On to kinetic energy, potential energy and work

Kinetic energy

$$T = \frac{1}{2}mv^2$$

$$\dot{T} = \frac{1}{2}m\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{m}{2}(\dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}}) = m(\dot{\mathbf{v}}\cdot\mathbf{v})$$

$$\dot{T} = \mathbf{F}\cdot\mathbf{v} = \frac{dT}{dt}$$

$$dT = \mathbf{F}\cdot d\mathbf{r}$$

Work done by force F

$$\int_1^2 \mathbf{F}\cdot d\mathbf{r} = \text{ Work done by force along path from position 1 to 2}$$

This work = net change in KE!

Let's go over example 4.1
and then problems 4.4, 4.8, all together

Conservatives forces

If (and only if) a force **f**:

1. Depends only on the particle's position, and not on any other variables
2. Does the same work as a particle moves from $\mathbf{r}_1$ to $\mathbf{r}_2$ for all paths between $\mathbf{r}_1$ to $\mathbf{r}_2$
3. Then we call the force conservative, and can define a potential energy U(**r**) associated to the force, and
4. Total energy E = T+U conserved

$$U(\mathbf{r}) = -\mathbf{Work}(\mathbf{r_0} \to \mathbf{r}) = -\int_{\mathbf{r_0}}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathbf{dr}'$$

Must define a reference point for potential energy!

How do we know that F is independent of path?

F is independent of path if and only if

$$\nabla \times \mathbf{F} = 0$$

Take my word (and Taylor's word) for it, or read in Div Grad Curl and all that (or try to work out problem 4.25, to start)

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Advantages to having conservatives forces

- The ever-important conservation of total energy (T+U) applies
- Can include multiple potential energy terms, and the total $E=T+U_1+U_2+\ldots U_N$ is conserved
- The change in total energy is the work done by the non-conservative forces

Let's do example 4.3 in textbook together

Relating force and potential energy

$$U(\mathbf{r}) = -\mathbf{Work}(\mathbf{r_0} \rightarrow \mathbf{r}) = -\int_{\mathbf{r_0}}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathbf{dr}'$$

?

$$\text{derivative } U(\mathbf{r}) = \text{derivative } \; -\int_{\mathbf{r_0}}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathbf{dr}'$$

$$\text{derivative } U(\mathbf{r}) = -\mathbf{F}(\mathbf{r})$$

Intuitively, this makes sense, but force is a vector, and any type of energy is a scalar. How to relate the two?

Relating force and potential energy

$$dW = W(\mathbf{r} \to \mathbf{r} + d\mathbf{r}) = \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

$$dW = F_x dx + F_y dy + F_z dz$$

$$dW = -dU = -[U(\mathbf{r} + d\mathbf{r}) - U(\mathbf{r})]$$

$$dW = -[U(x + dx, y + dy, z + dz) - U(x, y, z)] =$$

$$dW = -\frac{\partial U}{\partial x}dx - \frac{\partial U}{\partial y}dy - \frac{\partial U}{\partial z}dz$$

