But what does this tell us about mechanics?

Define the Lagrangian as:

Note minus sign

$$\mathcal{L} = T - U(x, y, z)$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{\mathbf{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

T has no x dependence

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x$$

U has no dependence on xdot

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}p_x = \dot{p}_x = F_x \rightarrow$$

Newton

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

But what does this tell us about mechanics?

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}$$

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}}$$

From last slide. Repeat to get the other two equations. These are in form of Euler-Lagrange equations!

Hamilton's principle:

$$S = \text{Action integral} = \int_{t_1}^{t_2} \mathcal{L} dt = \int_{t_1}^{t_2} (T - U) dt \text{ is stationary}$$

Keys to using Lagrange's Equations

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$
$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}$$
$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}}$$

Can use more **generalized coordinates** →

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}$$
$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}$$
$$\frac{\partial \mathcal{L}}{\partial q_3} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_3}$$

q_i (generalized coordinates) can really be any coordinates in any frame, but remember to write down Lagrangian in an inertial frame

Why is this?

Keys to using Lagrange's Equations

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}$$
$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}$$
$$\frac{\partial \mathcal{L}}{\partial q_3} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_3}$$

$$\frac{\partial \mathcal{L}}{\partial p_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{p}_1}$$
$$\frac{\partial \mathcal{L}}{\partial p_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{p}_2}$$
$$\frac{\partial \mathcal{L}}{\partial p_3} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{p}_3}$$

Additional generalized coordinates for every new particle. Can use, for example, center of mass position and relative position from CM

Some examples (following Taylor)

A particle moves in a conservative force field in two dimensions. What are the equations of motion, using Cartesian coordinates?

$$\mathcal{L} = \mathcal{L}(x, y, \dot{x}, \dot{y}) = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \rightarrow F_x = m\ddot{x}$$

$$\text{similarly } F_y = m\ddot{y}$$

From last slide

i^{th} component of generalized force

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

Generalize

$$\frac{\partial \mathcal{L}}{\partial q_i}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

i^{th} component of generalized momentum

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

And Lagrange says the generalized force = rate of change of generalized momentum

Another examples (following Taylor)

A particle moves in a conservative force field in two dimensions. What are the equations of motion, using Polar coordinates?

$$\mathcal{L} = \mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \phi)$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$mr\dot{\phi}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}m\dot{r} = m\ddot{r}$$

$$mr\dot{\phi}^2 + F_r = \frac{d}{dt}m\dot{r} = m\ddot{r}$$

$$F_r = m(\ddot{r} - r\dot{\phi}^2)$$

$F_r = ma_r$

Continuing on

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$I\omega = L$$

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt}(mr^2\dot{\phi})$$

$$\mathbf{F} = \nabla U$$

$$F_\phi = (\nabla U)_\phi$$

$$\nabla U = \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\phi}$$

$$(\nabla U)_\phi = \frac{1}{r}\frac{\partial U}{\partial \phi}$$

$$\frac{\partial U}{\partial \phi} = r(\nabla U)_\phi$$

$$\frac{\partial U}{\partial \phi} = r(\nabla U)_\phi = -rF_\phi$$

$$rF_\phi = \frac{d}{dt}I\omega = \frac{d}{dt}L$$

$$\text{Torque} = \Gamma = \frac{dL}{dt}$$

A useful thing to have spotted

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

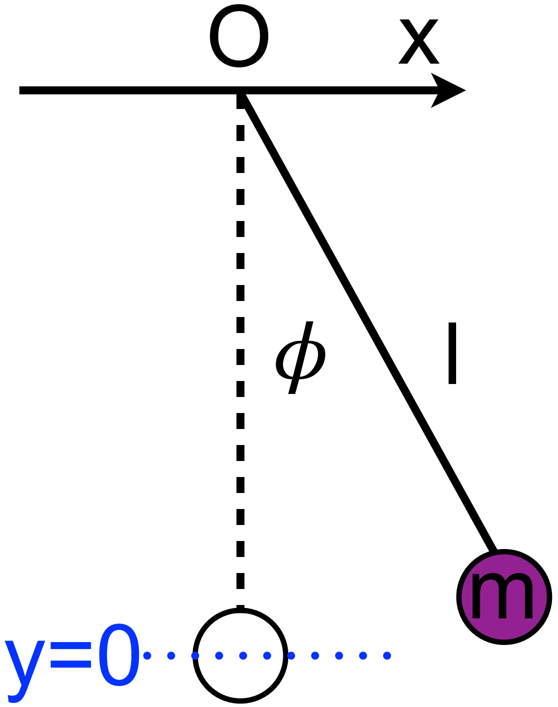
$$\frac{\partial \mathcal{L}}{\partial q_i} = 0 \rightarrow$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \text{ conserved}$$

