y^2}{1 - \epsilon^2} = c^2/(1 - \epsilon^2)$$

$$(x + d)^2 + \frac{y^2}{1 - \epsilon^2} = c^2/(1 - \epsilon^2) + d^2$$

Where completing the square means

$$d = \frac{c\epsilon}{1 - \epsilon^2}$$

Rewriting the solution (Problem 8.16)

$$(x + \frac{c\epsilon}{1-\epsilon^2})^2 + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2} + \frac{c^2\epsilon^2}{(1-\epsilon^2)^2}$$

$$(x + \frac{c\epsilon}{1-\epsilon^2})^2 + \frac{y^2}{1-\epsilon^2} = \frac{c^2 - c^2\epsilon^2 + c^2\epsilon^2}{(1-\epsilon^2)^2}$$

$$(x + \frac{c\epsilon}{1-\epsilon^2})^2 + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{(1-\epsilon^2)^2}$$

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

With constants:

$$a = \frac{c}{1-\epsilon^2}, b = \frac{c}{\sqrt{1-\epsilon^2}}, d = \frac{c\epsilon}{1-\epsilon^2}$$

$$a = \frac{c}{1-\epsilon^2}, b = \frac{c}{\sqrt{1-\epsilon^2}}, d = a\epsilon$$

Looking at the solution

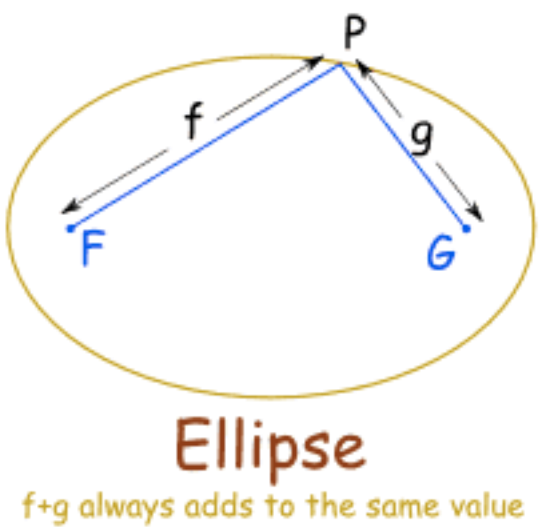
$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

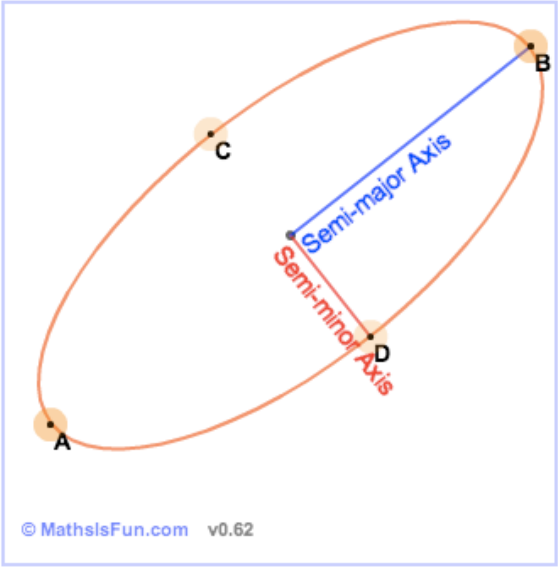
$$d = \frac{c\epsilon}{1-\epsilon^2}$$

$$a = \frac{c}{1-\epsilon^2}, b = \frac{c}{\sqrt{1-\epsilon^2}}, d = a\epsilon$$

Distance from origin to sun

This is an ellipse, with the center of the ellipse offset by d along the x axis from the origin.





A reminder about ellipses

Definition of eccentricity e of an ellipse

$$e = \frac{\sqrt{a^2 - b^2}}{a}$$

Quick check:
What do we get for a circle?

$$e = \frac{\sqrt{\frac{c^2}{(1-\epsilon^2)^2} - \frac{c^2}{1-\epsilon^2}}}{\frac{c}{1-\epsilon^2}}$$

$$e^2 = \frac{\frac{c^2}{(1-\epsilon^2)^2} - \frac{c^2}{1-\epsilon^2}}{\frac{c^2}{(1-\epsilon)^2}}$$

So ε is the eccentricity

$$e^2 = \frac{\frac{c^2}{(1-\epsilon^2)^2} - \frac{c^2-\epsilon^2 c^2}{(1-\epsilon^2)^2}}{\frac{c^2}{(1-\epsilon^2)^2}}$$