$$F_x = -\frac{\partial U}{\partial x}$$

$$F_y = -\frac{\partial U}{\partial y}$$

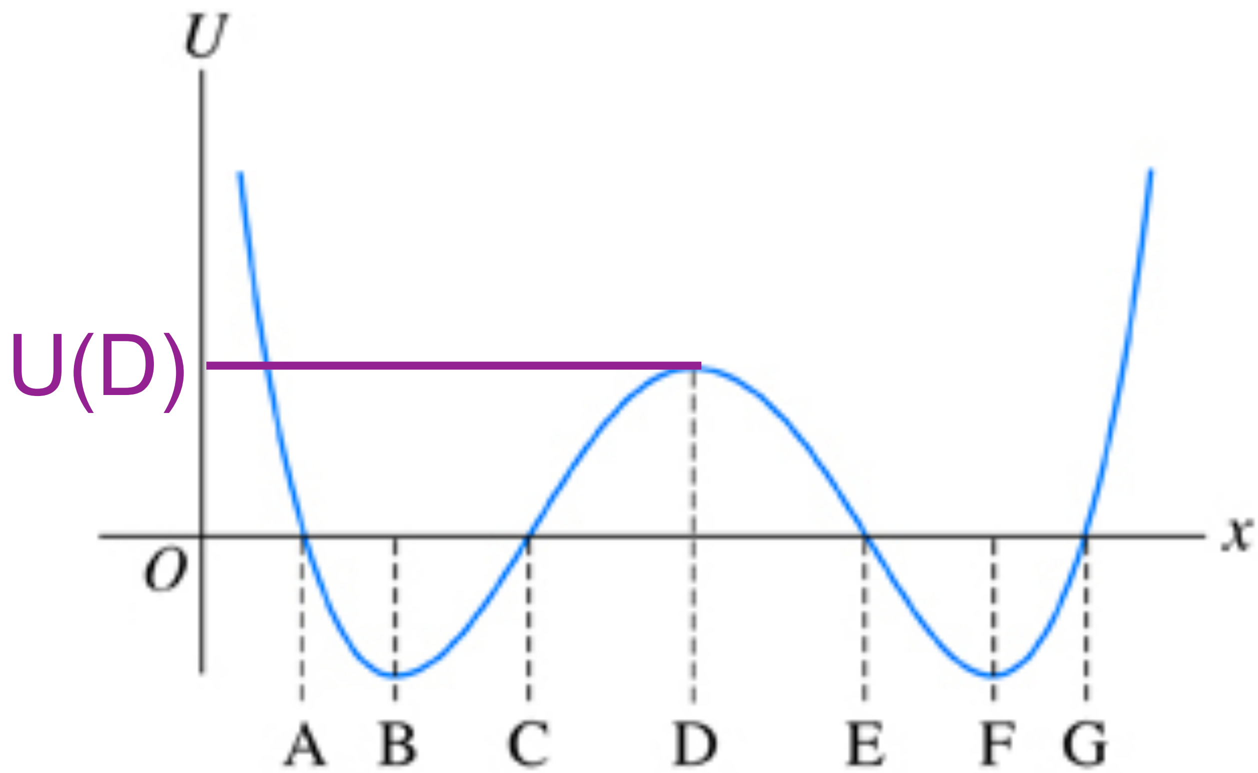
$$F_z = -\frac{\partial U}{\partial z}$$

If you haven't seen the gradient before, you will in E&M in the future. Very useful to know

$$\mathbf{F} = -(\hat{\mathbf{x}}\frac{\partial U}{\partial x} + \hat{\mathbf{y}}\frac{\partial U}{\partial y} + \hat{\mathbf{z}}\frac{\partial U}{\partial z}) = -\nabla U$$

Potential energy in one dimension

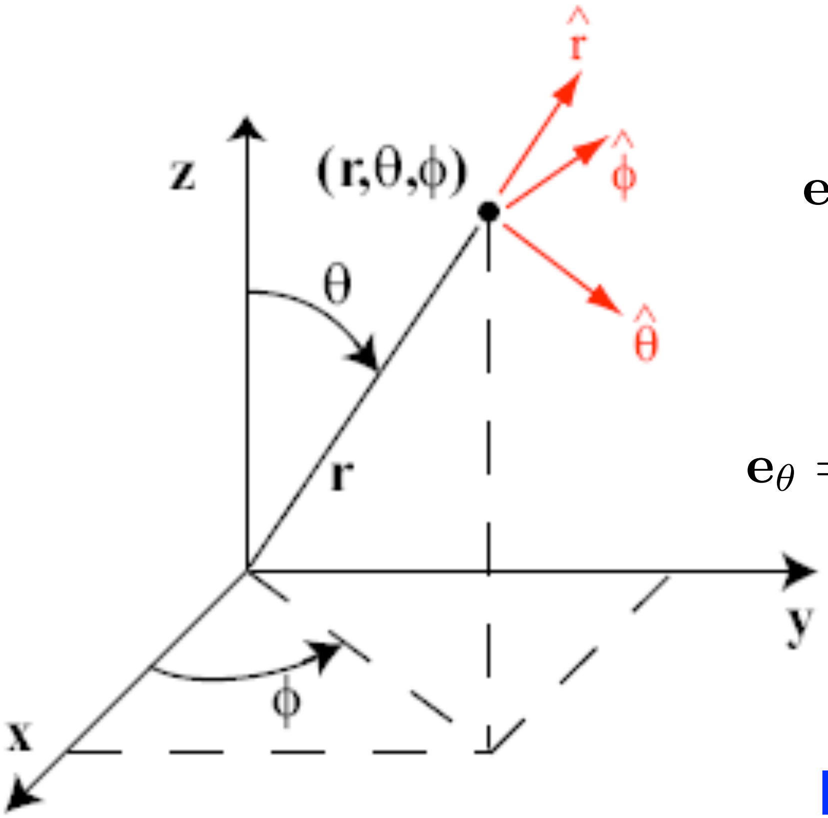
- $dU/dx = 0$ at x = B, D, F
  - x=B, F are stable points ($d^2U/dx^2 > 0$)
  - x=D is unstable ($d^2U/dx^2 < 0$)
- Particle at A or E experiences force to larger x
- Particle at C or G experience force to smaller x
- If total energy < U(D), particle can be stuck inside one of two wells



