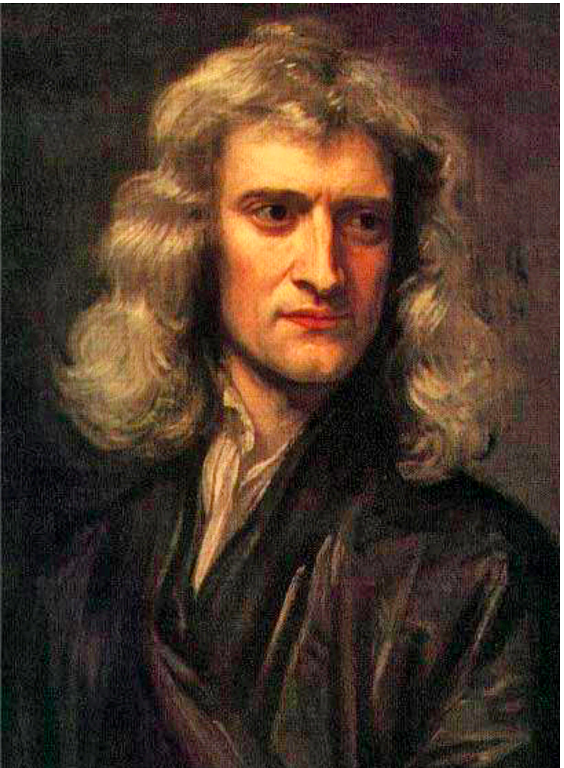
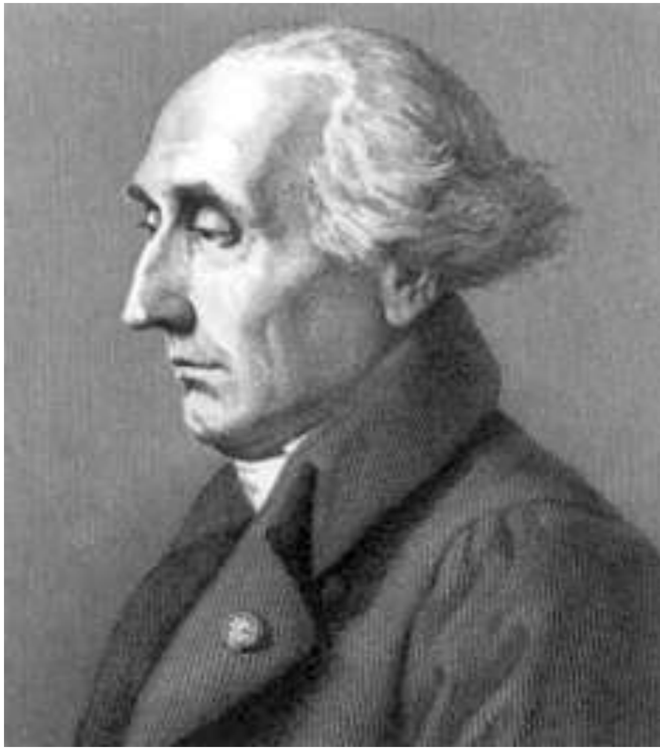
Now completely switching topics

Going to get away from Newtonian derivation of classical mechanics to a new version, given by Lagrange... But first we need a mathematical sidebar (ie Chapter 6)

