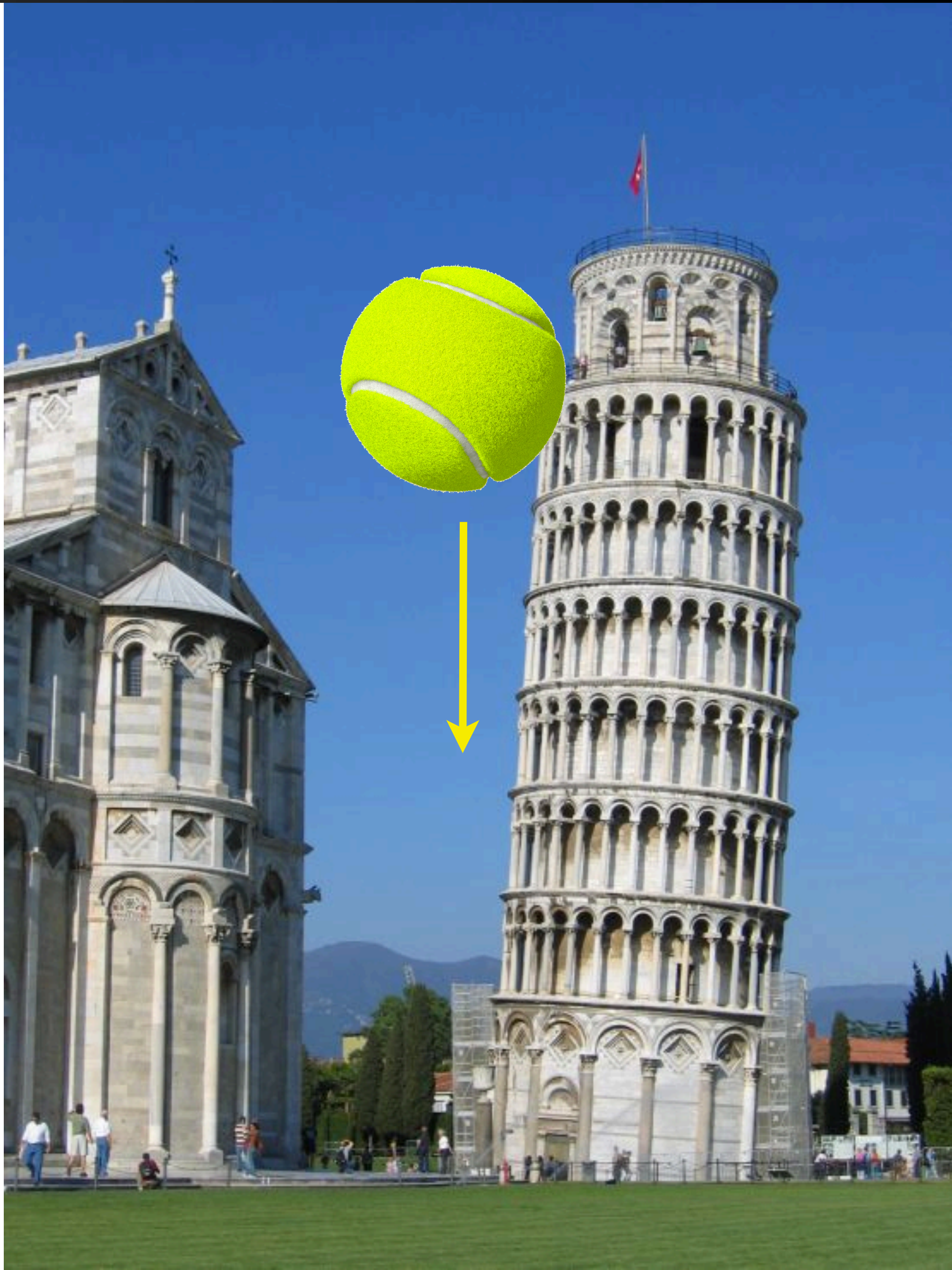
Back to our fun example

We know that this tennis ball will not accelerate forever - friction must slow down the acceleration!