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a = \frac{c}{1-\epsilon^2}, b = \frac{c}{\sqrt{1-\epsilon^2}}, d = a\epsilon$$

$$e^2 = \frac{c^2 - c^2 + \epsilon^2 c^2}{c^2}$$

$$e^2 = \epsilon^2$$

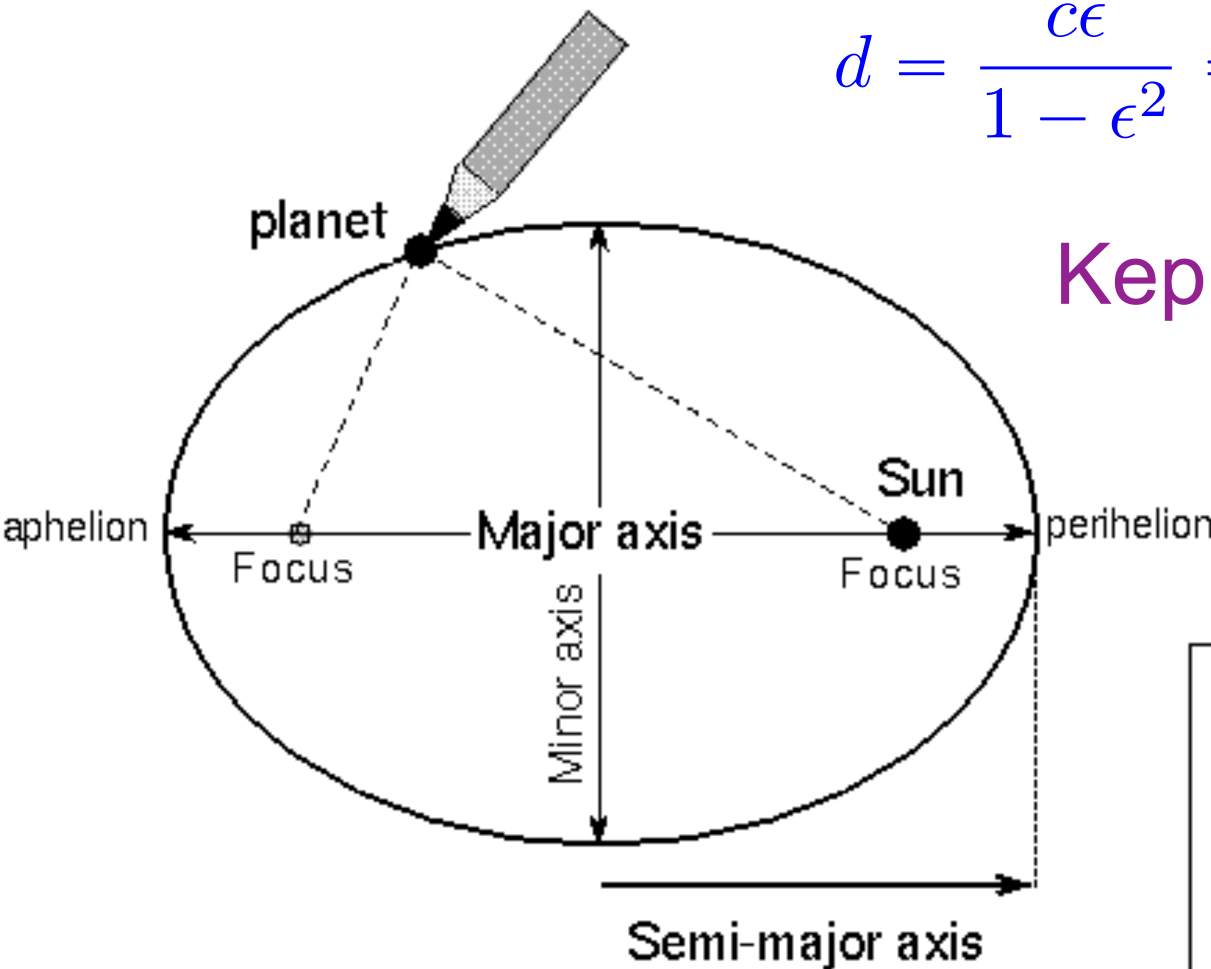
$$e = \epsilon$$

A reminder about ellipses

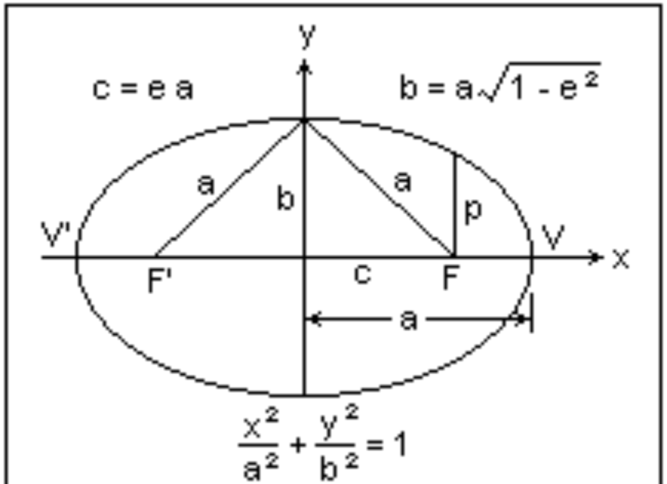
$$d = \frac{c\epsilon}{1-\epsilon^2} = a\epsilon = \text{ellipse focus}$$

Kepler's First Law

As expected,
ε=0→d=0