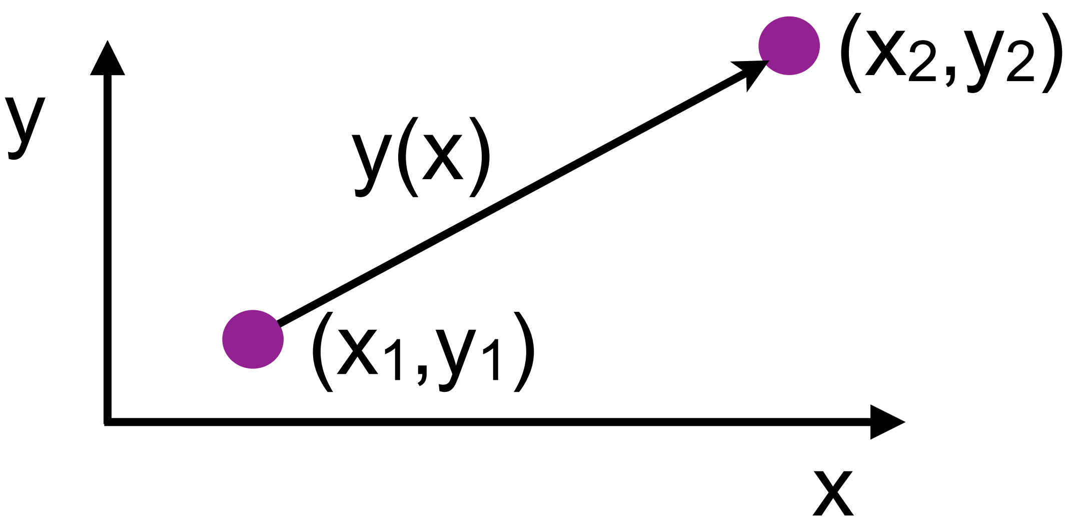


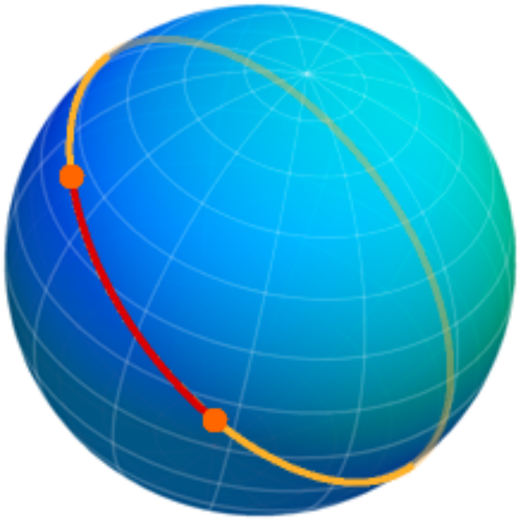


A few seemingly simple questions

What is the shortest path (ie the shortest function) connecting (x_1,x_1) and (x_2,y_2)? A straight line... but how do you prove this?

And what is the shortest path between two points on a sphere?