Some things to think about re: air resistance

- Typically assume that air resistance (ie drag) is along the direction of the velocity vector (typically pointing away from it)
  - As mentioned in Taylor, not always true, but largely true, and simplifies equations

$$\mathbf{f}_{\text{drag}} = -f(v)\mathbf{v}$$

$$f(v) = bv + cv^2$$

Related to viscosity of medium (→ bv)

Related to density (inertia) of medium (→ cv^2)

Equations of motion for projectile with linear air resistance

$$m\ddot{\mathbf{r}} = m\mathbf{g} - b\mathbf{v} = m\dot{\mathbf{v}}$$

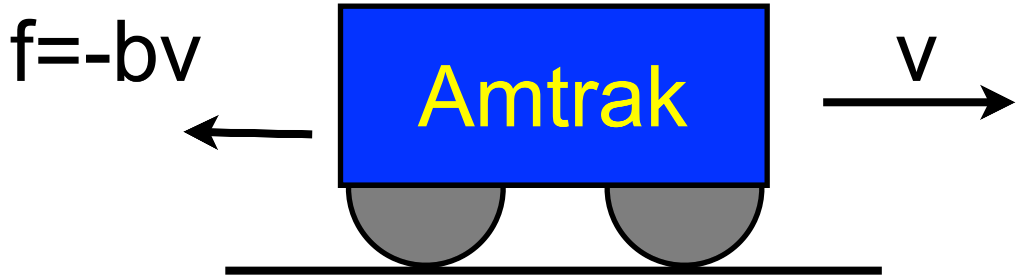
$$m\dot{v}_x = -bv_x$$

$$m\dot{v}_y = mg - bv_y$$

$$m\dot{v}_z = -bv_z$$

Starting off in easier dimension (no gravity or other F_{ext})

$$m\dot{v}_x = -bv_x$$



We will solve lots of differential equations in this course. Exponentials are a good guess for solutions when derivatives of a function are some constant times itself

$$\dot{v}_x = -kv_x, k = b/m$$

$$v_x(t) = Ae^{-kt}$$

$$v_x(0) = v_0 = Ae^0 = A$$

$$v_x(t) = v_0 e^{-kt} = v_0 e^{-t/\tau} (\tau = 1/k)$$

Starting off in easier dimension (no gravity or other F_{ext})

$$v_x(t) = v_0 e^{-kt} = v_0 e^{-t/\tau} = dx/dt$$

$$dx = v_0 e^{-t/\tau} dt$$

$$\int dx = \int v_0 e^{-t'/\tau} dt'$$

$$x - x_0 = v_0 \int_0^t e^{-t'/\tau} dt'$$

Always get a constant of integration

Starting off in easier dimension (no gravity or other F_{ext})

$$x - x_0 = v_0 \int_{t'=0}^{t'=t} e^{-t'/\tau} dt'$$

$$q = -t'/\tau, dq = -dt'/\tau \rightarrow dt' = -\tau dq$$

$$x - x_0 = -v_0\tau \int_{q=0}^{q=t/\tau} e^q dq$$

$$x - x_0 = -v_0\tau [e^q]_{q=0}^{q=-t/\tau}$$

$$x - x_0 = -v_0\tau [e^{-t/\tau} - 1]$$

$$x = x_0 + v_0\tau[1 - e^{-t/\tau}]$$

We will change lots of variables in integrations this semester :)