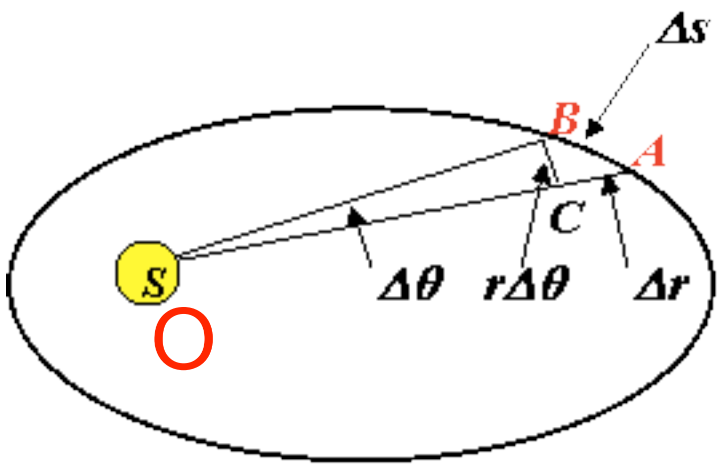


Drawing an **ellipse**: loop string around thumb tacks at each **focus** and stretch string tight with a pencil while moving the pencil around the tacks. The Sun is at one focus.



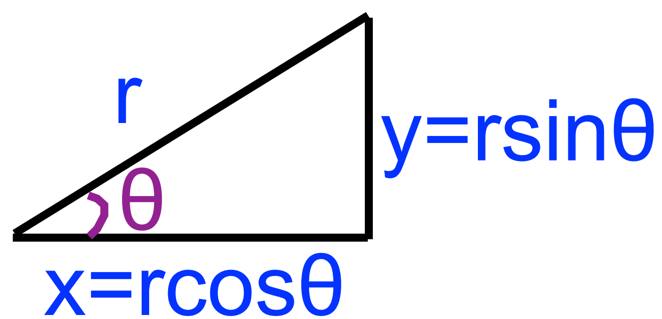
Ellipse

A brief step back to Section 3.4 (Kepler's 2nd law)

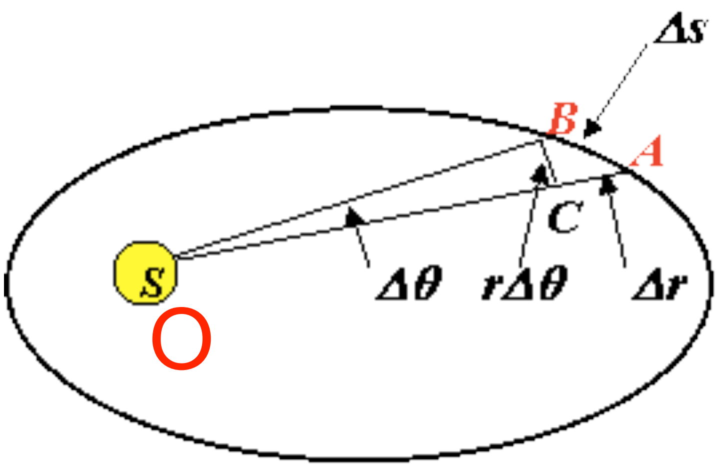


As a planet moves around the sun, what can we say about the area swept out by the orbit?