Let's start off in two dimensions in flat space

Infinitesimal path length

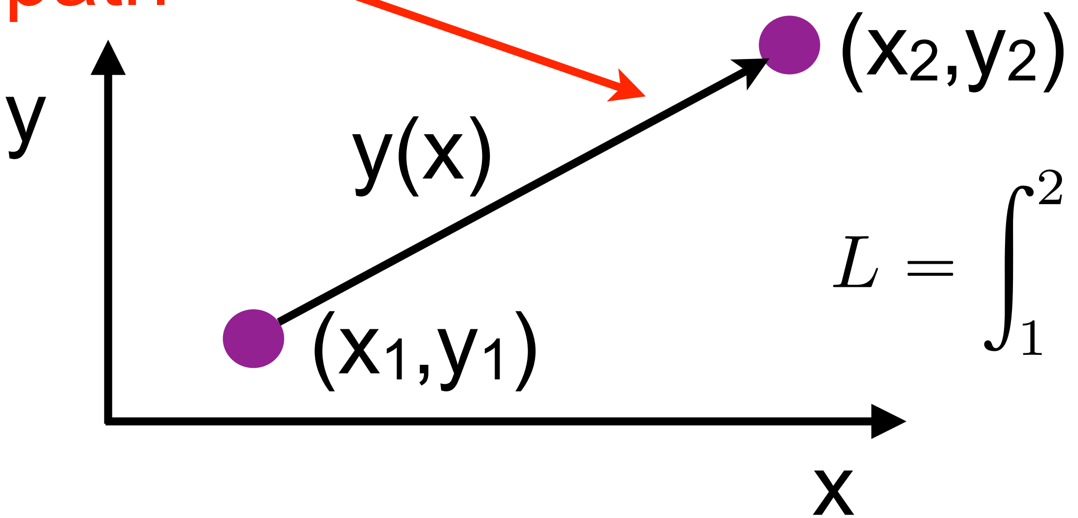
$$ds = \sqrt{dx^2 + dy^2}$$

$$dy = \frac{dy}{dx} dx$$

$$ds = \sqrt{dx^2 + dx^2 \frac{dy^2}{dx^2}}$$

$$ds = \sqrt{dx^2 \left(1 + \frac{dy^2}{dx^2}\right)} = dx\sqrt{1 + y'(x)^2}$$

One possible path



$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2} dx$$

Let's start off in two dimensions

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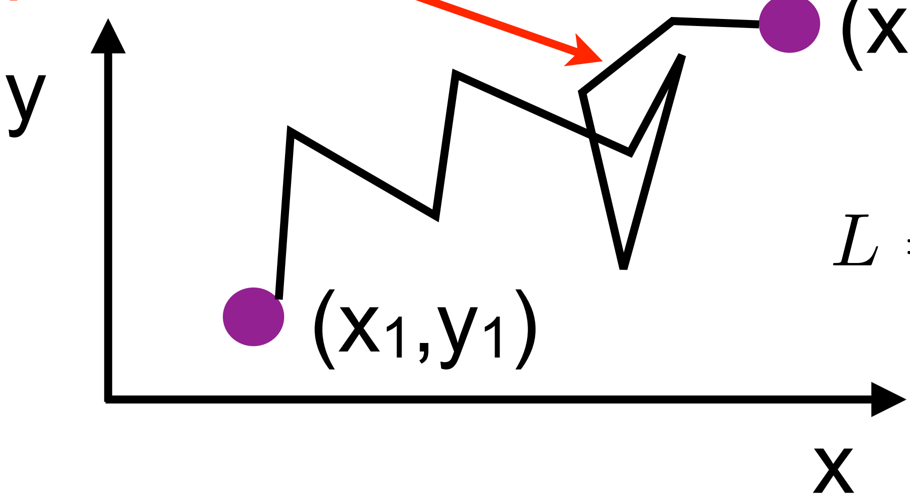
Another possible path



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Let’s start off in two dimensions

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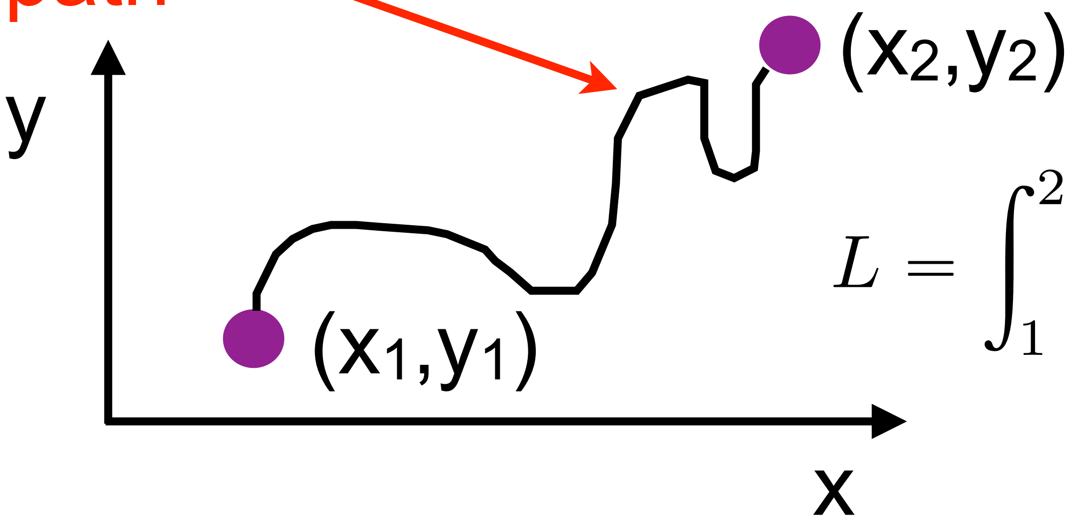
Another possible path