Looking at the solution some more

First of many
Taylor expansions

$$x = x_0 + v_0\tau[1 - e^{-t/\tau}]$$

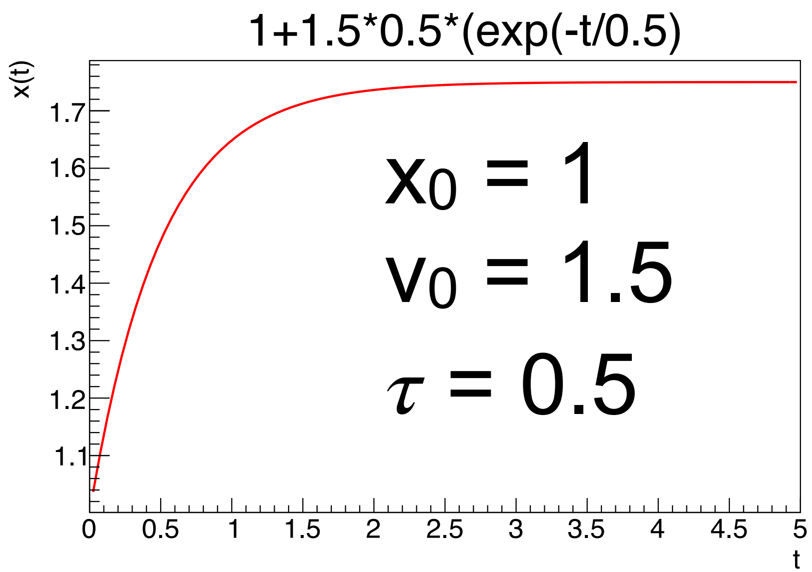
when $t \to \infty$, exponential small

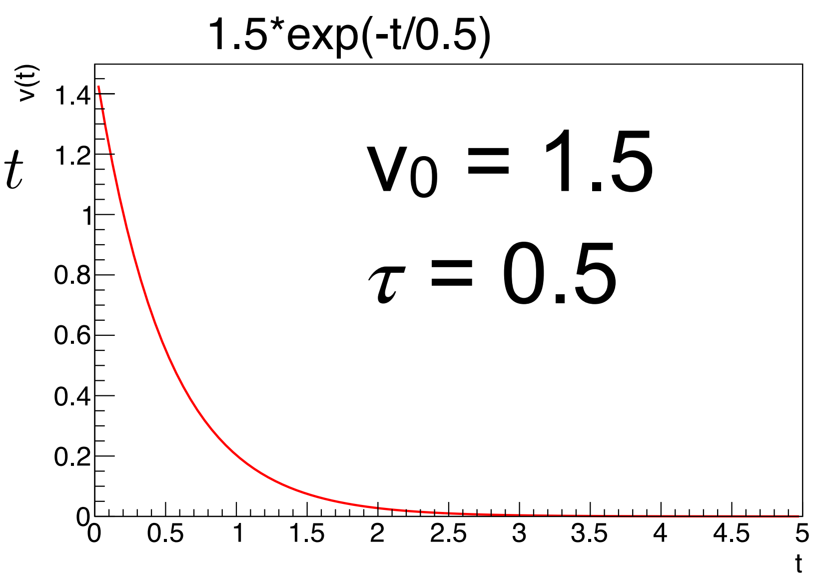
$$x \sim x_0 + v_0\tau$$

when drag small, $b \sim 0$, $\tau \to \infty$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$x \sim x_0 + v_0\tau[1 - (1 - t/\tau)] \sim x_0 + v_0 t$$





Good to observe expected behavior in limits!