Examples

Let's go over Example 4.7 and problems 4.23 and 4.36 together, and then work on 4.13 in small groups or by yourself

Gradients in other coordinate systems



$$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$$

$$\mathbf{r} = r\hat{\mathbf{r}} + \phi\hat{\phi} + \theta\hat{\theta}$$

$$\mathbf{r} = r\sin\theta\cos\phi\hat{\mathbf{x}} + r\sin\theta\sin\phi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}}$$

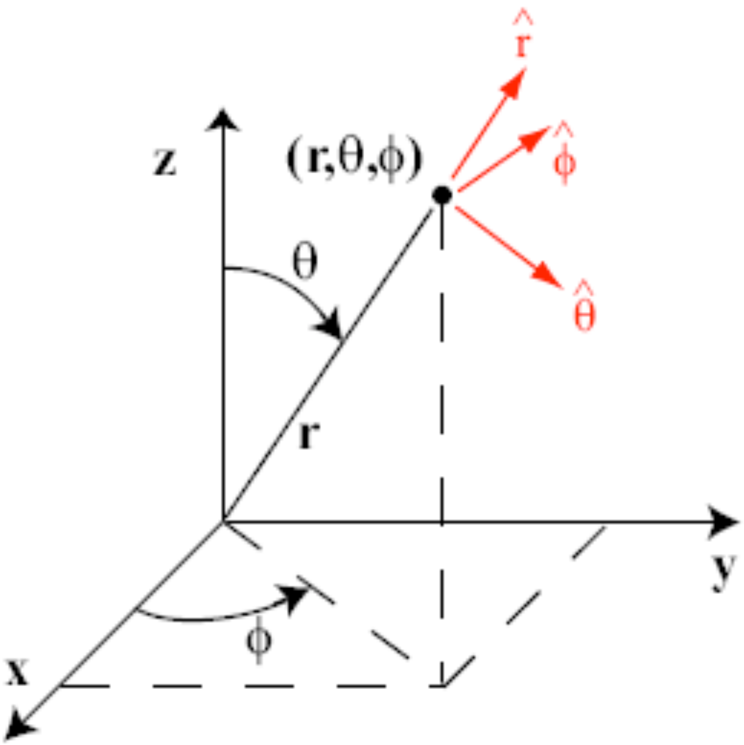
$$\mathbf{e}_r = \frac{\partial\mathbf{r}}{\partial r} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$\mathbf{e}_\phi = \frac{\partial\mathbf{r}}{\partial\phi} = -r\sin\theta\sin\phi\hat{\mathbf{x}} + r\sin\theta\cos\phi\hat{\mathbf{y}}$$

$$\mathbf{e}_\theta = \frac{\partial\mathbf{r}}{\partial\theta} = r\cos\theta\cos\phi\hat{\mathbf{x}} + r\cos\theta\sin\phi\hat{\mathbf{y}} - r\sin\theta\hat{\mathbf{z}}$$

3d polar coordinates are messy now... but often make you life a lot easier later on down the road. Try and follow this (Taylor skips over it)