In a small time Δt, the area swept out is the area of the triangle OAB

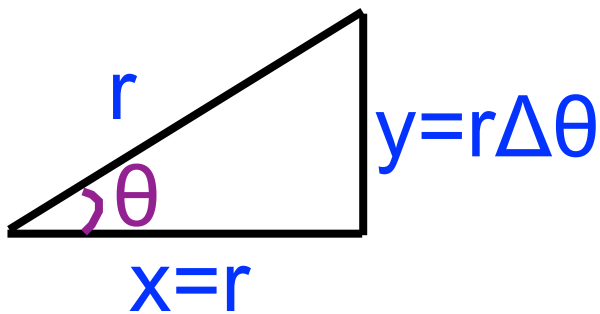


A brief step back to Section 3.4 (Kepler's 2nd law)



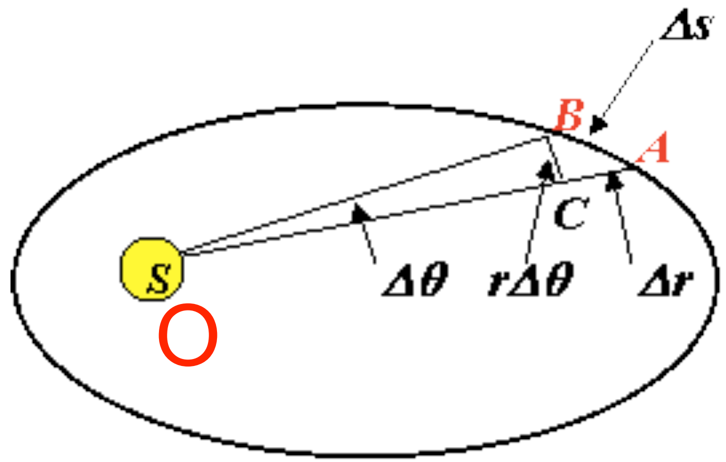
As a planet moves around the sun, what can we say about the area swept out by the orbit?

In a small time Δt, the area swept out is the area of the triangle OAB... but Δt is small, so Δθ is small. $\cos(\Delta\theta) \sim 1$, $\sin(\Delta\theta) \sim \Delta\theta$



$dA = 0.5xy = 0.5r^2\Delta\theta$

A brief step back to Section 3.4 (Kepler's 2nd law)



$$A = \frac{1}{2}\int_{\theta_0}^{\theta_1} r^2 d\theta$$

$$\text{Recall } l = mr^2\dot{\theta} = mr^2\frac{d\theta}{dt}$$

$$d\theta = \frac{l\,dt}{mr^2}$$

$$A = \frac{1}{2}\int_{t(\theta_0)}^{t(\theta_1)} r^2 \frac{l\,dt}{mr^2}$$

$$A = \frac{l}{2m}\int_{t(\theta_0)}^{t(\theta_1)} dt$$

As a planet moves around the sun, what can we say about the area swept out by the orbit?

Constant for this system! →

$$\frac{dA}{dt} = \frac{l}{2m} = \frac{mr^2\dot{\theta}}{2m} = \frac{r^2\omega}{2}$$

On to Kepler's third law

$$\frac{dA}{dt} = \frac{r^2\omega}{2}$$

$$A = \pi ab$$

$$\tau = \frac{A}{dA/dt} = \frac{2\pi ab}{r^2\omega}$$

$$\tau^2 = \frac{4\pi^2 a^2 b^2}{r^4\omega^2}$$

$$\tau^2 = \frac{4\pi^2 a^4(1-\epsilon^2)}{r^4\omega^2}$$

$$\tau^2 = \frac{4\pi^2 a^4(1-\epsilon^2)}{r^4(c\gamma)/(\mu r^4)}$$

$$\tau^2 = \frac{4\mu\pi^2 a^3}{\gamma}$$

Recall:

$$a = \frac{c}{1-\epsilon^2}, b = \frac{c}{\sqrt{1-\epsilon^2}}$$

$$(b/a) = \sqrt{1-\epsilon^2}$$

$$b^2 = a^2(1-\epsilon^2)$$

$$c = \frac{l^2}{\mu\gamma} = \frac{\mu^2 r^4\omega^2}{\mu\gamma} = \frac{\mu r^4\omega^2}{\gamma}$$

$$\omega^2 = \frac{c\gamma}{\mu r^4}$$

Continuing with Kepler's third law

$$\tau^2 = \frac{4\mu\pi^2 a^3}{\gamma}$$

$$\gamma = G m_e m_s, \mu = \frac{m_e m_s}{m_e + m_s}$$

$$\frac{\mu}{\gamma} = \frac{m_e m_s}{(m_e + m_s)(G m_e m_s)} = \frac{1}{G(m_e + m_s)}$$

