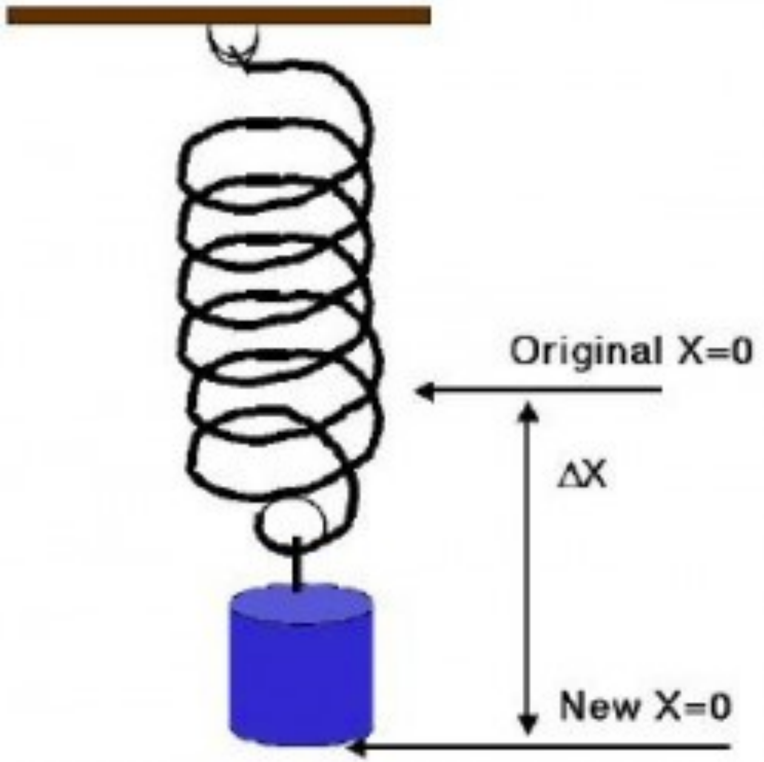
Begin our discussion of oscillations with Hooke's Law in one dimension



$$F(x) = -kx$$

$$U(x) = -\int F(x)dx = \frac{1}{2}kx^2$$

As Taylor (the author) points out, this is the expected form. Taylor (the mathematician!) expanding about an equilibrium point ...

$$U(x) = U(0) + U'(0)x + \frac{1}{2}U''(0)x^2 + \frac{1}{6}U'''(0)x^3 + \dots$$

Mr. Robert Hooke

Wikipedia:
“An artist's impression of Robert Hooke. No authenticated contemporary likenesses of Hooke survive.”

What does Newton say to Hooke (1D)?

$$F = ma = m\ddot{x} = -kx \rightarrow \ddot{x} = -\omega^2 x$$

$$\omega = \sqrt{k/m}$$

As always, try exponential functions

$$\ddot{x} = \frac{d^2x}{dt^2} = -\omega^2 x$$

$$x(t) = Ae^{Bt}, \dot{x} = ABe^{Bt}, \ddot{x} = AB^2e^{Bt} = B^2x$$

$$B^2x = -\omega^2 x$$

$$B = i\omega$$

$$x(t) = x_0 e^{i\omega t}$$

$$x(t) = x_0 e^{-i\omega t}$$

2nd order differential eqn, this is second solution

Briefly going back to Sec 2.5-2.6

$$e^z = \sum_{q=0}^{q=\infty} \frac{z^q}{q!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

$$e^{i\theta} = \sum_{q=0}^{q=\infty} \frac{(i\theta)^q}{q!} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots$$

$$i^0 = 1, i^1 = i, i^2 = -1, i^3 = i^2 \cdot i = -i, i^4 = i^2 \cdot i^2 = 1$$

n is integer

$$i^{0+4n} = i^0 i^{4n} = i^{4n} = (i^4)^n = 1^n = 1$$

$$i^{1+4n} = i^1 i^{4n} = i \cdot i^{4n} = i(i^4)^n = i1^n = i$$

$$i^{2+4n} = i^2 i^{4n} = -1^{4n} = -(i^4)^n = -1 \cdot 1^n = -1$$

$$i^{3+4n} = i^3 i^{4n} = -i^{4n} = -i \cdot (i^4)^n = -i \cdot 1^n = -i$$

Odd powers of i are imaginary, even powers are real, alternating between positive and negative

Briefly going back to Sec 2.5-2.6

$$e^{i\theta} = \sum_{q=0}^{q=\infty} \frac{(i\theta)^q}{q!} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots$$

$$e^{i\theta} = \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right]$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Can work out Taylor expansion for cosine and sine (see problem 2.18)

$$\eta(t) = Ae^{\omega t}$$

$$A = ae^{i\delta} \rightarrow$$

$$\eta(t) = ae^{i\delta}e^{\omega t} = ae^{i\delta + \omega t}$$

A complex

a,δ real

Electric particle moving in constant B field

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = m\dot{\mathbf{v}}$$

$$m\dot{v}_x = qBv_y$$

$$m\dot{v}_y = -qBv_x$$

$$m\dot{v}_z = 0 \rightarrow v_z = \text{const}$$

Particle with electric charge q moving in magnetic field. Let's align the magnetic field along the z-direction, $\mathbf{B} = B\hat{\mathbf{z}}$

$$\omega = \frac{qB}{m}$$

$$\dot{v}_x = \omega v_y$$

$$\dot{v}_y = -\omega v_x$$

Coupled equations!

$$\eta = v_x + iv_y$$

$$\dot{\eta} = \dot{v}_x + i\dot{v}_y = \omega v_y - i\omega v_x$$

This we know how to solve!

$$\dot{\eta} = -i\omega\eta$$

Electric particle moving in constant B field

$$\dot{\eta} = -i\omega\eta$$

$$\eta(t) = Ae^{-i\omega t} = \dot{x}(t) + i\dot{y}(t)$$

$$\dot{v_z} = \text{const} \rightarrow z(t) = z_0 + v_z t$$

$$\xi(t) = x(t) + iy(t) = \int \eta(t)dt = \int Ae^{-i\omega t}dt$$

$$q = -i\omega t, dq = -i\omega dt, dt = dq/(-i\omega)$$

$$i^{-1} = 1 \cdot i^{-1} = i^4 \cdot i^{-1} = i^3 = -i$$

$$dt = idq/\omega$$

$$\xi(t) = \frac{iA}{\omega}\int e^q dq = \frac{iA}{\omega}e^{-i\omega t} + \text{constant}$$

$$x(t) + iy(t) = \frac{iA}{\omega}e^{-i\omega t}$$

$$x(0) + iy(0) = \frac{iA}{\omega}$$

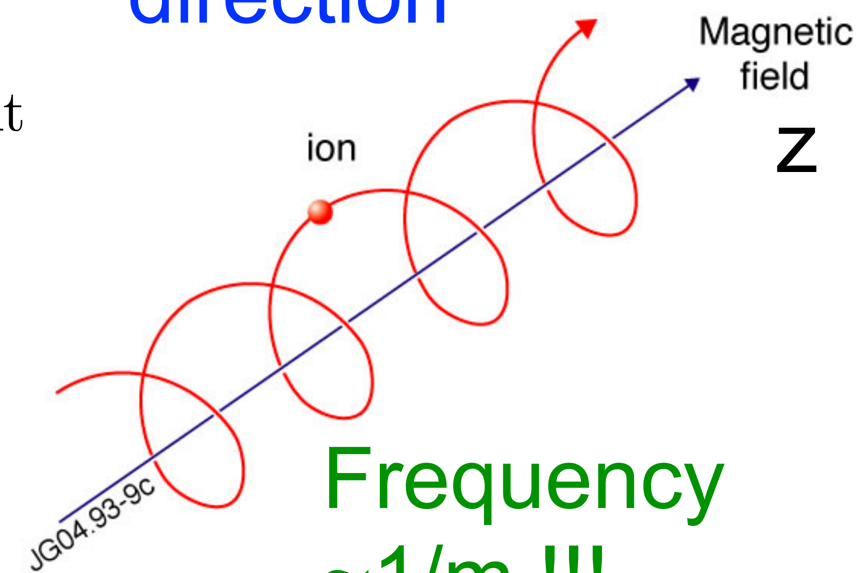
$$x(t) + iy(t) = [x(0) + iy(0)]\, e^{-i\omega t}$$