3d polar coordinates



$$\mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial r} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$\mathbf{e}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = -r\sin\theta\sin\phi\hat{\mathbf{x}} + r\sin\theta\cos\phi\hat{\mathbf{y}}$$

$$\mathbf{e}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = r\cos\theta\cos\phi\hat{\mathbf{x}} + r\cos\theta\sin\phi\hat{\mathbf{y}} - r\sin\theta\hat{\mathbf{z}}$$

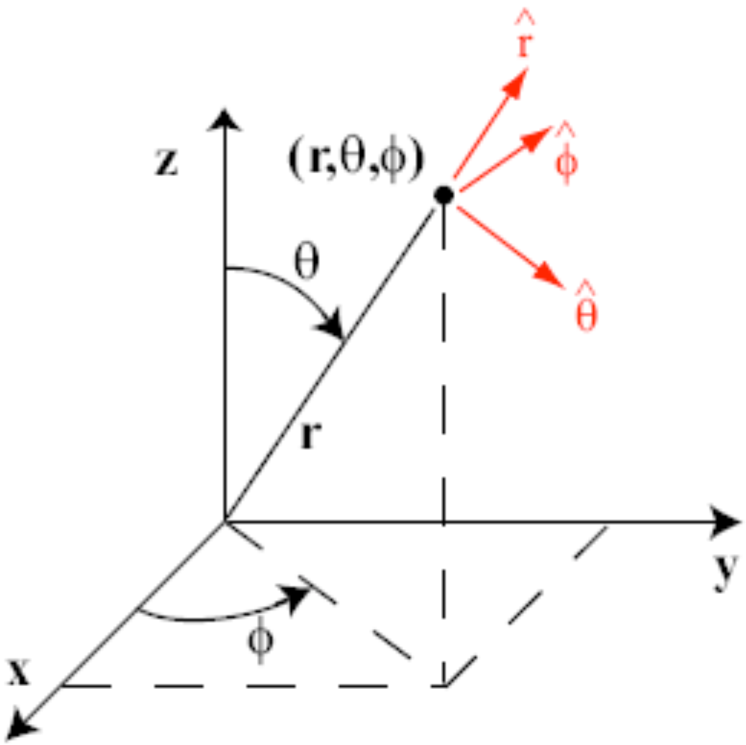
$$\hat{\mathbf{r}} = \hat{\mathbf{e}}_r = \frac{\mathbf{e}_r}{|\hat{\mathbf{e}}_r|} = \frac{\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}}$$

$$\hat{\phi} = \hat{\mathbf{e}}_\phi = \frac{\mathbf{e}_\phi}{|\hat{\mathbf{e}}_\phi|} = \frac{-r\sin\theta\sin\phi\hat{\mathbf{x}} + r\sin\theta\cos\phi\hat{\mathbf{y}}}{r\sqrt{\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi}}$$

$$\hat{\theta} = \frac{\mathbf{e}_\theta}{|\hat{\mathbf{e}}_\theta|} = \frac{r\cos\theta\cos\phi\hat{\mathbf{x}} + r\cos\theta\sin\phi\hat{\mathbf{y}} - r\sin\theta\hat{\mathbf{z}}}{r\sqrt{\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta}}$$

As before in 2d, unit vectors are not constant

3d polar coordinates



$$\hat{\mathbf{r}} = \frac{\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}}$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