Go ahead

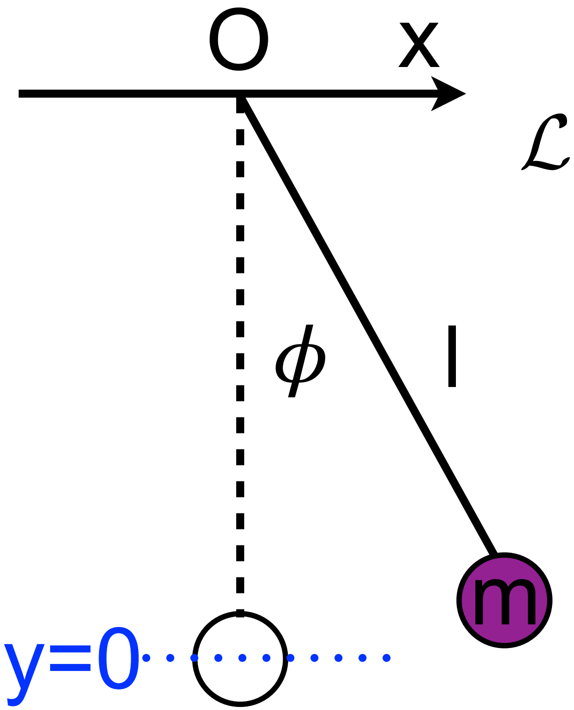
And work on problem 7.1 yourself, and then we'll work on 7.2 and 7.17 together

More examples from Taylor (chapter 7.2)



Two dimensional system, but really need only one coordinate to describe the system due to constraints of the rod (since rod length is constant)

More examples from Taylor (chapter 7.2)



$$\mathcal{L} = T - U = \frac{1}{2}m(l^2\dot{\phi}^2) - mgy$$

$$\mathcal{L} = T - U = \frac{1}{2}m(l^2\dot{\phi}^2) - mgl(1 - \cos\phi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$-mgl\sin\phi = \frac{d}{dt}(ml^2\dot{\phi}) = ml^2\ddot{\phi}$$

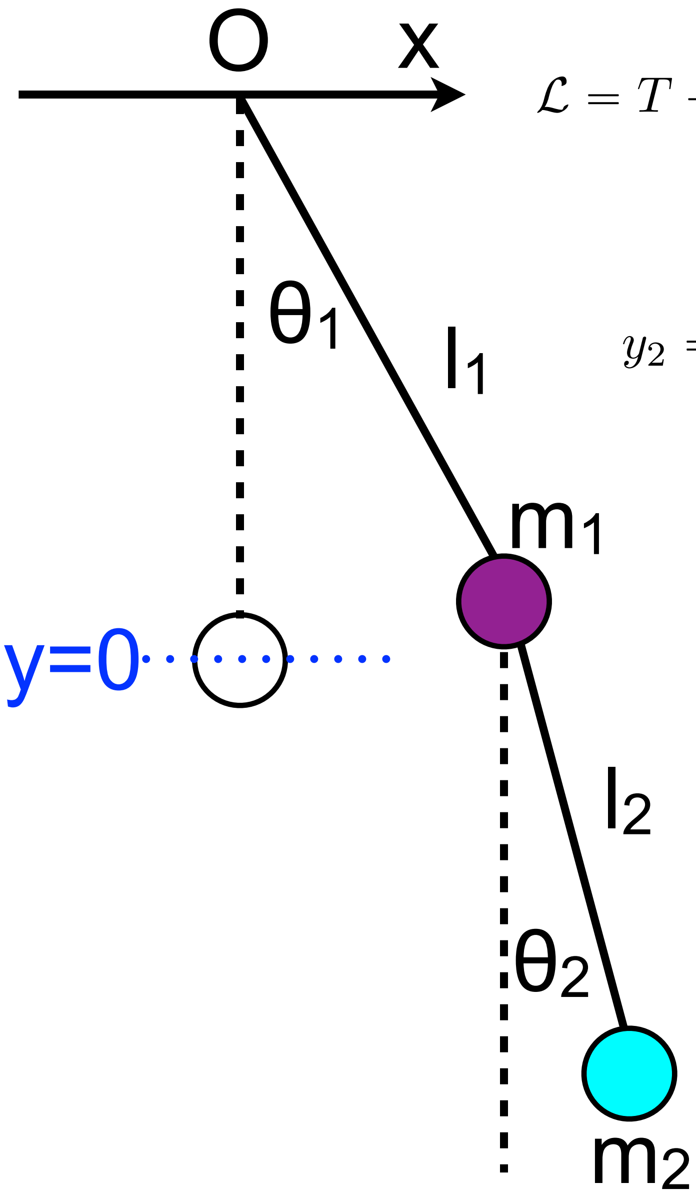
Torque on mass about O

$$ml^2 = I$$

$$\ddot{\phi} = \alpha$$

$$\Gamma = I\alpha$$

More examples from Taylor (chapter 7.3), just to see...