$$x(t) \sim \cos(\omega t + \delta)$$

$$y(t) \sim \sin(\omega t + \delta)$$

$$z(t) = z_0 + v_z t$$

Particle moves in a helix in x-y direction

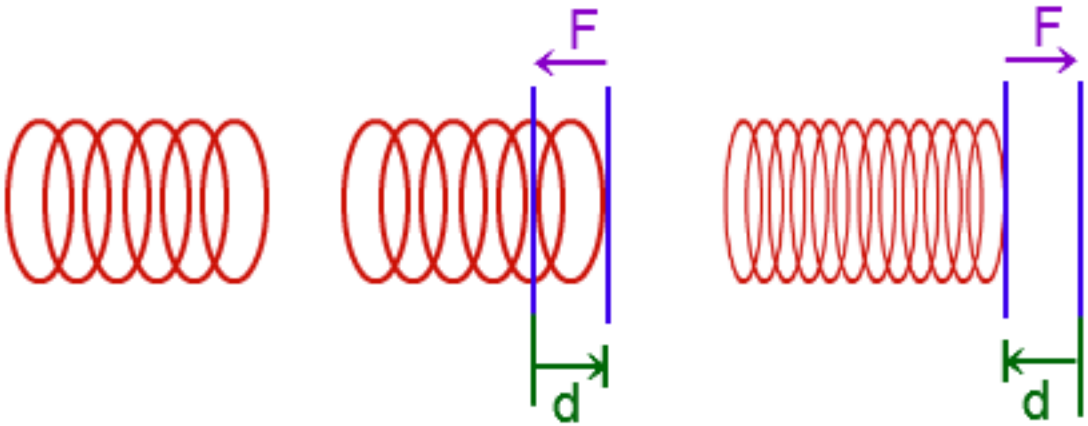


Frequency ~1/m !!!

Now back to Hooke + Newton

$$F = ma = m\ddot{x} = -kx \rightarrow \ddot{x} = -\omega^2 x$$

$$\omega = \sqrt{k/m}$$



$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

$$x(t) = A\left[\cos(\omega t) + i\sin(\omega t)\right] + B\left[\cos(-\omega t) + i\sin(-\omega t)\right]$$

$$x(t) = A\left[\cos(\omega t) + i\sin(\omega t)\right] + B\left[\cos(\omega t) - i\sin(\omega t)\right]$$

$$x(t) = (A+B)\cos(\omega t) + i(A-B)\sin(\omega t)$$

$$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t), \text{ with } C_1 = (A+B), C_2 = i(A-B)$$

Alternate forms of solution

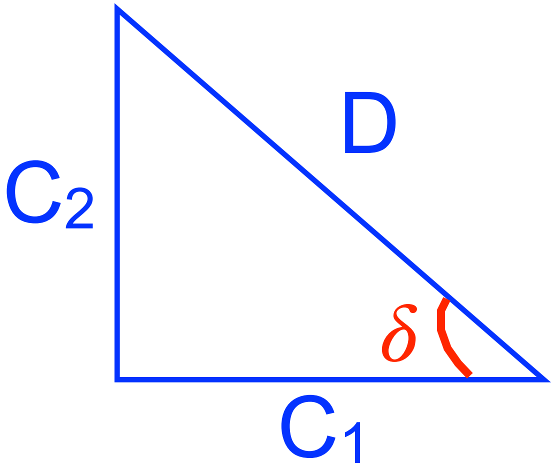
$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t), \text{ with } C_1 = (A+B), C_2 = i(A-B)$$

$$D = \sqrt{C_1^2 + C_2^2}$$

$$x(t) = D\left[\frac{C_1}{D}\cos(\omega t) + \frac{C_2}{D}\sin(\omega t)\right]$$

$$x(t) = D\left[cos\delta \cos(\omega t) + \sin\delta \sin(\omega t)\right]$$

$$x(t) = D\cos(\omega t - \delta)$$



$$C_1 = D\cos\delta$$

$$C_2 = D\sin\delta$$

$$\cos Y \cos Z + \sin Y \sin Z = \cos(Y - Z)$$

Yet other alternate forms of solution

Same solution, but good to get comfortable with all these forms!

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t), \text{ with } C_1 = (A+B), C_2 = i(A-B)$$

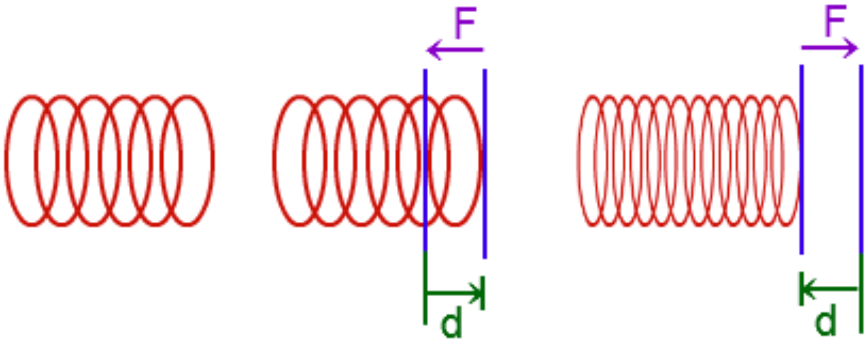
$$\rightarrow A = \frac{1}{2}(C_1 - iC_2), B = \frac{1}{2}(C_1 + iC_2)$$

$$\rightarrow B = A^*, \text{ with } z^* = x - iy \text{ if } z = x + iy$$

$$x(t) = Ae^{i\omega t} + A^* e^{-i\omega t} = 2 \cdot \text{Real part of} \left[Ae^{i\omega t}\right]$$

Obvious why?

Total energy of harmonic oscillator



$$x(t) = A\cos(\omega t - \delta)$$
$$x^2(t) = A^2\cos^2(\omega t - \delta)$$
$$\dot{x}(t) = -A\omega\sin(\omega t - \delta)$$
$$\dot{x}^2(t) = A^2\omega^2\sin^2(\omega t - \delta)$$
$$U = \frac{1}{2}kx^2 = \frac{k}{2}A^2\cos^2(\omega t - \delta)$$
$$T = \frac{1}{2}mv^2 = \frac{m}{2}A^2\omega^2\sin^2(\omega t - \delta)$$
$$\omega^2 = k/m \rightarrow$$

Total E does not depend on time (as expected!)

$$E = U + T = \frac{k}{2}A^2\cos^2(\omega t - \delta) + \frac{k}{2}A^2\sin^2(\omega t - \delta)$$
$$E = \frac{kA^2}{2}\left[\cos^2(\omega t - \delta) + \sin^2(\omega t - \delta)\right] = kA^2/2$$

Some problems

Problem 5.9 in small groups or by yourself, then
Problem 5.3 together
Then Problem 5.1 in small groups again or by yourself

What about oscillators in >1 dimension?

$$\mathbf{F} = -k\mathbf{r}, \omega^2 = k/m$$
$$\ddot{x} = -\omega^2 x$$
$$\ddot{y} = -\omega^2 y$$
$$x(t) = A_x \cos(\omega t)$$
$$y(t) = A_y \cos(\omega t - \delta)$$

How does 2d-motion change depending on the phase?