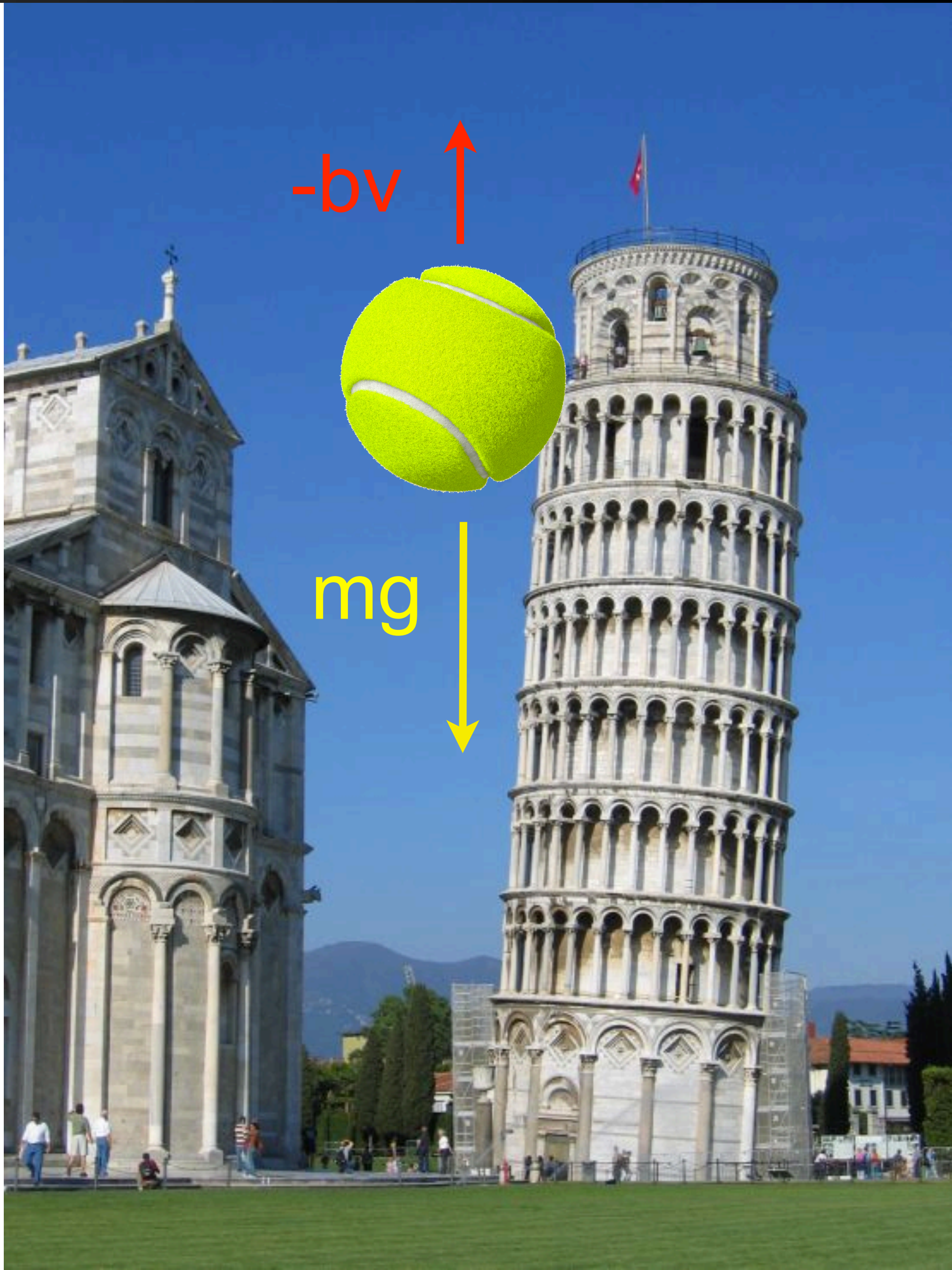
Now let's add gravity

$$m\dot{v} = mg - bv$$

$$\dot{v} = g - (b/m)v$$

$$\dot{v} = 0 \rightarrow v_{\text{terminal}} = \frac{mg}{b}$$

Pay attention to sign of v!
Gravity accelerates the ball until:
drag force = grav. force
At that point,
a terminal velocity is reached



Now let's add gravity

$$m\dot{v} = mg - bv$$

$$m\dot{v} = -b(v - v_{\text{ter}})$$

$$\text{let } u = (v - v_{\text{ter}}), \dot{u} = \dot{v} \text{ so}$$

$$m\dot{u} = -bu$$

We just solved this differential equation!