$$\mathcal{L} = T - U = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - m_1 g y_1 - m_2 g y_2$$

$$x_1 = l_1 \sin\theta_1, y_1 = l_1(1 - \cos\theta_1)$$

$$x_2 = x_1 + l_2 \sin\theta_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

$$y_2 = y_1 + l_2(1 - \cos\theta_2) = l_1(1 - \cos\theta_1) + l_2(1 - \cos\theta_2)$$

$$\dot{x_1} = l_1 \cos\theta_1 \dot{\theta}_1$$

$$\dot{y_1} = l_1 \sin\theta_1 \dot{\theta}_1$$

$$\dot{x_1}^2 + \dot{y_1}^2 = l_1^2 \dot{\theta}_1^2$$

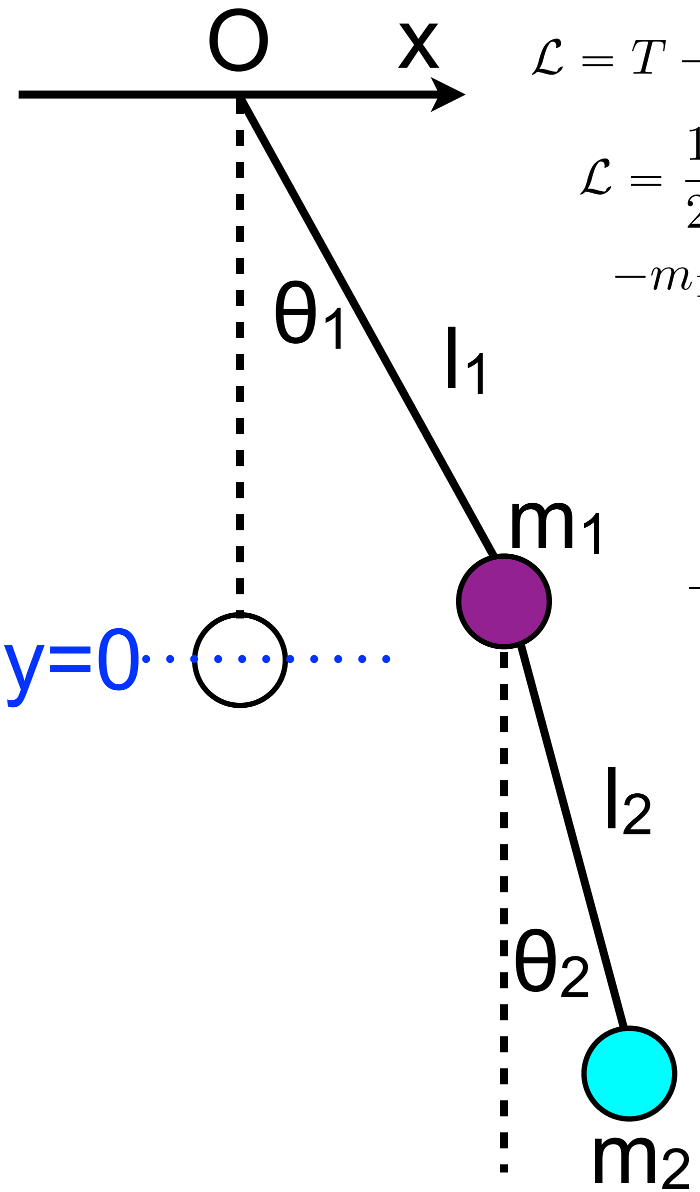
$$\dot{x_2} = l_1 \cos\theta_1 \dot{\theta_1} + l_2 \cos\theta_2 \dot{\theta_2}$$

$$\dot{y_2} = l_1 \sin\theta_1 \dot{\theta_1} + l_2 \sin\theta_2 \dot{\theta_2}$$

$$\dot{x_2}^2 + \dot{y_2}^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta_1} \dot{\theta_2} (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)$$

$$\dot{x_2}^2 + \dot{y_2}^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta_1} \dot{\theta_2} \cos(\theta_1 - \theta_2)$$

More examples from Taylor (chapter 7.3)



$$\mathcal{L} = T - U = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - m_1 g y_1 - m_2 g y_2$$

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right)$$
$$-m_1 g l_1(1-\cos\theta_1) - m_2 g\left(l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)\right)$$