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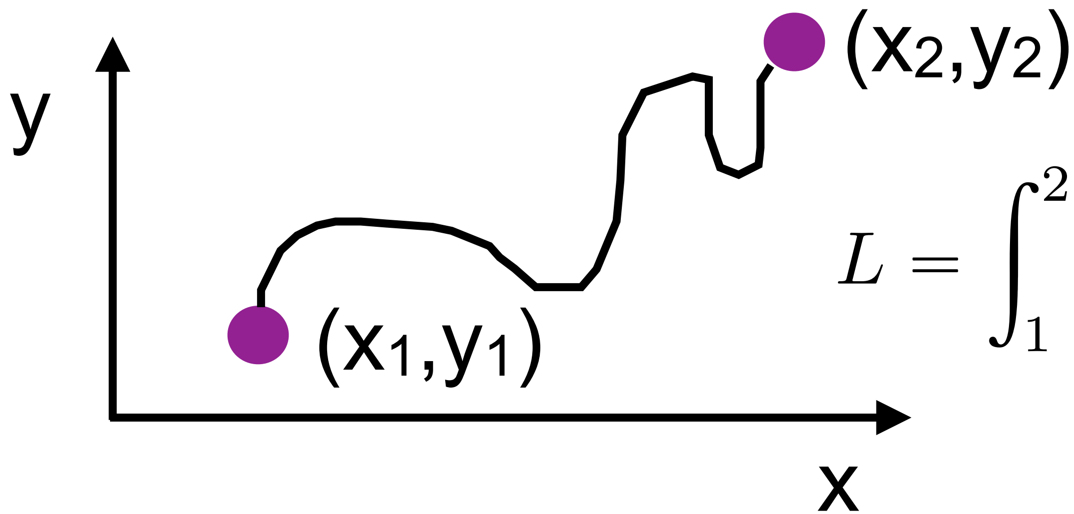
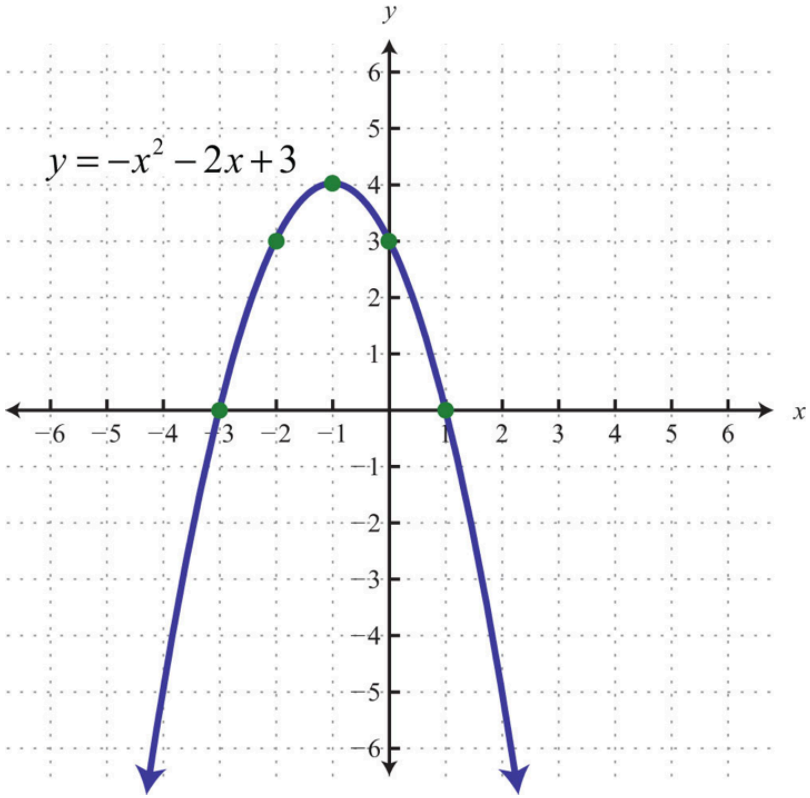
All paths start at (x_1,y_1) and end at (x_2,y_2)



$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}dx$$

Want to minimize the length of all possible paths



We know that calculus already tells us how to find the minimum/maximum of a single function: dy/dx = 0 gives us a stationary point.