Let's plot solutions where A_x or A_y = 0, and where they are not zero, and with various phrases (think about what the phases mean)

What if restoring force is not equal in both directions?

$$F_x = -k_x x, \omega_x^2 = k_x/m$$

$$F_y = -k_y y, \omega_y^2 = k_y/m$$

$$\ddot{x} = -\omega_x^2 x$$

$$\ddot{y} = -\omega_y^2 y$$

$$x(t) = A_x \cos(\omega_x t)$$

$$y(t) = A_y \cos(\omega_y t - \delta)$$

How does 2d-motion change depending on the two restoring forces (ie the relationship between the ω)? Easier to plot in 1 dimension on top of each other first

Damped oscillations (back to 1D)

$$m\ddot{x} + b\dot{x} + kx = 0$$

ma from Newton

-bv damping force

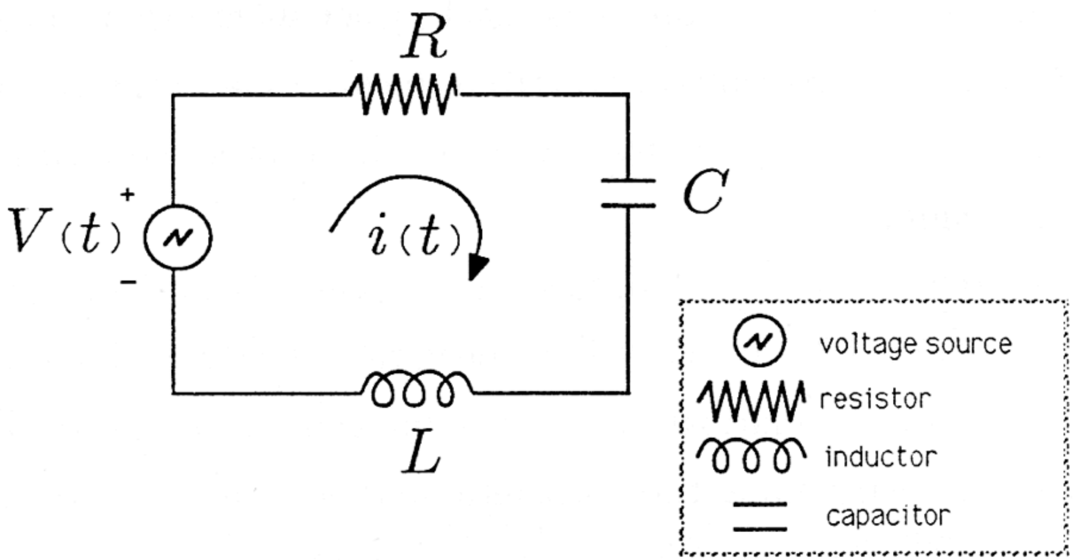
-kx from Hooke's Law

Note the assumption of linear damping force to make problem much simpler

Damped oscillations (a comparison)

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$$



Back to our classical mechanics problem

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$m\ddot{x} + b\dot{x} + kx = 0$$
$$\ddot{x} + (b/m)\dot{x} + (k/m)x = 0$$
$$\frac{b}{m} = 2\beta$$
$$\omega_0^2 = \frac{k}{m}$$
$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

As is very often the case, let's see if an exponential solution solves this differential equation

Back to our classical mechanics problem

Guess at exponential

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$x(t) = Ae^{rt}$$

$$\dot{x} = Are^{rt} = rx$$

$$\ddot{x} = Ar^2e^{rt} = r^2x$$

$$r^2x + 2\beta rx + \omega_0^2 x = 0$$

$$r^2 + 2\beta r + \omega_0^2 = 0$$

Back to our classical mechanics problem

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$r^2 + 2\beta r + \omega_0^2 = 0$$

$$r = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2}$$

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

$$x(t) = Ae^{rt}$$

$$x(t) = C_1 e^{\left(-\beta + \sqrt{\beta^2 - \omega_0^2}\right)t} + C_2 e^{\left(-\beta - \sqrt{\beta^2 - \omega_0^2}\right)t}$$

$$x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2 - \omega_0^2}\,t} + C_2 e^{-\sqrt{\beta^2 - \omega_0^2}\,t}\right)$$

How to interpret this?

$$x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}t}\right)$$

As always, let's check ranges/extremes for the new variables we've introduced.

No damping means β=0, so

$$x(t) = C_1 e^{\sqrt{-\omega_0^2}t} + C_2 e^{-\sqrt{-\omega_0^2}t}$$

$$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$

Get back (as expected) undamped harmonic oscillator we previously studied

Now let's allow (small) damping

$$x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}t}\right)$$

$$\text{if } \beta < \omega_0, \beta^2 - \omega_0^2 < 0, \sqrt{\beta^2 - \omega_0^2} = i\omega_1$$

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$$

$$x(t) = e^{-\beta t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right)$$

$$x(t) = e^{-\beta t}\cos(\omega_1 t - \delta)$$

x(t) is the product of two terms. What does this product look like? Let's plot this...

But what happens as we increase damping (ie increase β)?

Now let's allow (very large) damping

$$x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}t}\right)$$

$$\text{if } \beta > \omega_0, \beta^2 - \omega_0^2 > 0$$

Both exponentials are real: no oscillations when we have overdamping. Let's also plot this

One special case

$$x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}t}\right)$$

$$\text{if } \beta = \omega_0 \ldots$$

$$x(t) = e^{-\beta t}$$

We only have one solution now! What's the second one? Taylor give the answer (a good guess)

$$x(t) = -te^{-\beta t}$$

$$\dot{x} = \frac{d}{dt}(-te^{-\beta t})$$

$$\dot{x} = -e^{-\beta t} + t\beta e^{-\beta t}$$

$$\ddot{x} = \frac{d}{dt}(-e^{-\beta t} + t\beta e^{-\beta t})$$

$$\ddot{x} = \beta e^{-\beta t} + \beta e^{-\beta t} - t\beta^2 e^{-\beta t}$$

$$\ddot{x} = 2\beta e^{-\beta t} - t\beta^2 e^{-\beta t}$$

Critical damping

$$x(t) = -te^{-\beta t}$$

$$\dot{x} = -e^{-\beta t} + t\beta e^{-\beta t}$$

$$\ddot{x} = 2\beta e^{-\beta t} - t\beta^2 e^{-\beta t}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$2\beta e^{-\beta t} - t\beta^2 e^{-\beta t} + 2\beta(-e^{-\beta t} + t\beta e^{-\beta t}) + \omega_0^2(-te^{-\beta t}) = 0$$

$$2\beta - t\beta^2 - 2\beta + 2t\beta^2 - t\omega_0^2 = 0$$

$$t(\beta^2 - \omega_0^2) = 0$$ But $\beta = \omega_0$

So general solution for critical damped case is

$$x(t) = C_1 e^{-\beta t} + C_2 te^{-\beta t}$$

Some problems

Let's work on problems 5.20, 5.23, then you work on 5.28

Damping oscillations

In what cases do we want to most quickly dampen oscillations?

My favorite case (though not the most important one)

Driven damped oscillations (a comparison)

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}(t)$$

$$f(t) = F(t)/m$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$

How about when we force oscillations?

Let's look at sinusoidal driving forces

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

Guess a cosine with the same frequency

Can use exponentials too (see Taylor)

$$x = C\cos(\omega t - \delta)$$

$$\dot{x} = -C\omega\sin(\omega t - \delta)$$

$$\ddot{x} = -C\omega^2\cos(\omega t - \delta)$$

$$-C\omega^2\cos(\omega t - \delta) - 2C\beta\omega\sin(\omega t - \delta) + \omega_0^2 C\cos(\omega t - \delta) = f_0\cos(\omega t)$$

Useful generic identities:

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

Plugging it in

$$-C\omega^2 \cos(\omega t - \delta) - 2C\beta\omega \sin(\omega t - \delta) + \omega_0^2 C \cos(\omega t - \delta) = f_0 \cos(\omega t)$$

$$-C\omega^2 \left[\cos(\omega t)\cos\delta + \sin(\omega t)\sin\delta\right] - 2C\beta\omega \left[\sin(\omega t)\cos\delta - \cos(\omega t)\sin\delta\right]$$

$$+ \omega_0^2 C \left[\cos(\omega t)\cos\delta + \sin(\omega t)\sin\delta\right] = f_0 \cos(\omega t)$$

For this to be true always, $\sin(\omega t)$ and $\cos(\omega t)$ terms must always balance:

$$C \cos(\omega t) \left[(\omega_0^2 - \omega^2)\cos\delta + 2\beta\omega \sin\delta - f_0/C\right] = 0$$

$$C \sin(\omega t) \left[(\omega_0^2 - \omega^2)\sin\delta - 2\beta\omega \cos\delta\right] = 0$$

Plugging it in

$$C\cos(\omega t)\left[(\omega_0^2-\omega^2)\cos\delta+2\beta\omega\sin\delta-f_0/C\right]=0$$

$$C\sin(\omega t)\left[(\omega_0^2-\omega^2)\sin\delta-2\beta\omega\cos\delta\right]=0$$

$$\frac{\sin\delta}{\cos\delta}=\frac{2\beta\omega}{\omega_0^2-\omega^2}$$

$$\boxed{\tan\delta=\frac{2\beta\omega}{\omega_0^2-\omega^2}}$$

$$(\omega_0^2-\omega^2)\cos\delta+2\beta\omega\left(\frac{2\beta\omega\cos\delta}{\omega_0^2-\omega^2}\right)=f_0/C$$

A useful identity: $\cos(q)=1/\sqrt{\tan^2(q)+1}\rightarrow$

$$\frac{\omega_0^2-\omega^2}{\sqrt{1+4\beta^2\omega^2/(w_0^2-\omega^2)^2}}+2\beta\omega\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\frac{1}{\sqrt{1+4\beta^2\omega^2/(w_0^2-\omega^2)^2}}\right)=f_0/C$$

$$\frac{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}=f_0/C$$

$$\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}=f_0/C$$

$$\boxed{C=\frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}}$$

Phew!