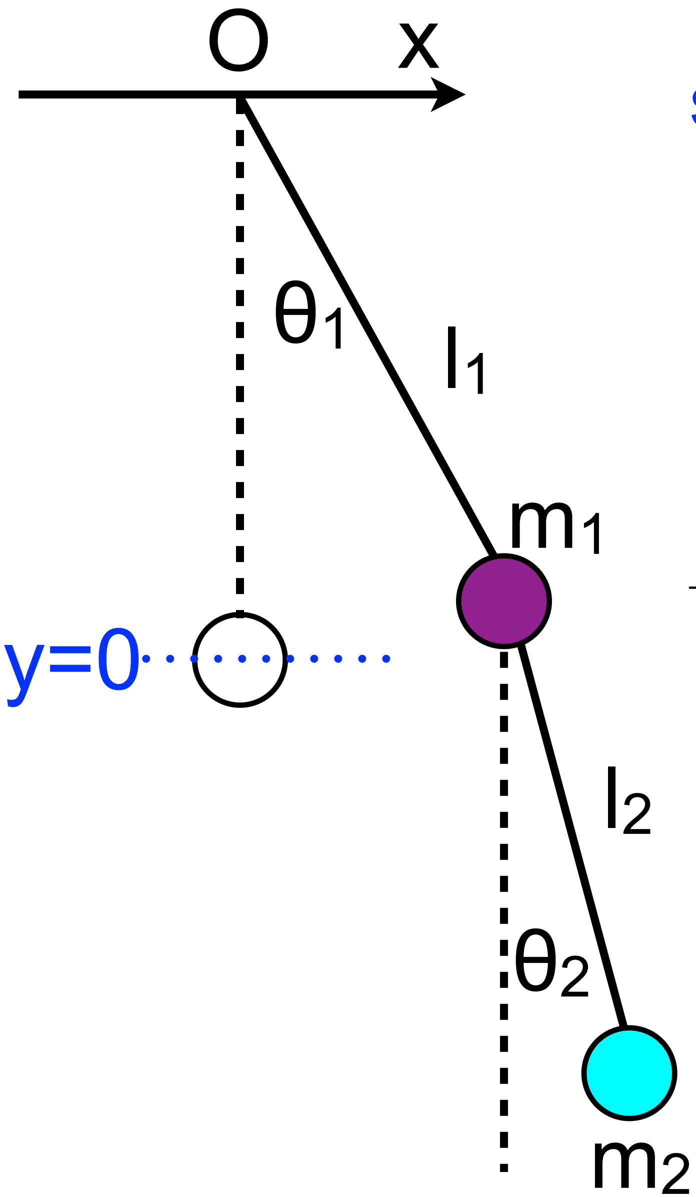
$$\frac{\partial\mathcal{L}}{\partial\theta_1} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}$$

$$-m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - (m_1+m_2)g l_1\sin\theta_1 =$$
$$\frac{d}{dt}\left[m_1 l_1^2\dot{\theta}_1 + m_2 l_1^2\dot{\theta}_1 + m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1-\theta_2)\right]$$

$$\frac{\partial\mathcal{L}}{\partial\theta_2} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}$$

$$m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2 g l_2\sin\theta_2 =$$
$$\frac{d}{dt}\left[m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos(\theta_1-\theta_2)\right]$$

More examples from Taylor (chapter 7.3)



No full solutions, but we'll get to small-angle solutions in Chapter 11. Imagine doing this with Newton, though?

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}$$

$$-m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin\theta_1 =$$

$$\frac{d}{dt}\left[m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right]$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}$$

$$m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2 =$$

$$\frac{d}{dt}\left[m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)\right]$$

Now for some terminology

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Let's say we chose
$q_1 = \theta_1$, $q_2 = \theta_2$