dy/dx = -2x-2=0 so x=-1 is the stationary point

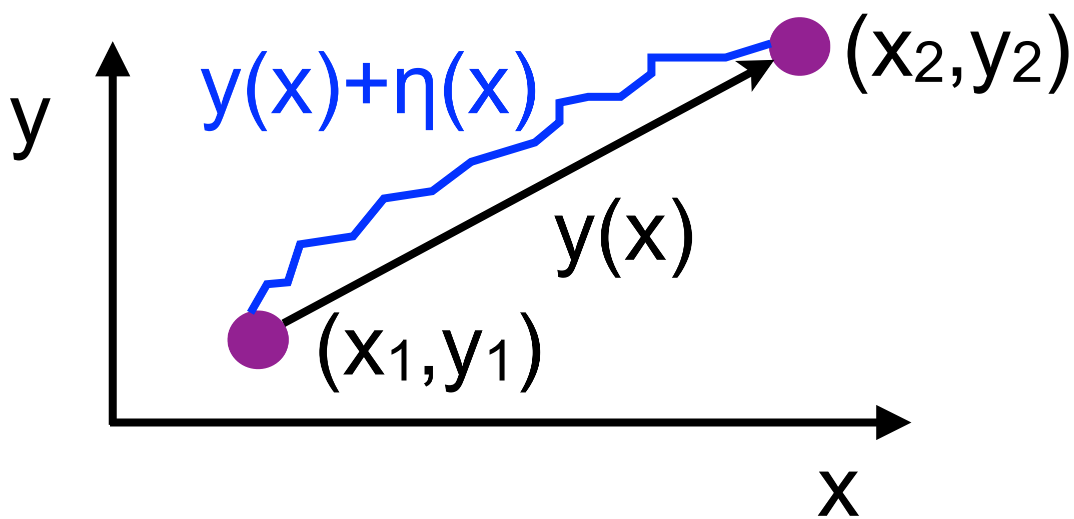
Wanted to find stationary point of

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2} dx$$

More generally find stationary point of

$$S = \int_{x_1}^{x_2} f[y, y', x] dx$$

Call correct solution y(x). Then Y(x) = y(x)+η(x) is another (incorrect) solution

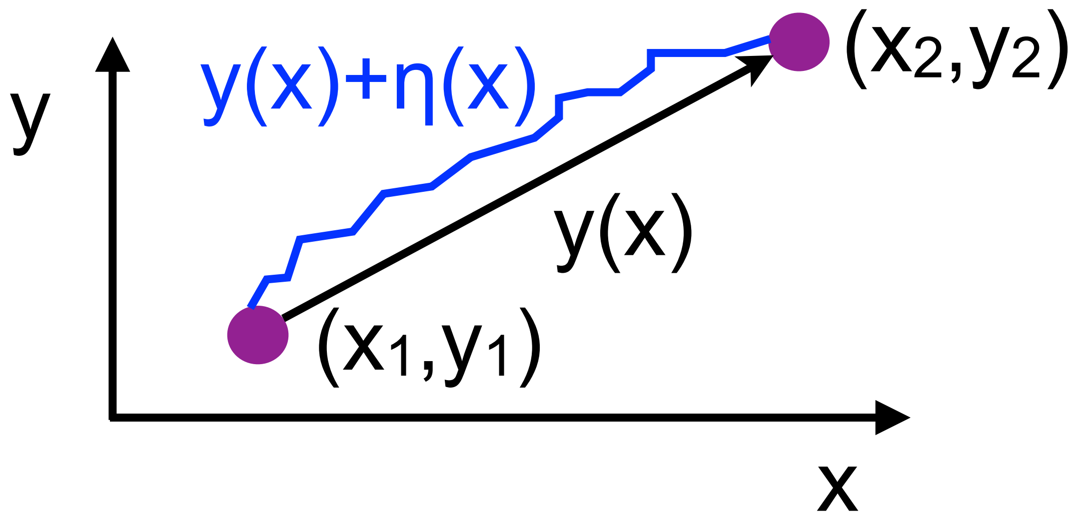


Calculus of variations

Call correct solution $y(x)$. Then $Y(x) = y(x)+\eta(x)$ is another (incorrect) solution. But we know $Y(x_1) = y_1$ and $Y(x_2) = y_2$, so

$$\eta(x_1) = \eta(x_2) = 0$$

$Y(x) = y(x)+ \alpha\eta(x)$ is another (incorrect) solution. The correct solution has $\alpha = 0$



Solution then has $dS/d\alpha = 0$

Calculus of variations

$$S[\alpha] = \int_{x_1}^{x_2} f[Y, Y', x]dx$$

$$Y = \alpha\eta, Y' = \alpha\eta'$$

$$S[\alpha] = \int_{x_1}^{x_2} f[y + \alpha\eta, y' + \alpha\eta', x]dx$$

Solution then has dS/dα = 0

$$\frac{d}{d\alpha} S[\alpha] = \int_{x_1}^{x_2} \frac{\partial}{\partial\alpha} f[y + \alpha\eta, y' + \alpha\eta', x]dx$$