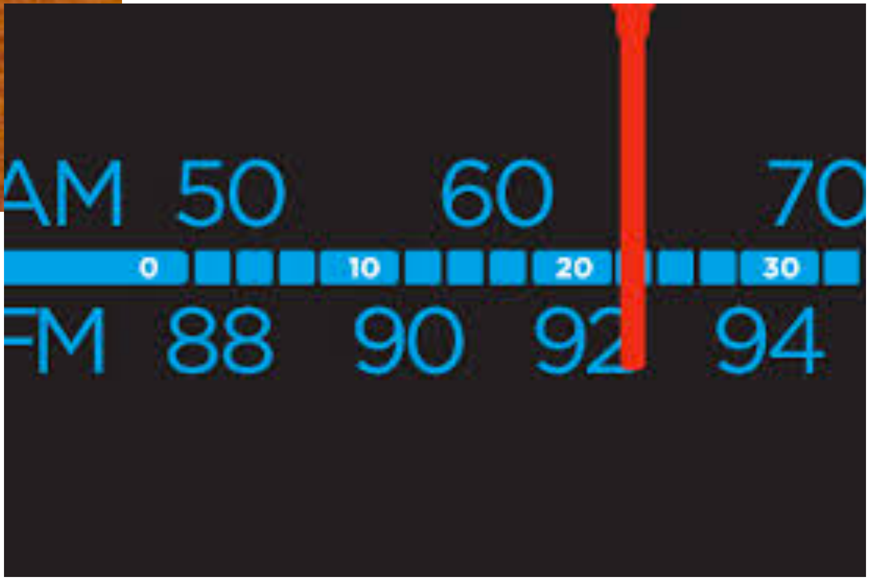
$$x(t) = C\cos(\omega t - \delta)$$

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

$$C = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

Let's examine some of the behavior of this solution. What happens when damping is small? What do we mean by a resonance?

Some examples of resonances



Some examples of resonances

Some examples of resonances

Where do we get a resonance?

$$C = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

C maximum when $\left[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\right]$ is a minimum

Fixed ω $\omega_0^{max} = \omega$

Fixed ω_0 $\omega^{max} = \omega_2 = \sqrt{\omega_0^2 - 2\beta^2}$

A nice summary from Taylor

$$\omega_0 = \sqrt{k/m} = \text{ natural frequency of undamped oscillator}$$

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} = \text{ frequency of damped oscillator}$$

$$\omega = \text{ frequency of driving force}$$

$$\omega_2 = \sqrt{\omega_0^2 - 2\beta^2} = \text{ value of } \omega \text{ at which response is maximum}$$

Quality factor

Let’s work out Problem 5.41 together

From Problem 5.41

$$\text{Full width at half maximum} = \text{FWHM} \sim 2\beta$$

$$\text{Quality factor } Q = \frac{w_0}{2\beta}$$

A word on transient solutions

$$x_{tr}(t) = e^{-\beta t}\left(C_{tr}^{1} e^{\sqrt{\beta^2-\omega_0^2}t} + C_{tr}^{2} e^{-\sqrt{\beta^2-\omega_0^2}t}\right)$$

We know the above is a solution to the undriven oscillator:

$$m\ddot{x} + b\dot{x} + kx = 0$$

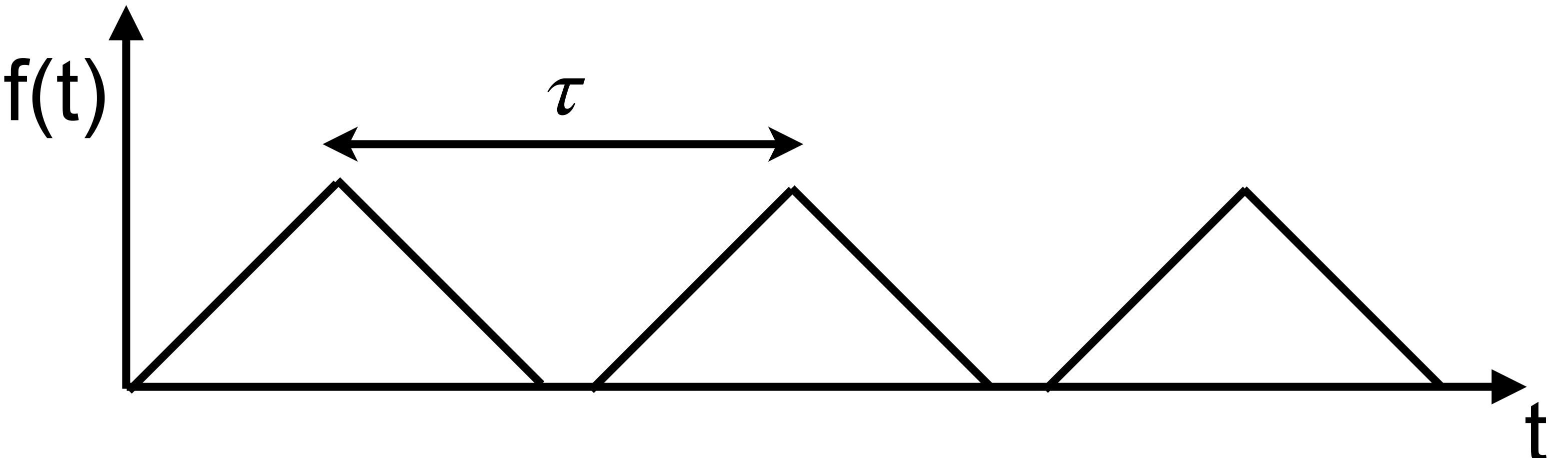
If we add $x_{tr}(t)$ to our nominal solution to the forced oscillator, we get back a new solution to the forced oscillator (since by definition in the differential equation is == 0)

The full solution is cos(ωt-δ)+the above transients (which die out over time and often are ignored)

Now slightly switching topics again

What does it mean for a function f to be periodic (with period τ)?

$$f(t + \tau) = f(t)$$



What about trigonometric functions?

cos(2πt/τ), sin (2πt/τ), cos(4πt/τ), sin (6πt/τ) have period τ, as do all:

n is integer

$$\cos(n\omega t), \sin(n\omega t)$$

$$\omega = 2\pi/\tau$$

And also this:

$$f(t) = \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

What Fourier tells us...

Any periodic function f(t) with period τ can be expressed as

$$f(t) = \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

In other words, any periodic function can be built up from an infinite series of cos and sin terms

We will study this because we now know how to solve problem of damped oscillators with sinusoidal driving forces. Also because Fourier series are incredibly useful in engineering, other areas of physics, information processing, etc...