$$\mathbf{r}_1 = x_1\hat{\mathbf{x_1}} + y_1\hat{\mathbf{y_1}}$$

$$\mathbf{r}_2 = x_2\hat{\mathbf{x_2}} + y_2\hat{\mathbf{y_2}}$$

with

$$x_1 = l_1 \sin\theta_1$$

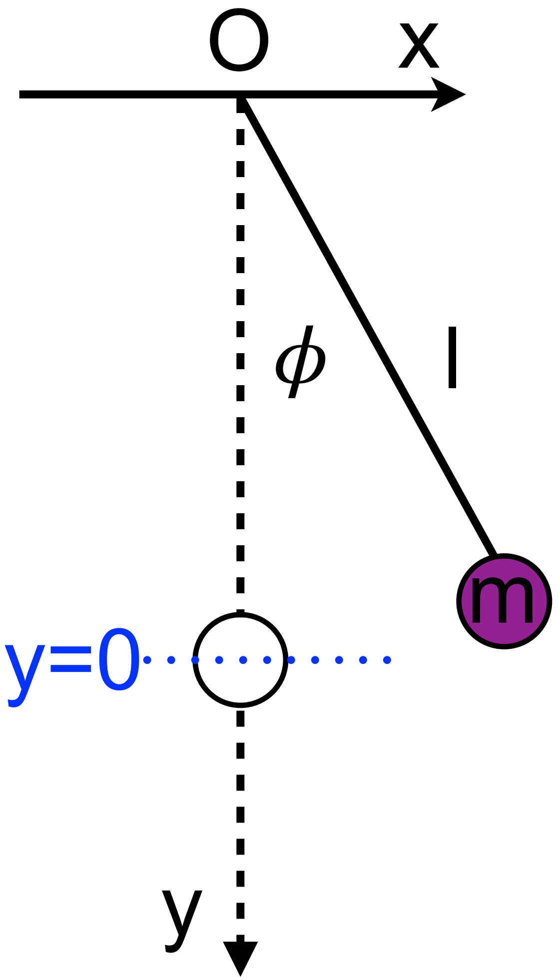
$$y_1 = l_1 \cos\theta_1$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

$$y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

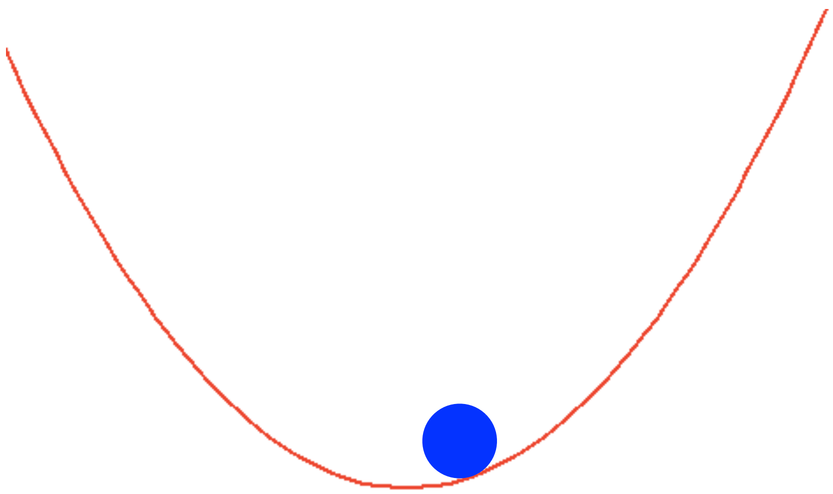
These relationships do not depend on time, so they are referred to as being natural

More terminology



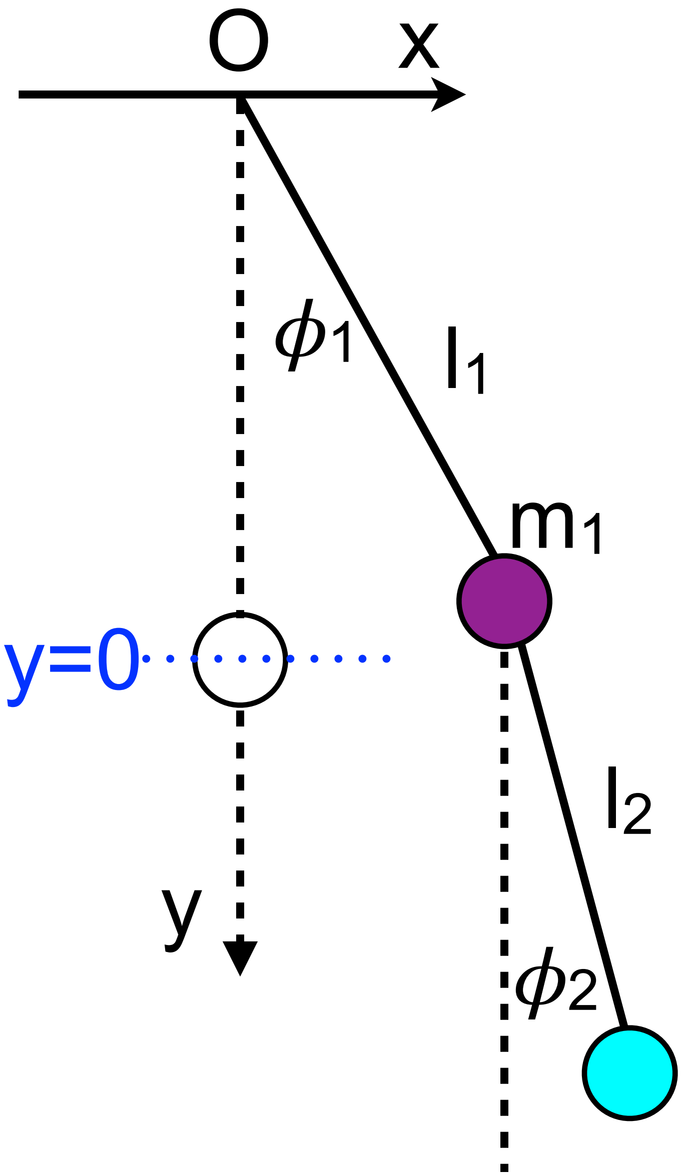
Two dimensional system, but really need only one coordinate to describe the system due to constraints of the rod. We call such systems constrained