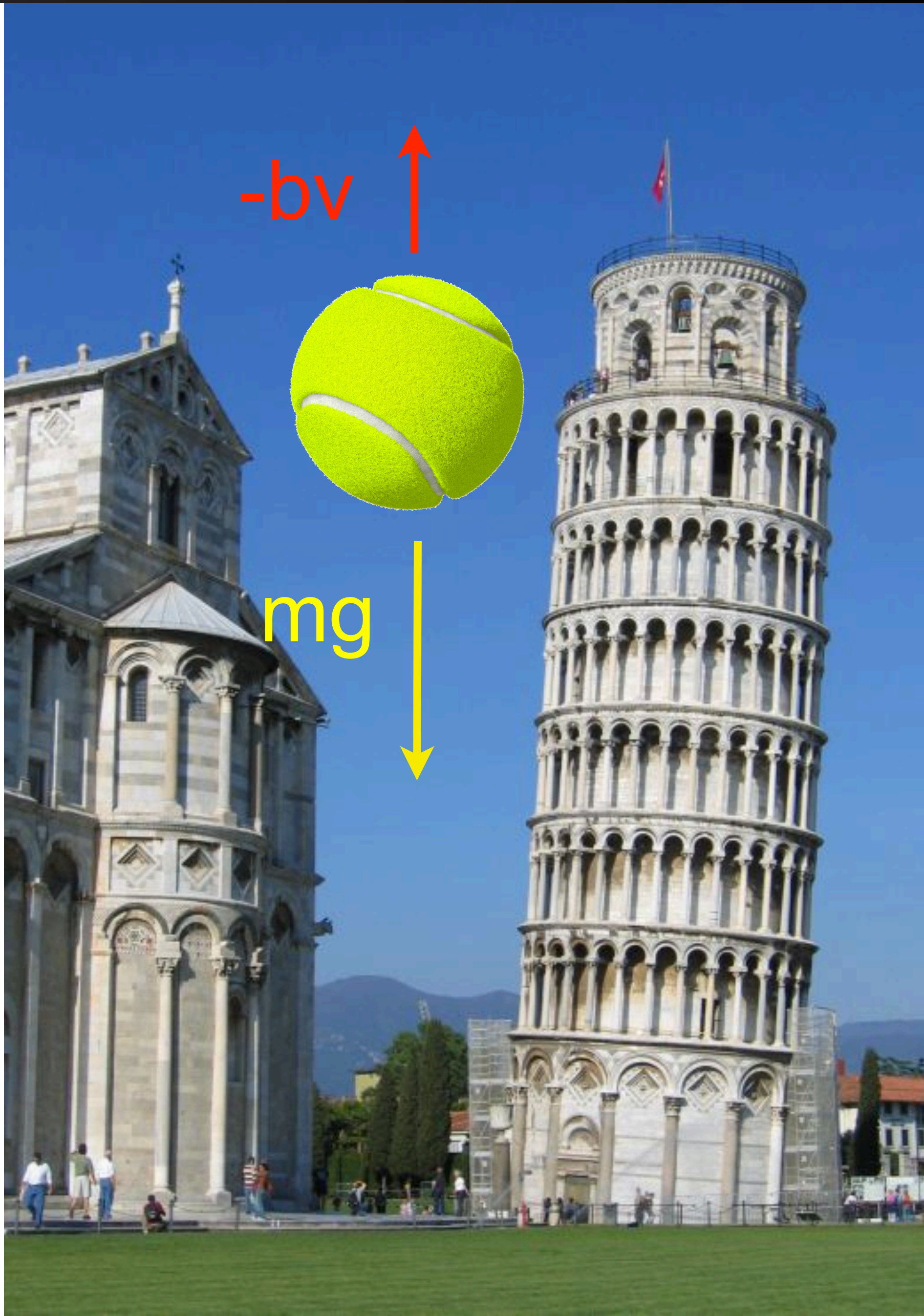
$$u(t) = u_0 e^{-t/\tau}$$

$$v(t) - v_{\text{term}} = Ae^{-t/\tau}$$

$$v(0) - v_{\text{term}} = Ae^0 = A \rightarrow A = v_0 - v_{\text{term}}$$

$$v(t) - v_{\text{term}} = (v_0 - v_{\text{term}})e^{-t/\tau}$$

$$v(t) = v_{\text{term}} + (v_0 - v_{\text{term}})e^{-t/\tau}$$



Plug and chug...