Fourier decomposition coefficients

$$f(t) = \sum_{n=0}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

$$a_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cos(n\omega t) dt$$ n>0

$$b_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \sin(n\omega t) dt$$ n>0

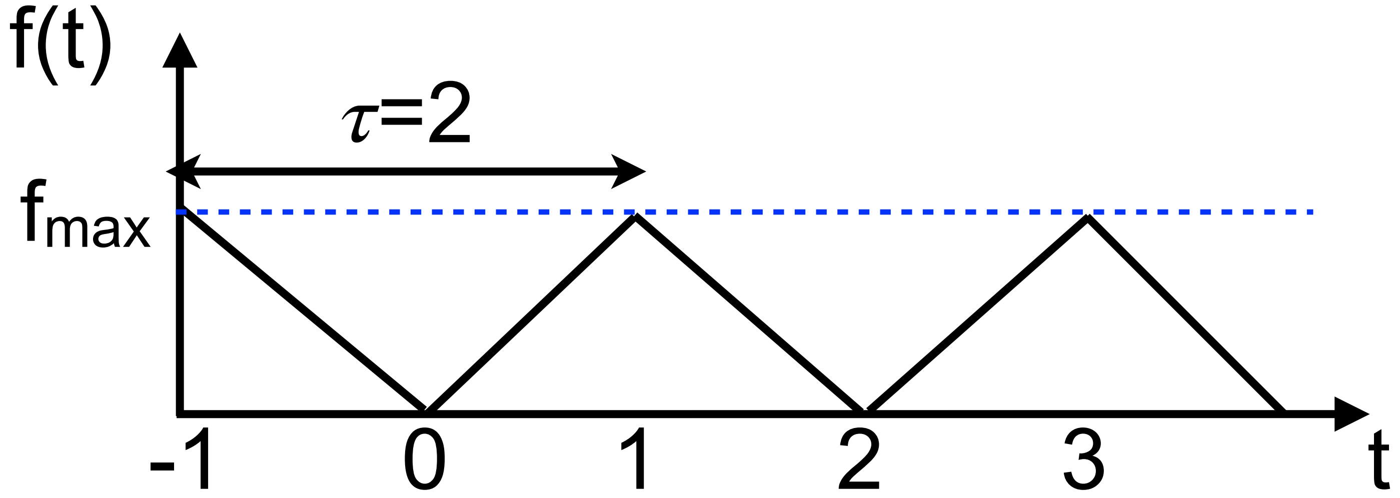
$$a_0 = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t) dt$$

$$b_0 = 0$$

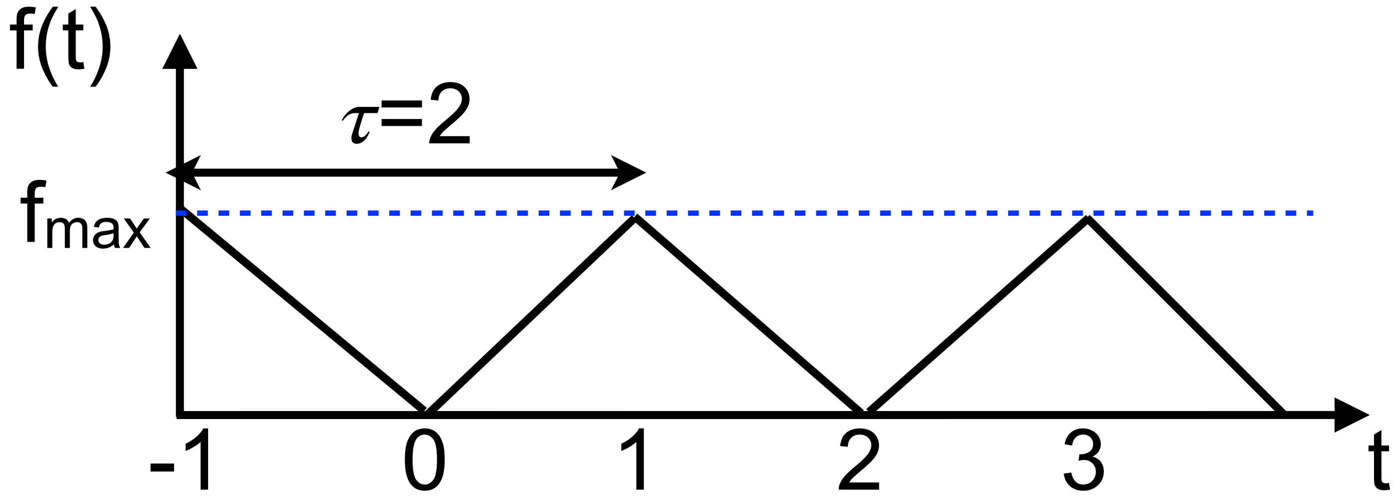
Want proof?
See problems 5.46-5.48

How to use?

Let's evaluate Fourier coefficients for our example from before (Problem 5.49):



The constant term



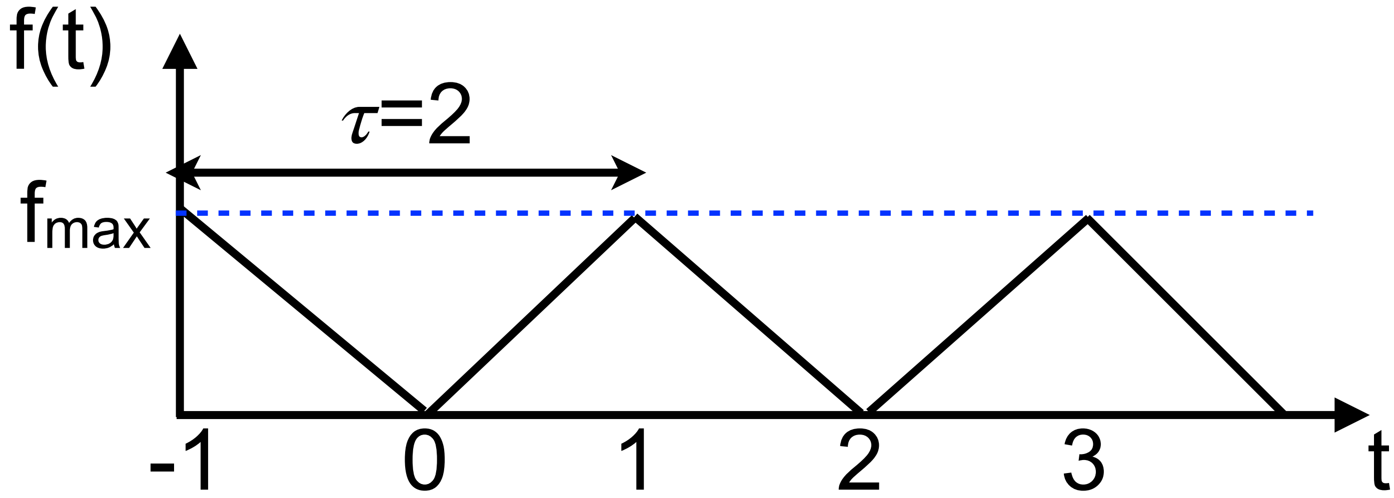
$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)dt = \frac{1}{2}\int_{-1}^{1} f(t)dt =$$

Because
f(x) = f(-x)

$$= \frac{2}{2}\int_{0}^{1} f(t)dt = \int_{0}^{1} f_{max} t dt =$$

$$f_{max}/2[t^2]_0^1 = f_{max}/2$$

What are the other “easy” terms?



$$b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin(n\omega t)dt$$

$$b_n = \frac{2}{\tau}\left[\int_{-\tau/2}^{0} f(t)\sin(n\omega t)dt + \int_{0}^{\tau/2} f(t)\sin(n\omega t)dt\right] = 0$$

f(x) = f(-x) but sin(x) = -sin(-x) so the two integrals have opposite sign and cancel

And the non-trivial terms?