More terminology



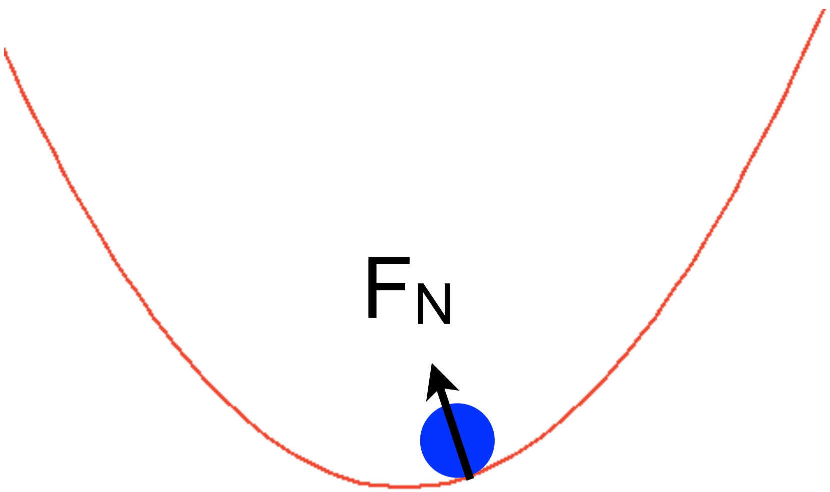
This system, with a ball moving along this bead, is also constrained

And yet a bit more terminology



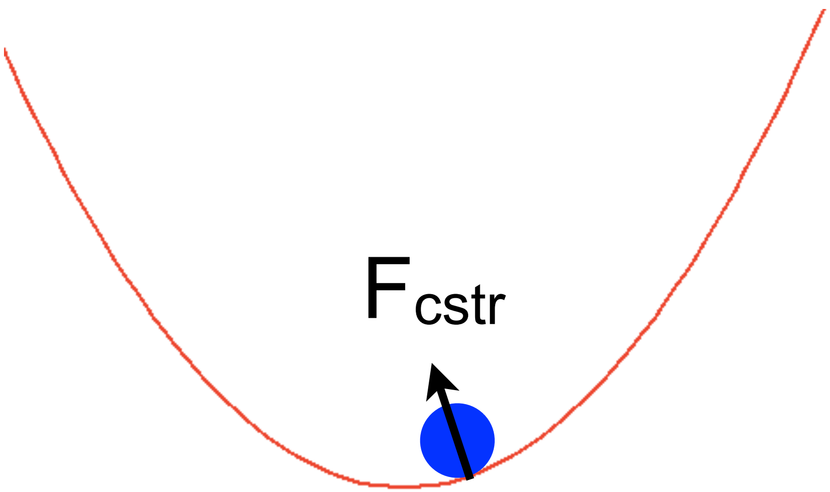
System has two degrees of freedom, and is described by two generalized coordinates. If these two numbers are equal, the system is holonomic

Why does this work for problems like this?



Ball has (for example) gravity pulling it down, but also a constraint force/normal force that keeps it moving along the parabola. Why doesn't this matter?

Lagrange’s Equations with constraints



Constraining force

Total force

Other forces

$$\mathbf{F}_{tot} = \mathbf{F}_{cstr} + \mathbf{F}$$

$$\mathbf{F} = -\nabla U$$

Other forces (ie gravity, etc) are conservative and derivable from a potential energy

$$\mathcal{L} = T - U$$

Follow similar strategy as before

Recall Hamilton's principle:

$$S = \text{Action integral} = \int_{t_1}^{t_2} \mathcal{L} dt = \int_{t_1}^{t_2} (T - U) dt \text{ is stationary}$$

Particle passes through **r**$_1$ and **r**$_2$ at t_1 and t_2

$$\mathbf{R}(t) = \mathbf{r}(t) + \epsilon(t)$$

Another (potentially wrong) path

Right path

path difference

$$\epsilon(t_1) = \epsilon(t_2) = 0$$

Define action for right/wrong paths

$$S_0 = \int_{t_1}^{t_2} T(\mathbf{r}, \dot{\mathbf{r}}, t) - U(\mathbf{r}, \dot{\mathbf{r}}, t) dt$$