$$\frac{\mu}{\gamma} \sim \frac{1}{G m_s}$$

$$\boxed{\tau^2 = \frac{4\pi^2}{G m_s} a^3}$$

For example, for earth and sun, where $m_s >> m_e$

Square of period proportional to cube of semimajor axis

Energy of the orbit

$$\frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) = E$$

$$\dot{r}(r_{min}) = 0 \rightarrow U_{eff}(r_{min}) = E$$

$$U(r_{min}) + \frac{l^2}{2\mu r_{min}^2} = E$$

$$-\frac{\gamma}{r_{min}} + \frac{l^2}{2\mu r_{min}^2} = E$$

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi}$$

$$r_{min} = \frac{c}{1+\epsilon}$$

$$r_{max} = \frac{c}{1-\epsilon}$$

$$c = \frac{l^2}{\mu\gamma}$$

Energy of the orbit

$$-\frac{\gamma}{r_{min}} + \frac{l^2}{2\mu r_{min}^2} = E$$

$$r_{min} = \frac{l^2}{\mu\gamma(1+\epsilon)}$$

$$-\frac{\gamma^2\mu(1+\epsilon)}{l^2} + \frac{\mu\gamma^2(1+\epsilon)^2}{2l^2} = E$$

$$E = \frac{\mu\gamma^2\left((1+\epsilon)^2 - 2(1+\epsilon)\right)}{2l^2}$$

$$\boxed{E = \frac{\mu\gamma^2(\epsilon^2-1)}{2l^2}}$$

E > 0 when ε > 1

E < 0 when ε < 1

E = 0 when ε=1

What is the orbit if ε=0?

$$r = \sqrt{x^2 + y^2}, \cos\phi = x/r$$

$$r = \frac{c}{1 + \cos\phi}$$

$$r(1 + x/r) = c$$

$$r + x = c$$

$$r = c - x$$

$$r^2 = (c - x)^2 = x^2 + y^2$$

$$c^2 + x^2 - 2xc = x^2 + y^2$$

$$c^2 - 2cx = y^2$$

Get a parabola when energy = 0

What is the orbit if ε>1?

$$r = \sqrt{x^2 + y^2}, \cos\phi = x/r$$

$$r = \frac{c}{1 + \epsilon\cos\phi}$$

$$r \to \infty \text{ when } \epsilon\cos\phi_{max} = -1$$

There is a maximum
angle that the satellite can reach!

Similar math to before for ε>1

$$r = \sqrt{x^2+y^2}, \cos\phi = x/r$$

$$r = \frac{c}{1+\epsilon\cos\phi}$$

$$r(1+\epsilon x/r) = c$$

$$r + \epsilon x = c$$

$$r = c - \epsilon x$$

$$r^2 = (c-\epsilon x)^2 = x^2 + y^2$$

$$c^2 + \epsilon^2 x^2 - 2c\epsilon x = x^2 + y^2$$

$$(1-\epsilon^2)x^2 + 2c\epsilon x + y^2 = c^2$$

Now $(1-\varepsilon^2) < 0$

$$x^2 + \frac{2c\epsilon}{1-\epsilon^2}x + \frac{y^2}{1-\epsilon^2} = c^2/(1-\epsilon^2)$$

Note sign swap compared to before

$$x^2 - \frac{2c\epsilon}{\epsilon^2-1}x - \frac{y^2}{\epsilon^2-1} = -c^2/(\epsilon^2-1)$$