Clear to everyone?

$$\frac{d}{d\alpha} S[\alpha] = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right] dx$$

Integration by parts

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta' dx = \left[\eta\frac{\partial f}{\partial y'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{d}{dx}\frac{\partial f}{\partial y'} dx$$

Euler-Lagrange Equation

$$\frac{d}{d\alpha}S[\alpha] = 0 = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right]dx$$

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\eta' dx = \left[\eta\frac{\partial f}{\partial y'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2}\eta\frac{d}{dx}\frac{\partial f}{\partial y'}dx$$

$$\frac{d}{d\alpha}S[\alpha] = 0 = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\eta - \eta\frac{d}{dx}\frac{\partial f}{\partial y'}\right]dx + \left[\eta\frac{\partial f}{\partial y'}\right]_{x_1}^{x_2}$$

But we know $\eta(x_1) = \eta(x_2) = 0$

$$\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\eta - \eta\frac{d}{dx}\frac{\partial f}{\partial y'}\right]dx = 0$$

$$\int_{x_1}^{x_2}\eta(x)\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right]dx = 0$$

Euler-Lagrange Equation

$$\int_{x_1}^{x_2} \eta(x) \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right] dx = 0$$

$$\rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

Because must hold for any η

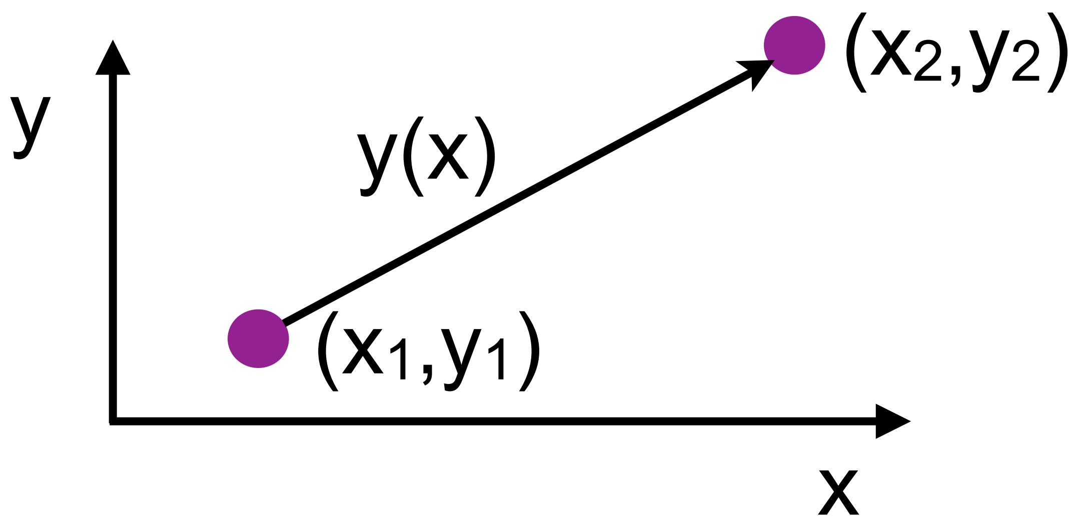
Euler-Lagrange Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

Starting with example 6.1 in book

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2} dx$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$



Starting with example 6.1 in book

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$f = \sqrt{1 + y'(x)^2}$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{2y'}{2}\left(1 + y'(x)^2\right)^{-0.5} = \frac{y'}{\sqrt{1 + y'^2}}$$

$$\frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} = 0 \rightarrow \frac{y'}{\sqrt{1 + y'^2}} = C \text{ (constant)}$$

$$y'^2 = C^2(1 + y'^2)$$

$$y'^2(1 - C^2) = C^2$$

$$y'^2 = \frac{C^2}{1 - C^2}$$

$$y' = dy/dx = \text{constant} \rightarrow y = ax + b$$

And now the famous brachistochrone (Ex 6.2)