→ action for the correct path

$$S_0 = \int_{t_1}^{t_2} \frac{m}{2}\dot{\mathbf{r}}^2 - U(\mathbf{r}, \dot{\mathbf{r}}, t) dt$$

$$S = \int_{t_1}^{t_2} T(\mathbf{R}, \dot{\mathbf{R}}, t) - U(\mathbf{R}, \dot{\mathbf{R}}, t) dt$$

→ action for arbitrary path

$$S = \int_{t_1}^{t_2} \frac{m}{2}\dot{\mathbf{R}}^2 - U(\mathbf{R}, \dot{\mathbf{R}}, t) dt$$

$$S = \int_{t_1}^{t_2} \frac{m}{2}(\dot{\mathbf{r}} + \dot{\epsilon})^2 - U(\mathbf{R}, \dot{\mathbf{R}}, t) dt$$

$$S - S_0 = \delta S = \int_{t_1}^{t_2} \frac{m}{2}\dot{\epsilon}^2 + m\dot{\mathbf{r}} \cdot \dot{\epsilon} - \Big(U(\mathbf{R}, \dot{\mathbf{R}}, t) - U(\mathbf{r}, \dot{\mathbf{r}}, t)\Big) \, dt$$

→ change in action between paths

Follow similar strategy as before

$$\delta S = \int_{t_1}^{t_2} \frac{m}{2}\dot{\epsilon}^2 + m\dot{\mathbf{r}}\cdot\dot{\epsilon} - \Big(U(\mathbf{R},\dot{\mathbf{R}} - U(\mathbf{r},\dot{\mathbf{r}},t)\Big),t)dt$$

Zero for small ε

$$U(\mathbf{R},\dot{\mathbf{R}},t) - U(\mathbf{r},\dot{\mathbf{r}},t) = U(\mathbf{r}+\epsilon,\dot{\mathbf{r}}+\dot{\epsilon},t) - U(\mathbf{r},\dot{\mathbf{r}},t)$$

$$U(\mathbf{R},\dot{\mathbf{R}},t) - U(\mathbf{r},\dot{\mathbf{r}},t) = \nabla U(\mathbf{r},\dot{\mathbf{r}},t)\cdot\epsilon$$

Recall definition of gradient

$$\delta S = \int_{t_1}^{t_2} \left[m\dot{\mathbf{r}}\cdot\dot{\epsilon} - \nabla U(\mathbf{r},\dot{\mathbf{r}},t)\cdot\epsilon\right]dt$$

And once again integrate by parts and use boundary conditions

$$\delta S = \int_{t_1}^{t_2} \left[m\dot{\mathbf{r}} \cdot \dot{\epsilon} - \nabla U(\mathbf{r}, \dot{\mathbf{r}}, t) \cdot \epsilon \right] dt$$

Integrate by parts:

$$\int_a^b u\,dv = [uv]_a^b - \int v\,du$$

$$u = m\dot{\mathbf{r}}, dv = \dot{\epsilon}dt$$

$$\int_{t_1}^{t_2} (m\dot{\mathbf{r}} \cdot \dot{\epsilon})dt = [m\dot{\mathbf{r}} \cdot \epsilon]_{t_1}^{t_2} - \int_{t_1}^{t_2} (m\epsilon \cdot \ddot{\mathbf{r}})dt$$

Zero because

$\varepsilon(t_1) = \varepsilon(t_2) = 0$

$$\delta S = \int_{t_1}^{t_2} \left[-m\epsilon \cdot \ddot{\mathbf{r}} - \nabla U \cdot \epsilon \right] dt$$

Working out the math

$$\delta S = \int_{t_1}^{t_2} \left[-m\epsilon \cdot \ddot{\mathbf{r}} - \nabla U \cdot \epsilon \right] dt$$

$$\delta S = -\int_{t_1}^{t_2} \epsilon \cdot \left[m\ddot{\mathbf{r}} + \nabla U \right] dt$$

$$m\ddot{\mathbf{r}} = \mathbf{F}_{tot} = \mathbf{F}_{cstr} + \mathbf{F} = \mathbf{F}_{cstr} - \nabla U$$