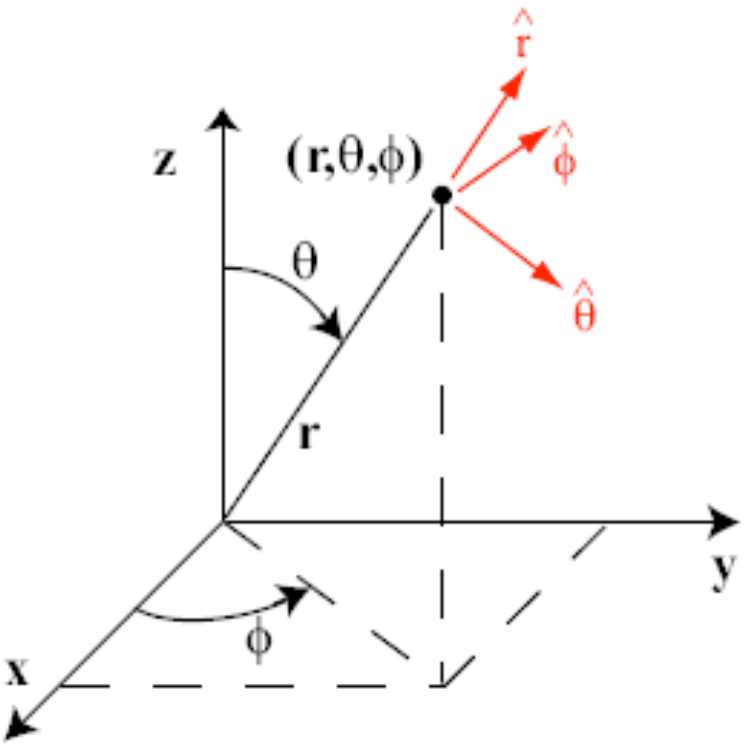
$$\hat{\phi} = \frac{-r\sin\theta\sin\phi\hat{\mathbf{x}} + r\sin\theta\cos\phi\hat{\mathbf{y}}}{r\sqrt{\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi}}$$

$$\hat{\phi} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$

$$\hat{\theta} = \frac{r\cos\theta\cos\phi\hat{\mathbf{x}} + r\cos\theta\sin\phi\hat{\mathbf{y}} - r\sin\theta\hat{\mathbf{z}}}{r\sqrt{\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta}}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$$

Unit vectors in 3d polar coordinates



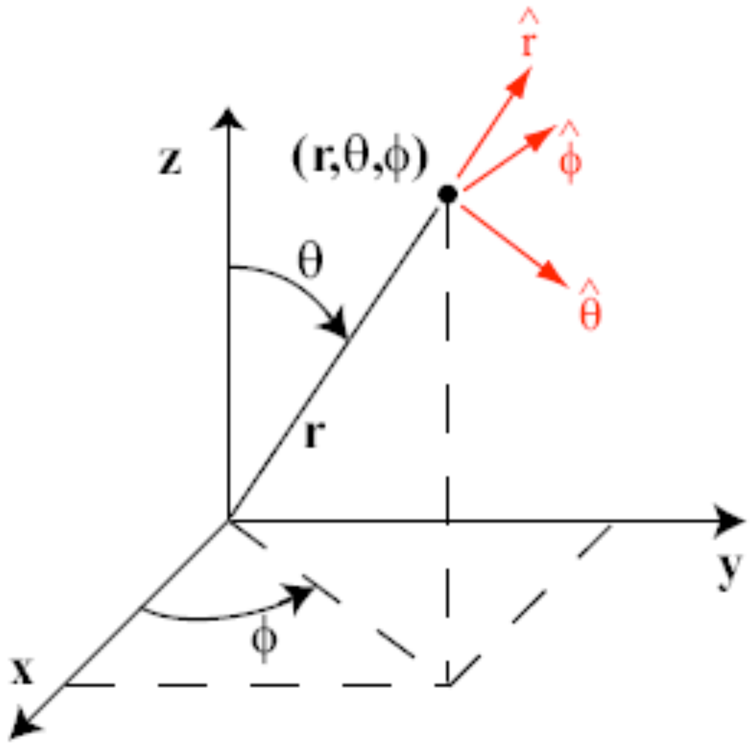
$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$

$$\hat{\phi} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$

Actually not so ugly in the end
- lots of terms have canceled

How does this help us find the gradient of a function f?



$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$\hat{\phi} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$$

$$df = [f(\mathbf{r} + d\mathbf{r}) - f(\mathbf{r})]$$

$$df = -\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right)$$

$$\nabla f = -\left(\frac{\partial U}{\partial x}\hat{\mathbf{x}} + \frac{\partial U}{\partial y}\hat{\mathbf{y}} + \frac{\partial U}{\partial z}\hat{\mathbf{z}}\right) \rightarrow$$