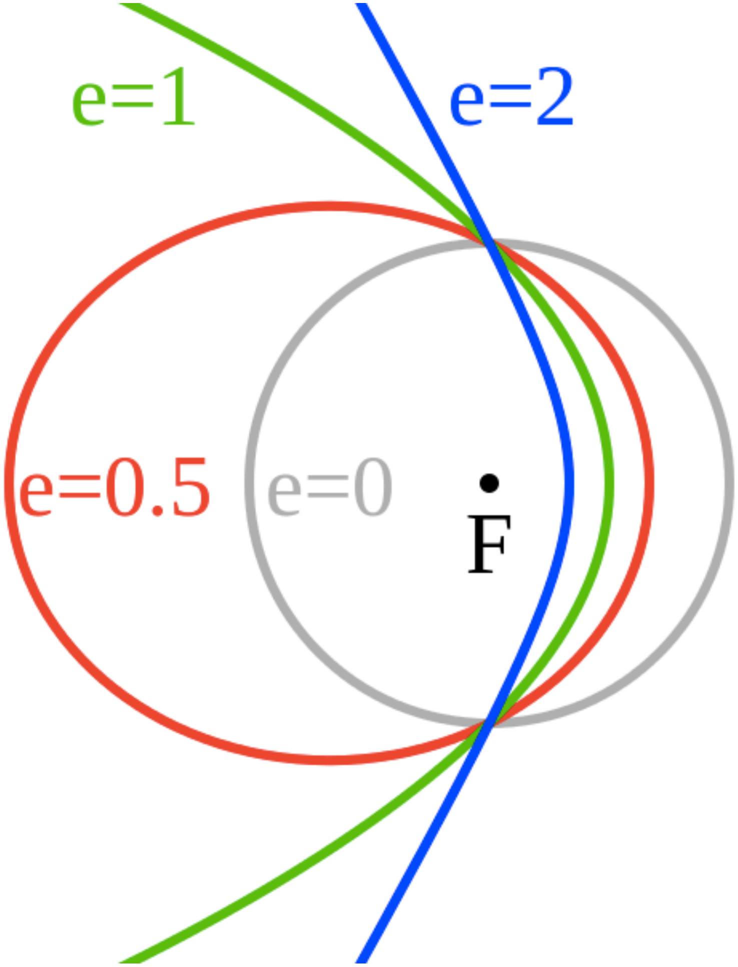
$$(x-d)^2 - \frac{y^2}{\epsilon^2-1} = -c^2/(1-\epsilon^2) + d^2$$

Result if ε>1

Hyperbola:

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$

Putting it all together



Let’s work on some problems

8.28 and 8.29 in small groups or by yourself

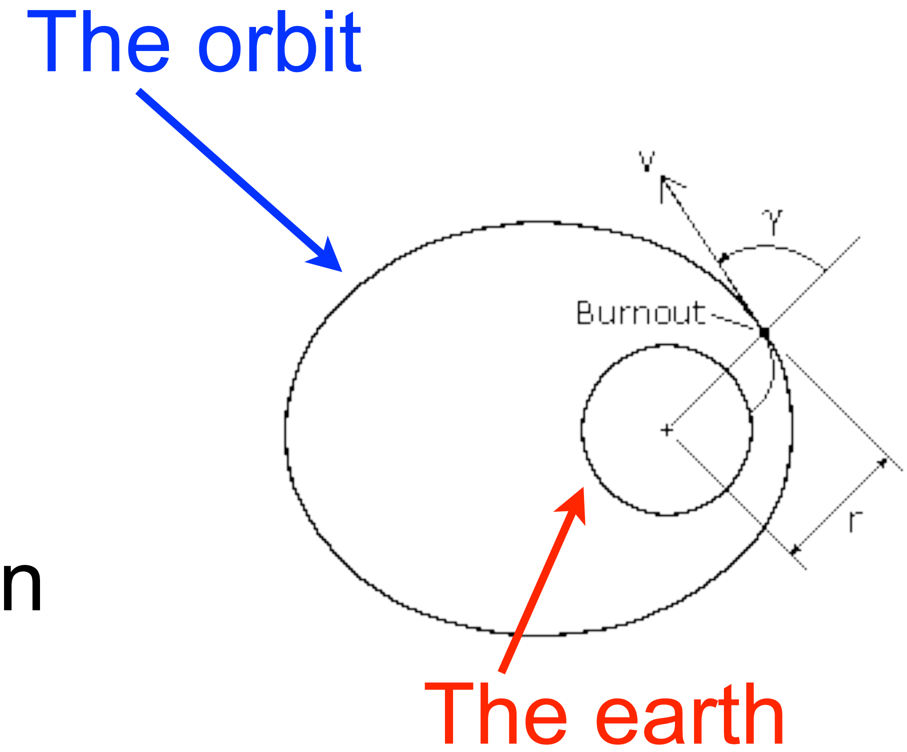
Launching a satellite (not in Taylor)

Given a satellite launch with burnout (when the rocket shuts down) at a certain angle, how can we calculate the eccentricity and apogee/perigee of the orbit?