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos(n\omega t)dt$$

$$a_n = \frac{4}{\tau}\int_{0}^{\tau/2} f(t)\cos(n\omega t)dt$$

$$a_n = \frac{4}{2}\int_{0}^{1} f(t)\cos(n\omega t)dt = 2\int_0^1 f_{max}t\cos(n\omega t)dt$$

$$a_n = 2f_{max}\int_0^1 t\cos(n\omega t)dt$$

Reminder of integration by parts...

$$\int udv = uv - \int vdu$$

$$u = t, du = dt, dv = \cos(n\omega t)dt, v = \frac{1}{n\omega}\sin(n\omega t)$$

$$a_n = 2f_{max}\left(\left[\frac{t}{n\omega}\sin(n\omega t)\right]_0^1 - \int_0^1 \frac{1}{n\omega}\sin(n\omega t)dt\right)$$

$$a_n = 2f_{max}\left[\frac{t}{n\omega}\sin(n\omega t) + \frac{1}{n^2\omega^2}\cos(n\omega t)\right]_{t=0}^{t=1}$$

$$a_n = 2f_{max}\left(\frac{1}{n\omega}\sin(n\omega) + \frac{1}{n^2\omega^2}\cos(n\omega) - \frac{1}{n^2\omega^2}\right)$$

But remember,

τ = 2, so ω=π

And the non-trivial terms?

$$a_n = 2f_{max}\left(\frac{1}{n\pi}\sin(n\pi) + \frac{1}{n^2\pi^2}\cos(n\pi) - \frac{1}{n^2\pi^2}\right)$$

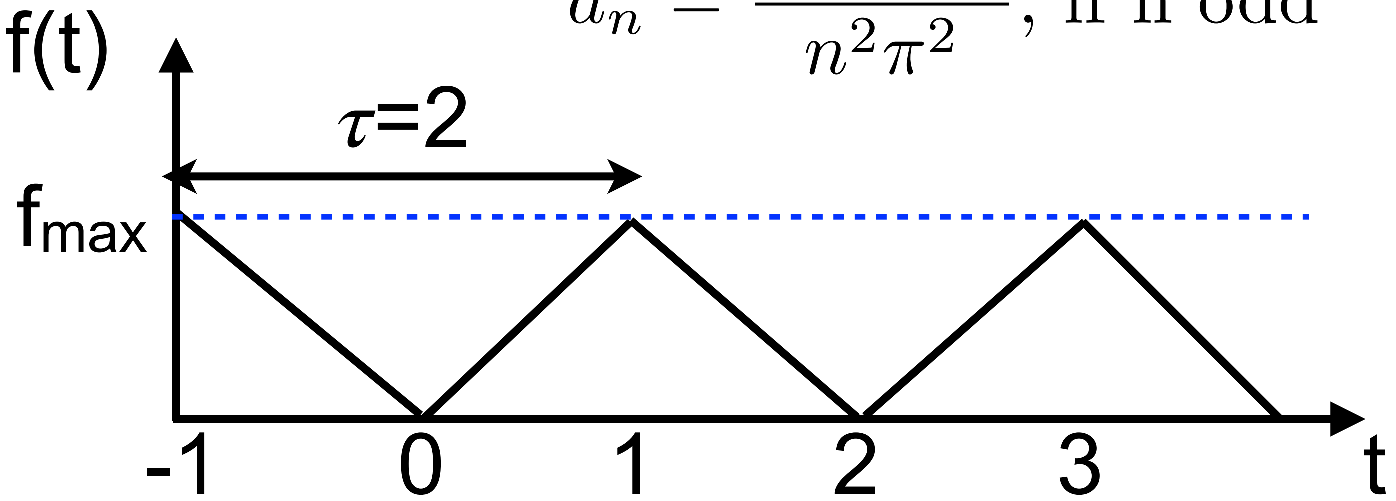
$$\sin(n\pi) = 0$$

$$\cos(n\pi) = +1, \text{ if n even}$$

$$\cos(n\pi) = -1, \text{ if n odd}$$

$$a_n = 0, \text{ if n even}$$

$$a_n = \frac{-4f_{max}}{n^2\pi^2}, \text{ if n odd}$$

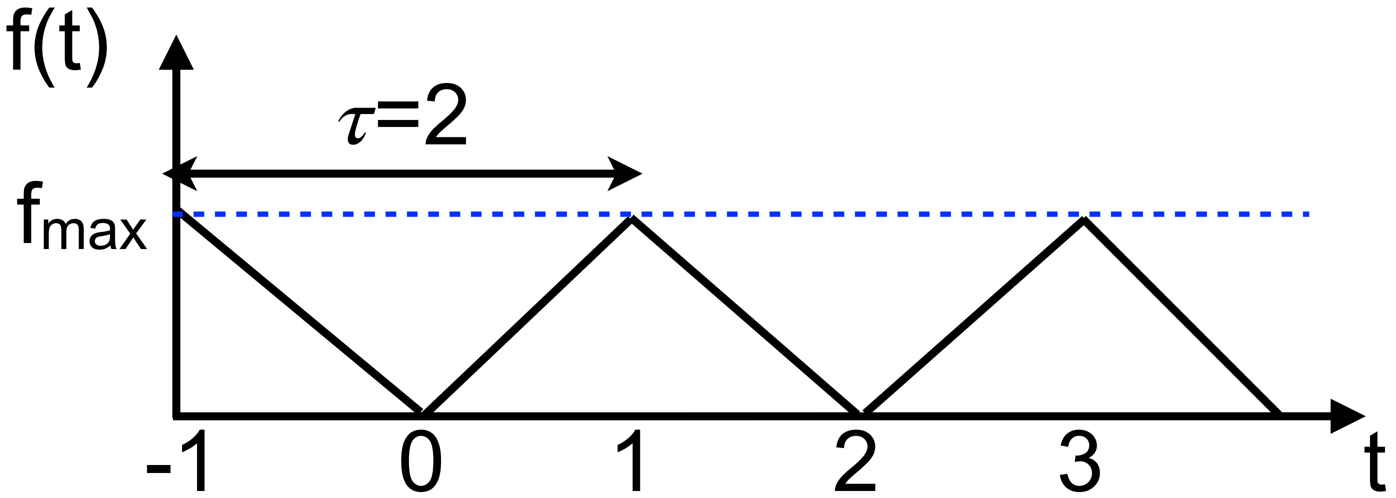


The answer

$$f(t) = f_{max}/2 + \sum_{n=1}^{\infty} [a_n \cos(n\pi t)]$$

$$a_n = \frac{-4 f_{max}}{n^2 \pi^2} \text{ for n odd}$$

$$a_n = 0 \text{ for n even}$$



How well does it work?

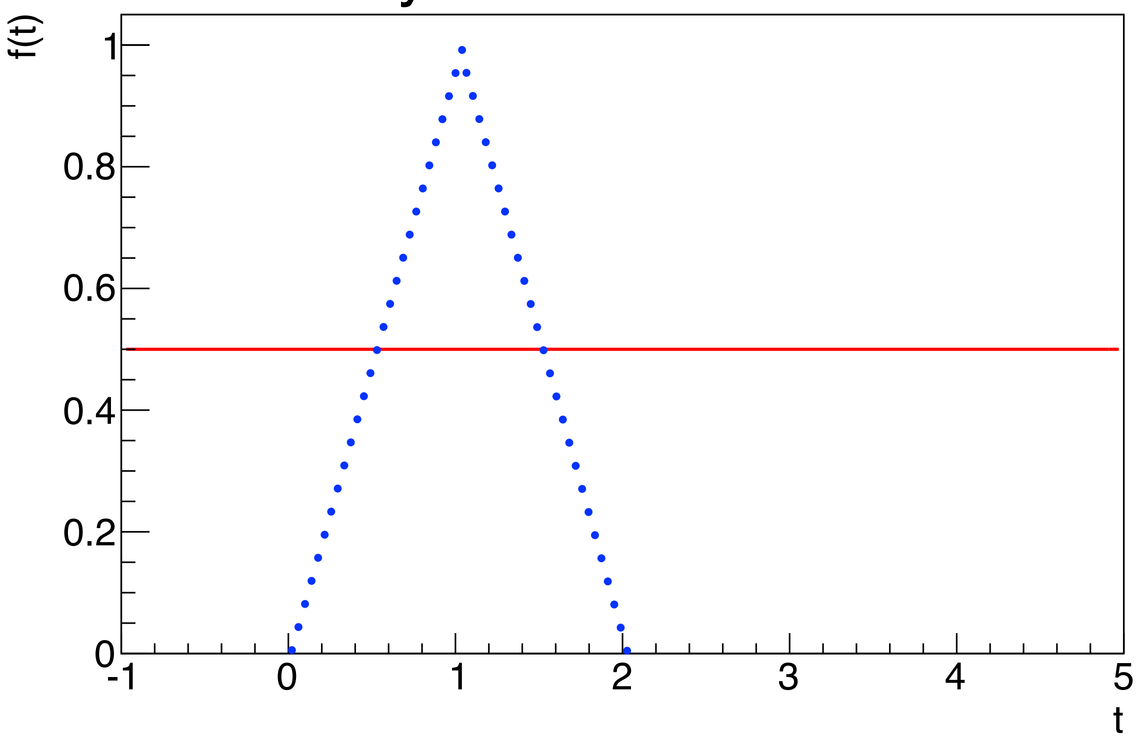
$$f(t) = f_{max}/2 + \sum_{n=1}^{\infty} [a_n \cos(n\pi t)]$$

$$a_n = \frac{-4 f_{max}}{n^2 \pi^2} \text{ for n odd}$$

$$a_n = 0 \text{ for n even}$$

Let's try this out!
Set fmax = 1 for simplicity

Only a0 term



Obviously not enough

How well does it work?