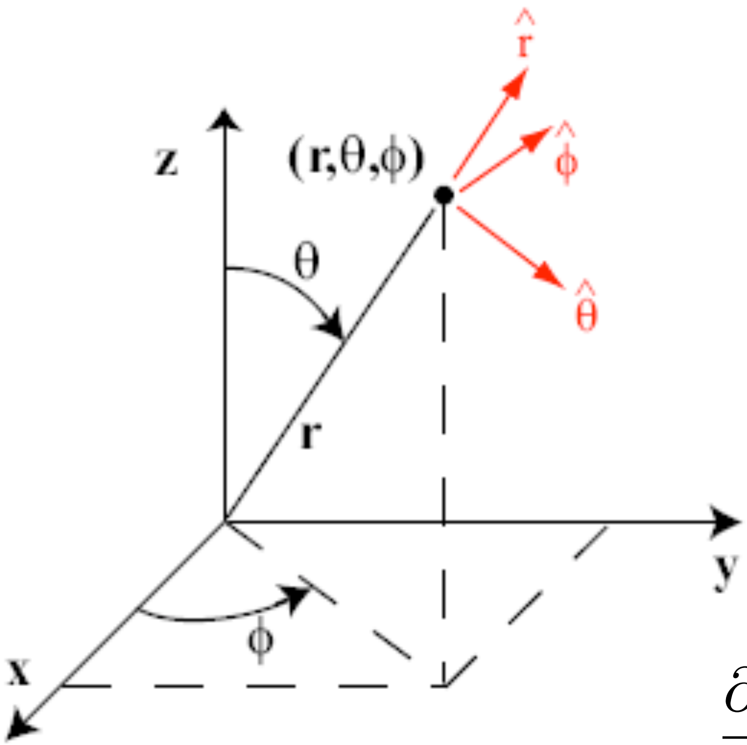
$$df = \nabla f \cdot d\mathbf{r}$$

But also ...

What is dr?

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi$$

How does this help us calculate the gradient?



$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$\hat{\phi} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$$

$$d\mathbf{r} = d(r\hat{\mathbf{r}}) = \hat{\mathbf{r}}dr + r\,d\hat{\mathbf{r}}$$

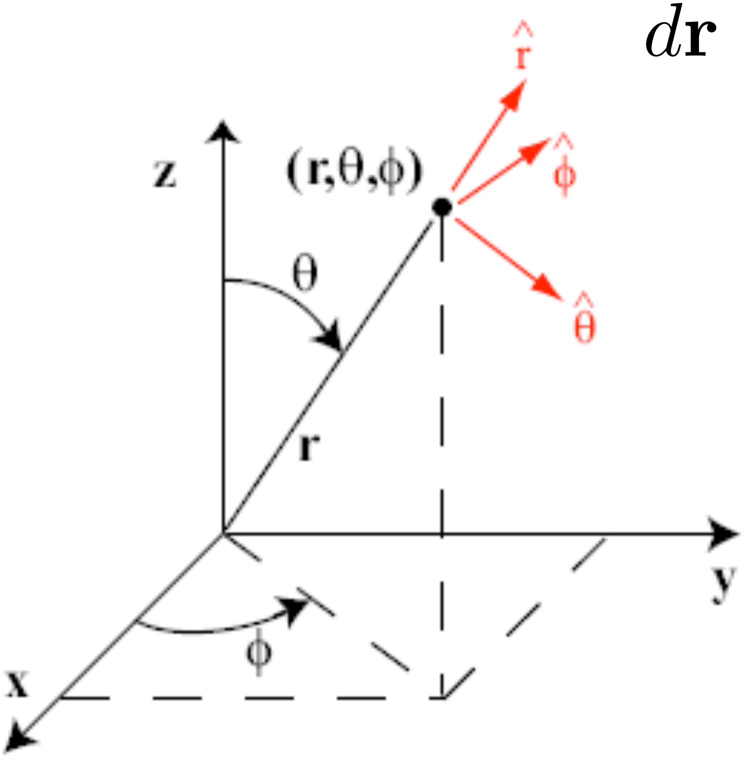
$$d\mathbf{r} = \hat{\mathbf{r}}dr + r\left(\frac{\partial\hat{\mathbf{r}}}{\partial r}dr + \frac{\partial\hat{\mathbf{r}}}{\partial\theta}d\theta + \frac{\partial\hat{\mathbf{r}}}{\partial\phi}d\phi\right)$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial r} = 0$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} = \hat{\theta}$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial\phi} = -\sin\theta\sin\phi\hat{\mathbf{x}} + \sin\theta\cos\phi\hat{\mathbf{y}} = \sin\theta\hat{\phi}$$