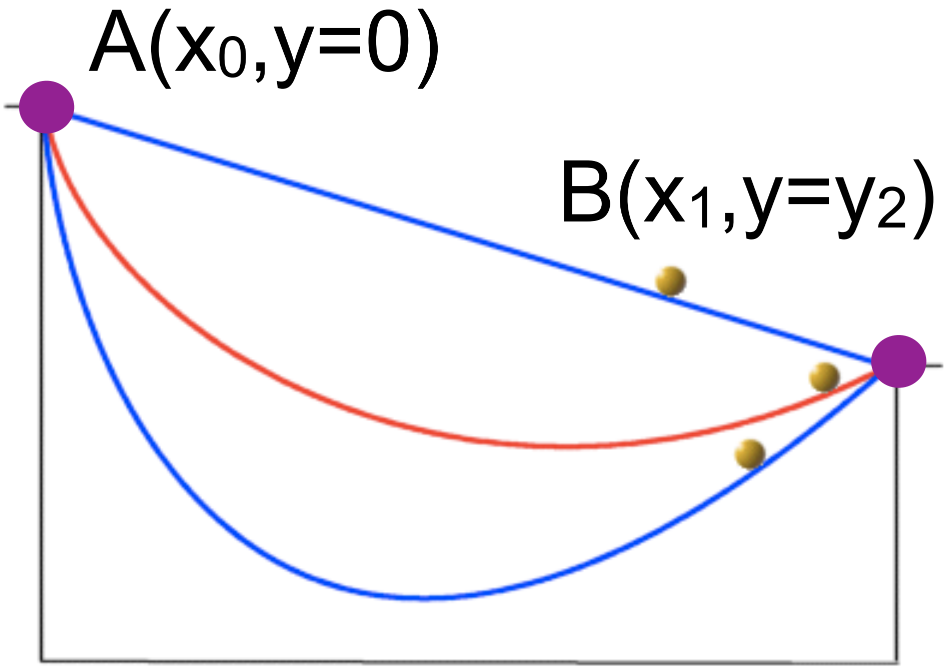
Assume no friction, and try to find the path that gets a roller coster from related from point A to point B in minimum time?

$$\text{time} = \int_1^2 \frac{ds}{v}$$

$$E = \frac{mv^2}{2} - mgy = \text{const}$$

$$E(t=0) = 0 - 0 = 0 \rightarrow mgy = \frac{mv^2}{2}$$

$$v = \sqrt{2gy}$$



And now the famous brachistochrone (Ex 6.2)

$$\text{time} = \int_1^2 \frac{ds}{v}$$

$$v = \sqrt{2gy}$$

$$ds = \sqrt{dx^2 + dy^2} = dy\sqrt{x'(y)^2 + 1}$$

$$x' = dx/dy$$

$$\text{time} = \frac{1}{\sqrt{2g}} \int_0^2 \frac{\sqrt{x'(y)^2 + 1}}{\sqrt{y}} dy$$

Note that we have reversed the roles of x and y in the integral for simplicity (we will often do this - change x for y for t, so just be careful!)

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0 \quad f = \sqrt{\frac{x'^2 + 1}{y}}$$

Working out the math

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0 \qquad f = \sqrt{\frac{x'^2+1}{y}}$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial x'} = \frac{1}{2}\sqrt{\frac{y}{x'^2+1}}2x' = \sqrt{\frac{x'^2 y}{x'^2+1}}$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial x'} = \frac{1}{2}\sqrt{\frac{y}{x'^2+1}}(2x'/y) = \sqrt{\frac{x'^2}{y(x'^2+1)}}$$

$$\frac{d}{dy}\sqrt{\frac{x'^2}{y(x'^2+1)}} = 0$$

$$\sqrt{\frac{x'^2}{y(x'^2+1)}} = \text{const}$$

$$\frac{x'^2}{y(x'^2+1)} = \text{const} = \frac{1}{2a}$$

And some algebra

$$\frac{x'^2}{y(x'^2+1)} = \text{const} = \frac{1}{2a}$$

$$2ax'^2 = yx'^2 + y$$

$$x'^2(2a-y) = y$$

$$x' = \sqrt{\frac{y}{2a-y}} = dx/dy$$

$$\int \sqrt{\frac{y}{2a-y}}dy = \int dx = x$$

Let $y = a(1-\cos\theta), dy = a\sin\theta d\theta$

$$x = \int \sqrt{\frac{a(1-\cos\theta)}{2a-a(1-\cos\theta)}}(a\sin\theta)d\theta$$