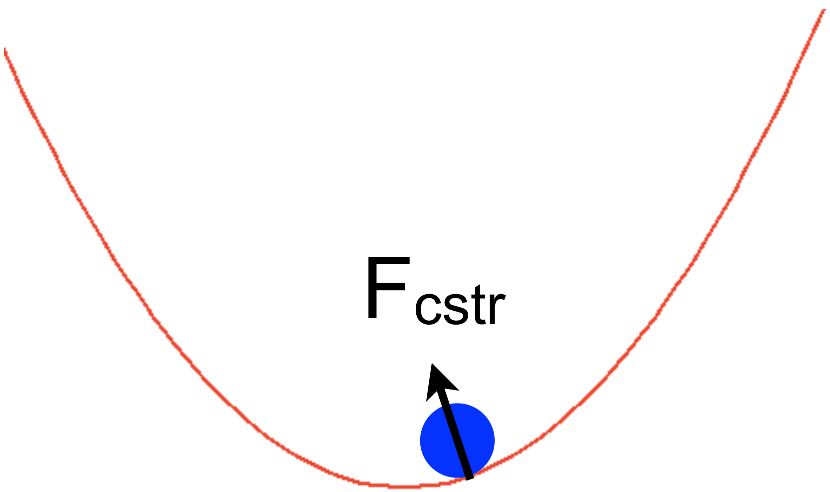
$$\delta S = -\int_{t_1}^{t_2} \epsilon \cdot \left[\mathbf{F}_{cstr} - \nabla U + \nabla U \right] dt$$

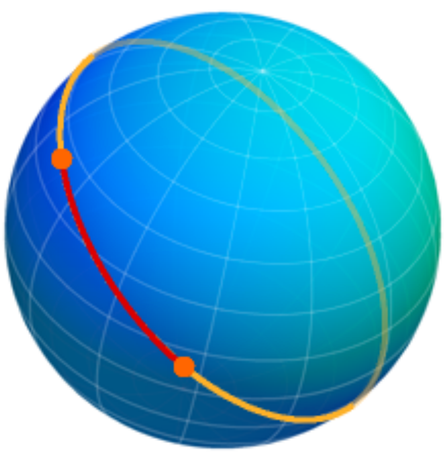
$$\delta S = -\int_{t_1}^{t_2} (\epsilon \cdot \mathbf{F}_{cstr}) dt$$

So we are left thinking about the dot product between ϵ and the constraining force

What are the potential variations in path?

So we are left thinking about the dot product between ϵ and the constraining force

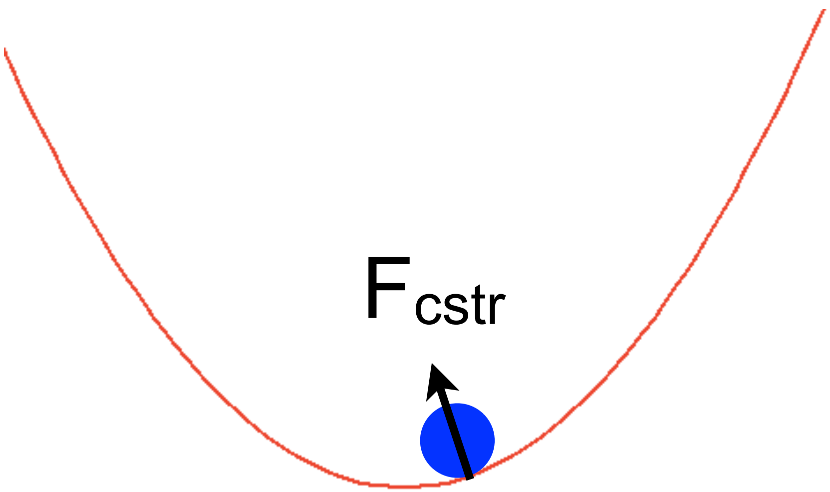


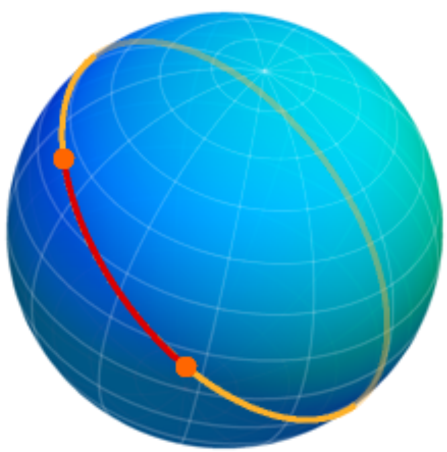


ϵ is a variation on the path that the particle could take. BUT it can only take a path along the surface allowed by the constraint

What does it mean to have a constraining force?

So we are left thinking about the dot product between ϵ and the constraining force





The constraining force is a **normal force**, ie perpendicular to the volume/surface in which the particle can move. So ...

$$\epsilon \cdot \mathbf{F}_{cstr} = 0 \rightarrow \delta S = 0$$

Putting it together

The action integral is stationary at the right path, which means that Euler-Lagrange equations still apply

Remember that this only works if the constraining force is a normal force to the surface of motion

More problems

You should try and become as comfortable as possible working out problems in the Lagrangian formalism, so let’s solve examples 7.4-7.7 on the board together, and then problems 7.16,7.20,7.23,7.34

Homework

Taylor 7.3, 7.8, 7.14, 7.18, 7.29