Almost there...



$$d\mathbf{r} = \hat{\mathbf{r}} dr + r \left(\frac{\partial \hat{\mathbf{r}}}{\partial r} dr + \frac{\partial \hat{\mathbf{r}}}{\partial \theta} d\theta + \frac{\partial \hat{\mathbf{r}}}{\partial \phi} d\phi \right)$$

$$d\mathbf{r} = \hat{\mathbf{r}} dr + r \left(\hat{\theta} d\theta + \sin\theta \hat{\phi} d\phi \right)$$

$$d\mathbf{r} = \hat{\mathbf{r}} dr + \hat{\theta} r d\theta + \hat{\phi} r \sin\theta d\phi$$

$$df = \nabla f \cdot d\mathbf{r}$$

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi$$

$$\nabla f \cdot d\mathbf{r} = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi$$

Equating our two definitions

$$d\mathbf{r} = \hat{\mathbf{r}}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$$

$$df = \nabla f \cdot d\mathbf{r}$$

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi = dr(\nabla f)_r + rd\theta(\nabla f)_\theta + r\sin\theta d\phi(\nabla f)_\phi$$

$$\boxed{\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}}$$

Hope you followed that. See Div Grad Curl and all That for more derivations. Wanted you to see it at least once

Gradient for Coulomb potential

$$U(r) = \frac{kqQ}{r} = \frac{C}{r}$$

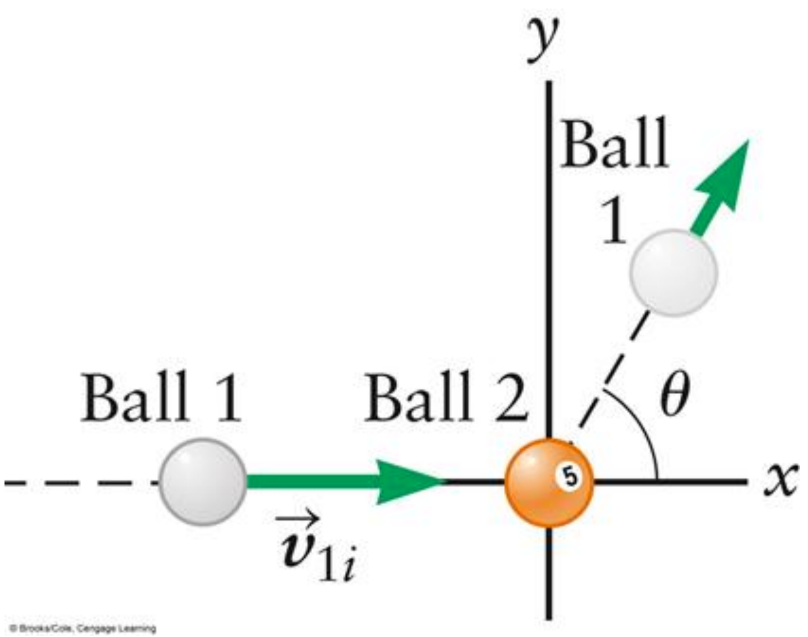
$$\nabla U = \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\phi}$$

$$\nabla U = \frac{\partial U}{\partial r}\hat{\mathbf{r}} = -\frac{C}{r^2}\hat{\mathbf{r}}$$

$$\mathbf{F}(\mathbf{r}) = -\nabla U = \frac{C}{r^2}\hat{\mathbf{r}}$$

Elastic collisions

- Elastic collision: collision via a conservative force with $U(|\mathbf{r}_1-\mathbf{r}_2|) \to 0$ (or constant) as $|\mathbf{r}_1-\mathbf{r}_2| \to \infty$
  - Conservative force implies conservation of energy, but since U is zero at large separation, initial and final kinetic energies are equal