$$x = \int \sqrt{\frac{a(1-\cos\theta)}{a+a\cos\theta}}(a\sin\theta)d\theta$$

$$x = a\int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sin\theta d\theta$$

$$x = a\int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sin\theta d\theta$$

$$x = a\int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sqrt{1-\cos^2\theta}d\theta$$

$$x = a\int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sqrt{(1+\cos\theta)(1-\cos\theta)}d\theta$$

$$x = a\int (1-\cos\theta)d\theta$$

$$x = a(\theta - \sin\theta) + const$$

The solution

$$x = a(\theta - \sin\theta) + \text{const}$$

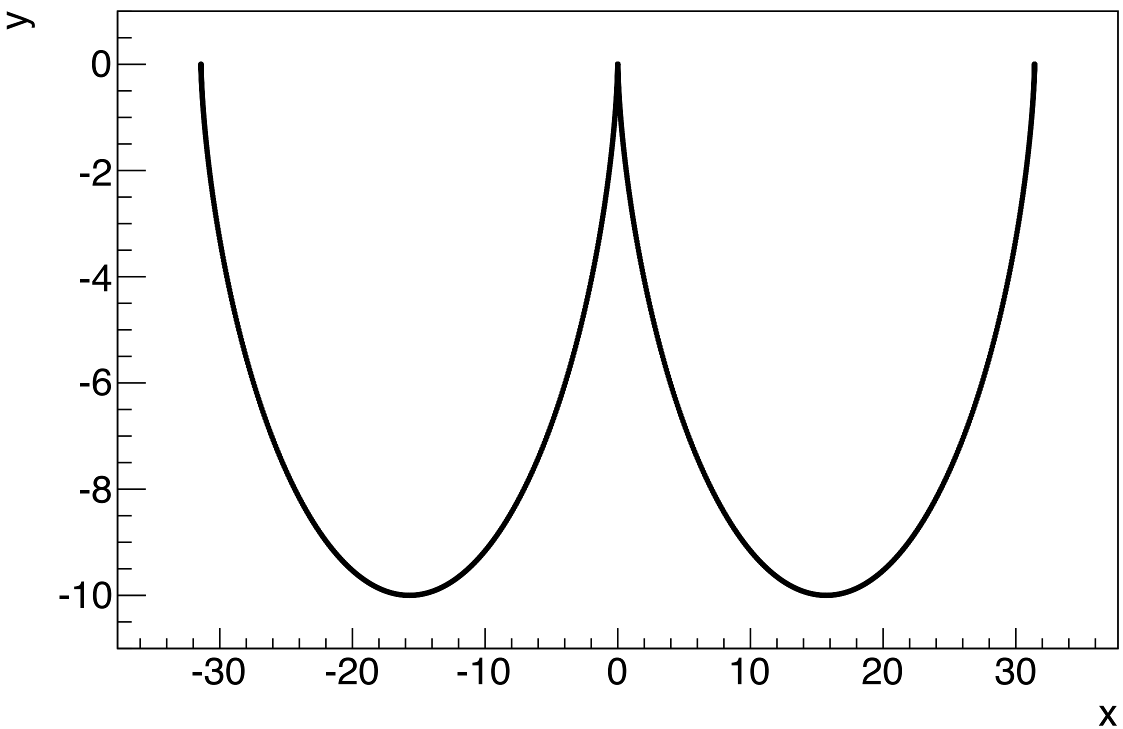
$$y = a(1 - \cos\theta)$$

curve passes through $(x, y) = (0, 0)$ at $\theta = 0 \rightarrow$

$$\text{const} = 0$$

$$x = a(\theta - \sin\theta)$$

$$y = a(1 - \cos\theta)$$



Euler-Lagrange with more than two variables

Find stationary point of:

$$S = \int_{u_1}^{u_2} f[y(u), y'(u), x'(u), x(u), u] du$$

$$x' = dx/du, y' = dy/du$$

Can repeat previous exercise to find:

$$\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0$$

$$\frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y'} = 0$$

Let's work on

Example 6.3, Problem 6.1
and then 6.16 together

Great circles as geodesics