Launching a satellite

Consider burnout a distance r_b from the center of the earth with velocity v_b

Call the angle between v_b and r_b to be γ



Angular momentum about center of earth is a constant = $mr_bv_b \sin\gamma$ = L. At perigee/apogee, mrv = L (since r and v are perpendicular)

Launching a satellite

$$\text{Energy at burnout} = \frac{m}{2}v_b^2 - \frac{GM_e m}{r_b} = \text{Constant}$$

$$\text{Energy later} = \frac{m}{2}v^2 - \frac{GM_e m}{r} = \frac{m}{2}v_b^2 - \frac{GM_e m}{r_b}$$

$$v^2 - v_b^2 = 2GM_e\left(\frac{1}{r} - \frac{1}{r_b}\right)$$

Launching a satellite

E conservation

$$v^2 - v_b^2 = 2GM_e \left(\frac{1}{r} - \frac{1}{r_b} \right)$$

L conservation

$$rv = r_b v_b \sin\gamma$$

$$v^2 = (r_b/r)^2 v_b^2 \sin^2\gamma$$

$$(r_b/r)^2 v_b^2 \sin^2\gamma - v_b^2 = 2GM_e \left(\frac{1}{r} - \frac{1}{r_b} \right)$$

$$\sin^2\gamma - \left(\frac{r^2}{r_b^2} \right) = \frac{2GM_e}{v_b^2} \left(\frac{r}{r_b^2} - \frac{r^2}{r_b^3} \right)$$

$$\sin^2\gamma - \left(\frac{r^2}{r_b^2} \right) = \frac{2GM_e}{v_b^2 r_b} \left(\frac{r}{r_b} - \frac{r^2}{r_b^2} \right)$$

$$\left(\frac{r}{r_b} \right)^2 \left(\frac{2GM_e}{r_b v_b^2} - 1 \right) - \frac{2GM_e}{r_b v_b^2} \left(\frac{r}{r_b} \right) + \sin^2\gamma = 0$$

Solving the quadratic equation

$$\left(\frac{r}{r_b}\right)^2\left(\frac{2GM_e}{r_b v_b^2}-1\right)-\frac{2GM_e}{r_b v_b^2}\left(\frac{r}{r_b}\right)+\sin^2\gamma=0$$

$$k=\frac{2GM_e}{r_b v_b^2}, x=r/r_b$$

$$x^2(k-1)-kx+\sin^2\gamma=0$$

$$x=\frac{k\pm\sqrt{k^2-4(k-1)\sin^2\gamma}}{2(k-1)}$$

Two solutions - smaller for perigee, larger for apogee

Example

$$\left(\frac{r}{r_b}\right)^2\left(\frac{2GM_e}{r_b v_b^2}-1\right)-\frac{2GM_e}{r_b v_b^2}\left(\frac{r}{r_b}\right)+\sin^2\gamma=0$$

$$k=\frac{2GM_e}{r_b v_b^2}, x=r/r_b$$

$$x^2(k-1)-kx+\sin^2\gamma=0$$

$$x=\frac{k\pm\sqrt{k^2-4(k-1)\sin^2\gamma}}{2(k-1)}$$

A satellite launched from earth burns out at a height of 300 km at 8,500 m/s and a zenith angle = 85 degrees. What are the orbit apogee, perigee and eccentricity?

Careful! r is height from center of earth (need 6.38e6 meters extra)