$$v(t) = dy/dt = v_{\mathrm{term}} + (v_0 - v_{\mathrm{term}})e^{-t/\tau}$$

$$\int_{y_0}^{y} dy' = \int_0^t v_{\mathrm{term}} + (v_0 - v_{\mathrm{term}})e^{-t'/\tau} dt'$$

$$y - y_0 = v_{\mathrm{term}}t + (v_0 - v_{\mathrm{term}}) \int_0^t e^{-t'/\tau} dt'$$

$$z = -t'/\tau, dz = -dt'/\tau, dt' = -\tau dz$$

$$y - y_0 = v_{\mathrm{term}}t + (v_0 - v_{\mathrm{term}})(-\tau) \int_0^{-t/\tau} e^z dz$$

$$y - y_0 = v_{\mathrm{term}}t + (v_0 - v_{\mathrm{term}})(-\tau)[e^z]_0^{-t/\tau}$$

$$y - y_0 = v_{\mathrm{term}}t + (v_0 - v_{\mathrm{term}})(-\tau)(e^{-t/\tau} - 1)$$

$$y = y_0 + v_{\mathrm{term}}t + (v_0 - v_{\mathrm{term}})\tau(1 - e^{-t/\tau})$$

Let's again consider what happens in extremes

$$v(t) = v_{\text{term}} + (v_0 - v_{\text{term}})e^{-t/\tau}$$

$$\text{when } t \to \infty, \text{ exponential small}$$

$$v(\infty) \sim v_{\text{term}}$$

Let's again consider what happens in extremes

$$y = y_0 + v_{\text{term}} t + (v_0 - v_{\text{term}})\tau(1 - e^{-t/\tau})$$

when drag small, $b \sim 0$, $\tau \to \infty$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$y = y_0 + v_{\text{term}} t + (v_0 - v_{\text{term}})\tau[1 - (1 - t/\tau + \frac{t^2}{2\tau^2})]$$

$$y = y_0 + v_{\text{term}} t + (v_0 - v_{\text{term}})\tau[t/\tau - \frac{t^2}{2\tau^2})]$$

$$y = y_0 + v_{\text{term}} t + (v_0 - v_{\text{term}})(t - \frac{t^2}{2\tau})$$

$$y = y_0 + v_0 t + \frac{t^2}{2\tau}(v_{\text{term}} - v_0)$$

for small drag, $v_0 \ll v_{\text{term}}$ and remember that $\tau = m/b, v_{\text{term}} = mg/b$

$$y = y_0 + v_0 t + \frac{bt^2}{2m}(mg/b) = y_0 + v_0 t + \frac{gt^2}{2}$$

What happens

If we throw the projectile with initial velocity downward greater than the terminal velocity? (Problem 2.5 in Taylor)

Projectiles

$$x(t) = v_0 \tau [1 - e^{-t/\tau}]$$

$$y(t) = v_{\rm term} t + (v_0 - v_{\rm term}) \tau (1 - e^{-t/\tau})$$

Let’s take a moment here to check units and see if that makes sense

Projectiles

$$x(t) = v_0\tau[1 - e^{-t/\tau}]$$

$$y(t) = v_{\text{term}}t + (v_0 - v_{\text{term}})\tau(1 - e^{-t/\tau})$$

As Taylor points out, for projectiles, it's perhaps better to flip the sign of v_t (is that obvious why?)

Projectiles

$$x(t) = v_{x0}\tau[1 - e^{-t/\tau}]$$

$$y(t) = v_{\text{term}}t + (v_{y0} - v_{\text{term}})\tau(1 - e^{-t/\tau})$$

$$y(x) = \frac{v_{y0} + v_{\text{term}}}{v_{x0}}x + v_{\text{term}}\tau\ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

Invert x(t) to solve for t(x) and then plug in, let's do these first steps together before some nasty algebra

As Taylor points out, for projectiles, it's perhaps better to flip the sign of v_t (is that obvious why?)