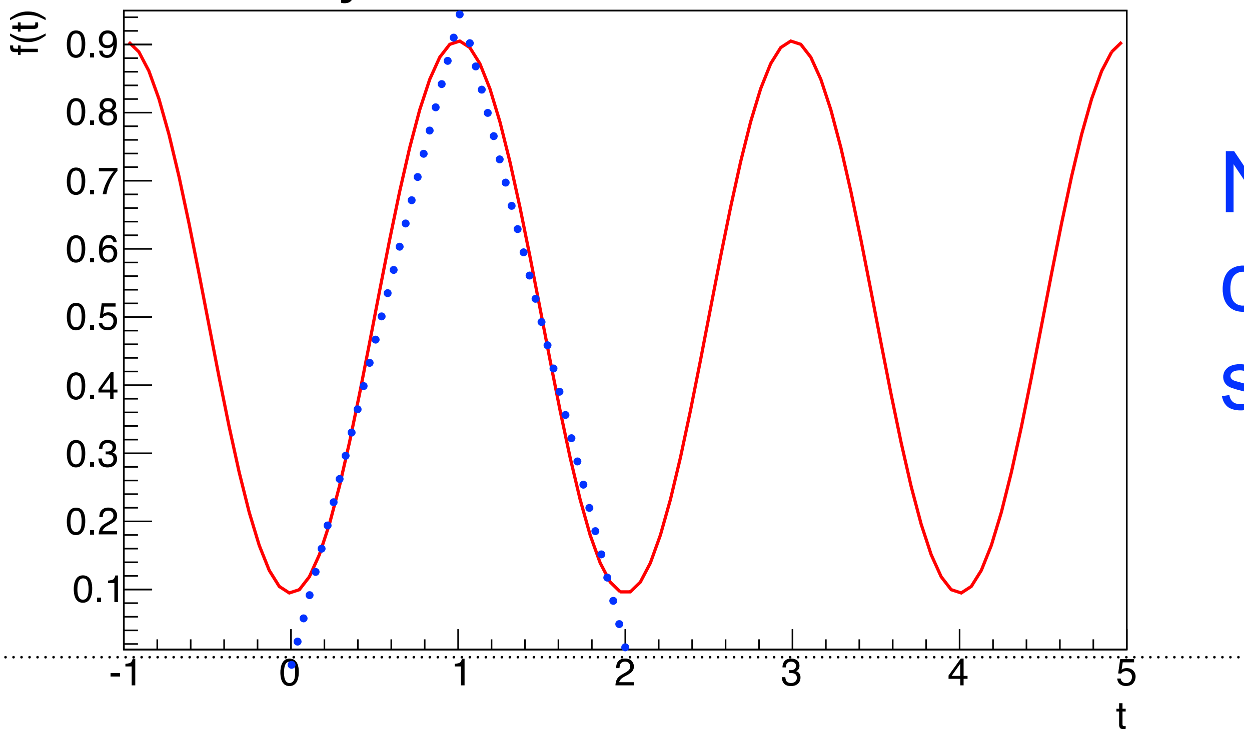
$$f(t) = f_{max}/2 + \sum_{n=1}^{\infty} [a_n \cos(n\pi t)]$$

$$a_n = \frac{-4 f_{max}}{n^2 \pi^2} \text{ for n odd}$$

$$a_n = 0 \text{ for n even}$$

Let's try this out! Set fmax = 1 for simplicity

Only a0,a1 terms



Not great, but can see this starting to work

How well does it work?

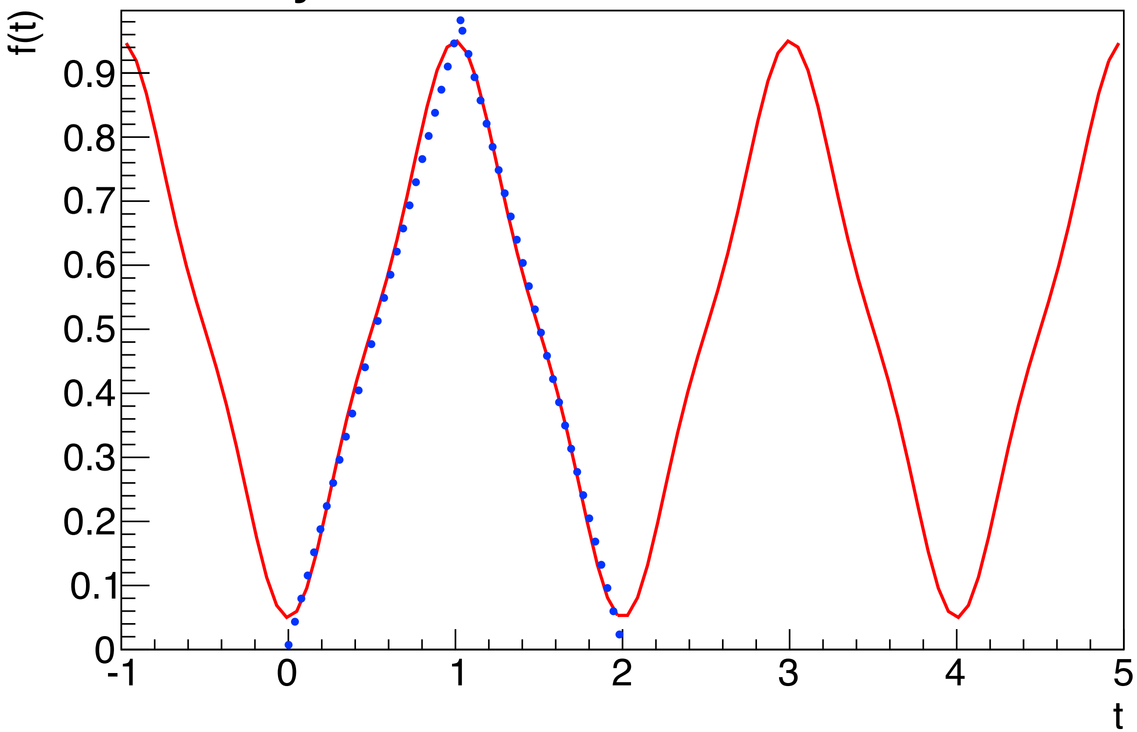
$$f(t) = f_{max}/2 + \sum_{n=1}^{\infty} [a_n \cos(n\pi t)]$$

$$a_n = \frac{-4 f_{max}}{n^2 \pi^2} \text{ for n odd}$$

$$a_n = 0 \text{ for n even}$$

Let's try this out!
Set fmax = 1 for simplicity

Only a0,a1,a3 terms



Can see this now?

How well does it work?