Solution

x = 0.979, 1.56

Altitude at perigee =
0.979(6.38e6 + 300e3) meters
= 160 km above the earth

Altitude at apogee =
1.56(6.38e6 + 300e3) meters
=4000 km above the earth

Solution

Altitude at perigee =
0.979(6.38e6 + 300e3) =
6540 km above earth center

$$r(\phi) = \frac{c}{1 + \epsilon \cos \phi}$$

Altitude at apogee =
1.56(6.38e6 + 300e3) =
10420 km above earth center

$$r_{min} = \frac{c}{1 + \epsilon}$$

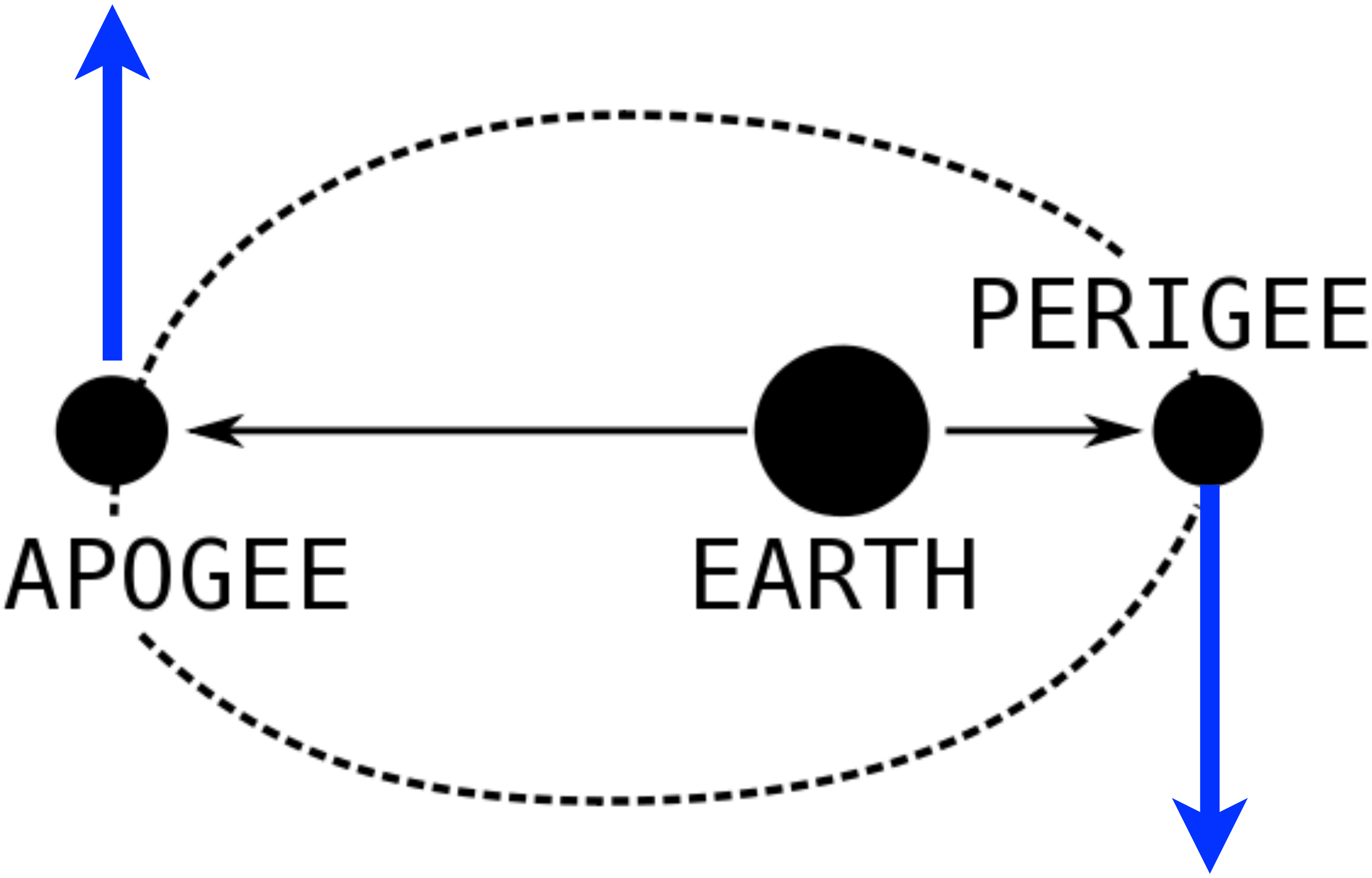
$$r_{max} = \frac{c}{1 - \epsilon}$$

rmin/rmax = (1-ε)/(1+ε) = 0.63
so eccentricity = 0.23

Change of orbit (back to Taylor)

Know that r of first orbit at given Φ = r of second orbit (after a thrust push)

A special case of thrust



Consider thrust at perigee/apogee in tangential direction (so direction of velocity doesn't change)

A special case of thrust

$$r(\phi) = \frac{c}{1 + \epsilon \cos \phi}$$

$$r_1(0) = c_1/(1 + \epsilon_1)$$

$$r_2(0) = c_2/(1 + \epsilon_2)$$

$$r_1(0) = r_2(0) \to \frac{c_1}{1 + \epsilon_1} = \frac{c_2}{1 + \epsilon_2}$$

Thrust factor

$$v_2 = \lambda v_1$$

$$l = \mu r v \text{ constant } \to l_2 = \lambda v_1$$

$$c = \frac{l^2}{\mu \gamma} \to c_2 = \lambda^2 c_1$$

A special case of thrust

$$\frac{c_1}{1+\epsilon_1} = \frac{c_2}{1+\epsilon_2}$$

$$c_2 = \lambda^2 c_1$$

$$\frac{c_1}{1+\epsilon_1} = \frac{c_1\lambda^2}{1+\epsilon_2}$$

$$1+\epsilon_2 = \lambda^2(1+\epsilon_1)$$

$$\epsilon_2 = \lambda^2\epsilon_1 + (\lambda^2 - 1)$$

If λ positive, what does that mean for eccentricity? Similarly, what happens if it is negative?

Let's go over

First part of example 8.6 together

Homework, as usual due in 1 week

8.3,8.12,8.15,8.17,8.18,8.33