Elastic collisions

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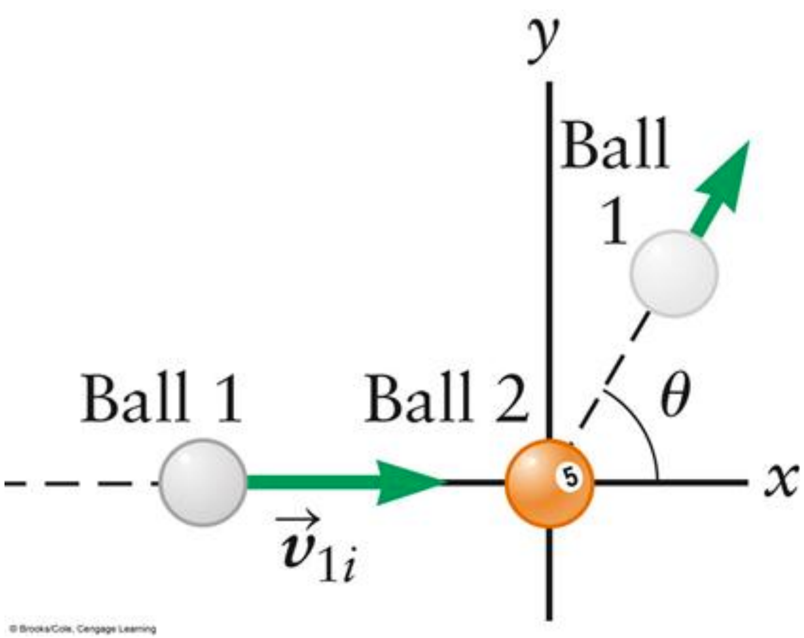


Let's work out Example 4.8 with $m_1 \neq m_2$ (ie problem 4.46)

Extension to large numbers of particles

- Make sure you read Taylor 4.9-4.10. Will not go over the derivations and just quote the (hopefully intuitive) results:
  - For N-particle system, can still define a potential U, and then the net force on particle $\alpha = -\nabla_\alpha U$, where ∇_α is the gradient with respect to the coordinates of particle α
  - Energy conservation still applies, as does the work-KE theorem

Let's work on Example 4.9 in book

Example 4.9 questions

- Why does normal force do no work?
- Why does friction do no work (hint, what does “without slipping” mean?)
- How do we calculate moment of inertia for a cylinder?

$$I = \int_V \rho r^2 dV$$

$$\rho = \mathrm{Mass/Volume} = M/(\pi R^2 Z)$$

$$dV = 2\pi r dr dz$$

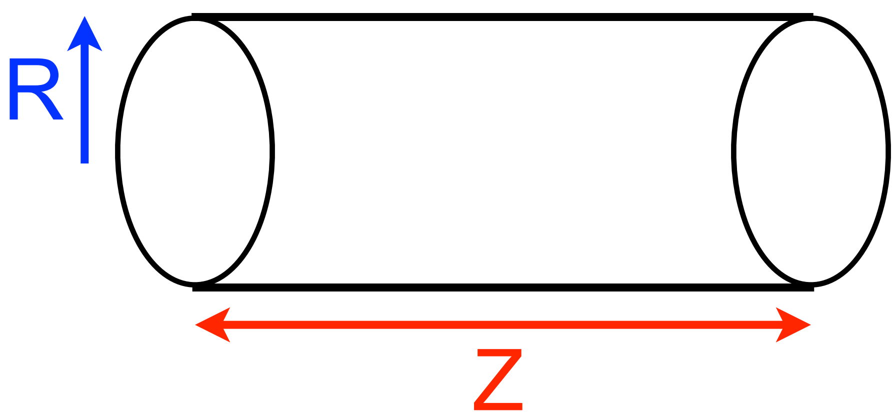
$$I = M/(\pi R^2 Z) \int_0^z dz \int_0^R 2\pi r r^2 dr$$

$$I = M/(\pi R^2) \int_0^R 2\pi r^3 dr$$

$$I = 2M/R^2 \int_0^R r^3 dr$$

$$I = \frac{2M}{4R^2}[r^4]_0^R$$

$$I = \frac{M}{2R^2}R^4 = \frac{MR^2}{2}$$



Homework (due as usual in 1 week)

Taylor 4.7, 4.12, 4.35, 4.41, 4.47, 4.48, 4.53