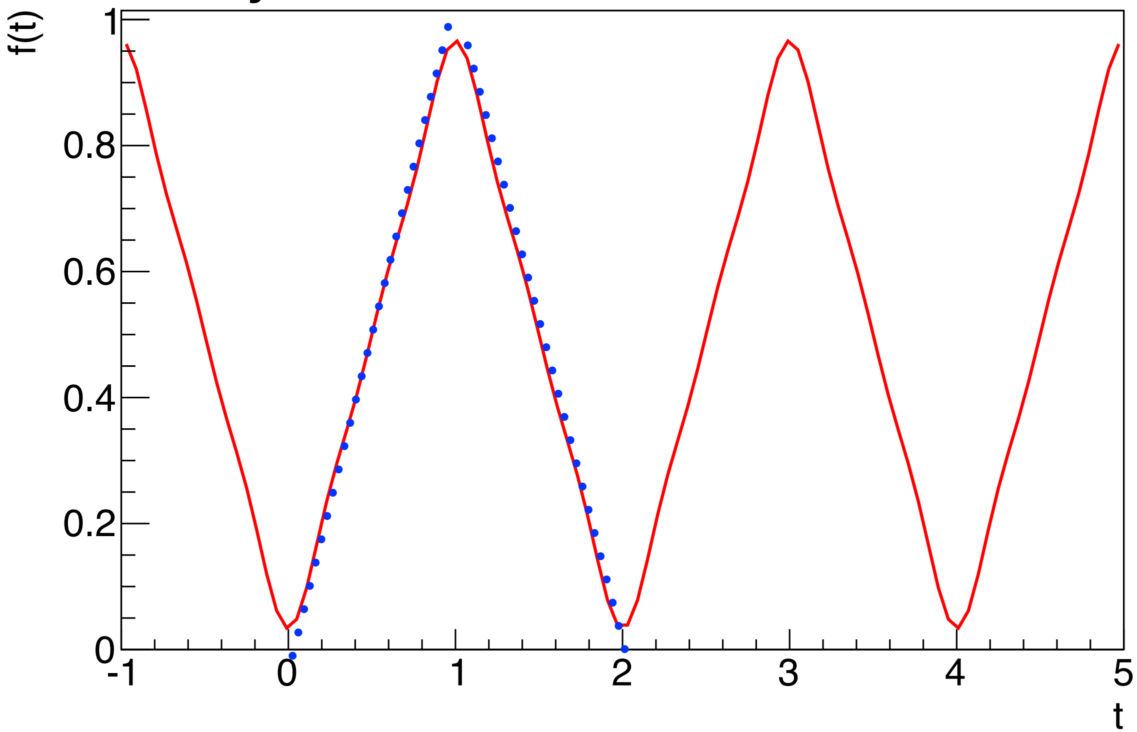
$$f(t) = f_{max}/2 + \sum_{n=1}^{\infty} [a_n \cos(n\pi t)]$$

$$a_n = \frac{-4f_{max}}{n^2\pi^2} \text{ for n odd}$$

$$a_n = 0 \text{ for n even}$$

Let's try this out!
Set fmax = 1 for simplicity

Only a0,a1,a3,a5 terms



With only 4 terms doing quite well

How well does it work?

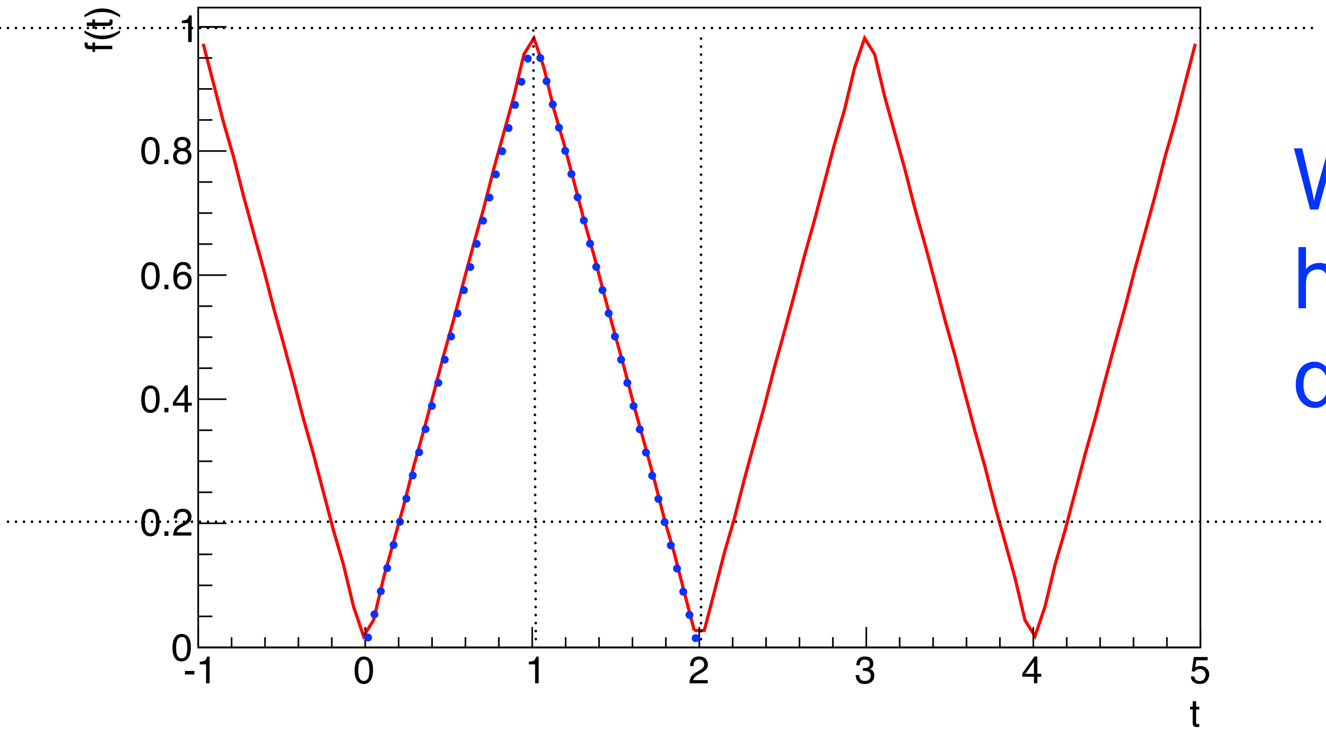
$$f(t) = f_{max}/2 + \sum_{n=1}^{\infty} [a_n \cos(n\pi t)]$$

$$a_n = \frac{-4 f_{max}}{n^2 \pi^2} \text{ for n odd}$$

$$a_n = 0 \text{ for n even}$$

Let's try this out! Set fmax = 1 for simplicity

Only a0,a1,a3,a5,a7,a9,a11 terms



With only handful terms doing quite well

How will we use this for driven oscillators?

We know now that any periodic function f(t) with period τ can be expressed as

$$\omega = 2\pi/\tau$$

$$f(x) = \sum_{n=0}^{\infty} [a_n \cos(n\omega x) + b_n \sin(n\omega x)]$$

And we want to solve: $m\ddot{x} + b\dot{x} + kx = f(x)$

Let's start by assuming we can break f(x) down into $f_1(x)$ and $f_2(x)$, and that we know the solutions x_1 and x_2 to:

$$m\ddot{x} + b\dot{x} + kx = f_1(x)$$

$$m\ddot{x} + b\dot{x} + kx = f_2(x)$$

How will we use this for driven oscillators?

$$m\ddot{x}_1 + b\dot{x}_1 + kx_1 = f_1(x)$$

$$m\ddot{x}_2 + b\dot{x}_2 + kx_2 = f_2(x)$$

$$x_3 = x_1 + x_2$$

$$kx_3 = kx_1 + kx_2$$

$$\dot{x}_3 = \dot{x}_1 + \dot{x}_2$$

$$\ddot{x}_3 = \ddot{x}_1 + \ddot{x}_2$$

$$m\ddot{x}_3 + b\dot{x}_3 + kx_3 = m(\ddot{x}_1 + \ddot{x}_2) + b(\dot{x}_1 + \dot{x}_2) + k(x_1 + x_2)$$

$$m\ddot{x}_3 + b\dot{x}_3 + kx_3 = m\ddot{x}_1 + b\dot{x}_1 + kx_1 + m\ddot{x}_2 + b\dot{x}_2 + kx_2$$

$$m\ddot{x}_3 + b\dot{x}_3 + kx_3 = f_1(x) + f_2(x) = f(x)$$

So if we can break up f(x) into pieces for which we know the solution, we can solve any periodic driven oscillator

How will we use this for driven oscillators?

$$\text{So if } f(x) = \sum_{n=0}^{\infty} f_n \cos(n\omega x)$$

$$x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta t)$$

$$A_n = \frac{f_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2 \omega^2}}$$

$$\tan \delta_n = \frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}$$

n^th Fourier coefficient

Let’s look at example 5.5 in Taylor together

And your homework, due in 1 week

5.2,5.10,5.11,5.26,5.43

Midterm, due in 1.5 weeks

Your midterm will be on this subject - let’s discuss it now