Projectiles

$$x(t) = v_{x0}\tau[1 - e^{-t/\tau}]$$

$$y(t) = v_{\text{term}}t + (v_{y0} - v_{\text{term}})\tau(1 - e^{-t/\tau})$$

$$y(x) = \frac{v_{y0} + v_{\text{term}}}{v_{x0}}x + v_{\text{term}}\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

What is the range (R) of a projectile? Find y(R) = 0. Are you ready? ...

Projectiles (lots of algebra here...)

$$y(x) = \frac{v_{y0} + v_{\text{term}}}{v_{x0}} x + v_{\text{term}} \tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right) = 0$$

$$\ln(1+z) = z - z^2/2 + z^3/3 - z^4/4 \to$$

$$\ln(1-z) = -z - z^2/2 - z^3/3 - \ldots = -(z + z^2/2 + z^3/3 + \ldots)$$

$$\frac{v_{y0} + v_{\text{term}}}{v_{x0}} R - v_{\text{term}}\tau \left[\frac{R}{v_{x0}\tau} + \frac{1}{2}\left(\frac{R}{v_{x0}\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{x0}\tau}\right)^3 + \ldots\right] = 0$$

$$\frac{v_{y0} + v_{\text{term}}}{v_{x0}} - v_{\text{term}}\tau \left[\frac{1}{v_{x0}\tau} + \frac{1}{2}\frac{R}{v_{x0}^2\tau^2} + \frac{1}{3}\frac{R^2}{v_{x0}^3\tau^3} + \ldots\right] = 0$$

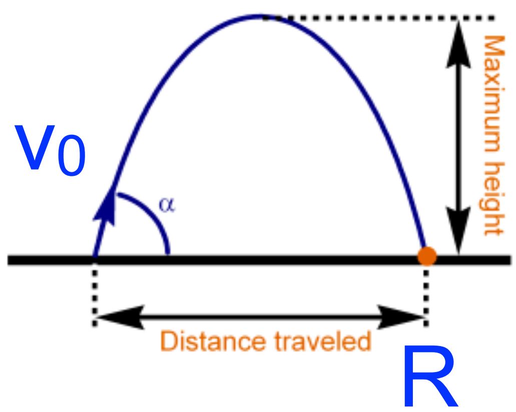
$$\frac{v_{y0} + v_{\text{term}}}{v_{x0}} - \frac{v_{\text{term}}\tau}{v_{x0}\tau} - v_{\text{term}}\tau \left[\frac{1}{2}\frac{R}{v_{x0}^2\tau^2} + \frac{1}{3}\frac{R^2}{v_{x0}^3\tau^3} + \ldots\right] = 0$$

$$\frac{v_{y0}}{v_{x0}} - v_{\text{term}}\tau \left[\frac{1}{2}\frac{R}{v_{x0}^2\tau^2} + \frac{1}{3}\frac{R^2}{v_{x0}^3\tau^3} + \ldots\right] = 0$$

$$\frac{v_{y0}}{v_{\text{term}}\tau v_{x0}} = \frac{1}{2}\frac{R}{v_{x0}^2\tau^2} + \frac{1}{3}\frac{R^2}{v_{x0}^3\tau^3}$$

$$R = \frac{3v_{x0}\tau}{2}\left[\frac{-1}{2} \pm \sqrt{\frac{1}{4} + \frac{4v_{y0}}{3v_{\text{term}}}}\right]$$

$$R = \frac{3v_{x0}\tau}{2}\left[\frac{-1}{2} + \sqrt{\frac{1}{4} + \frac{4v_{y0}}{3v_{\text{term}}}}\right]$$



Solve for R = range

R=0 (trivial)

Solve quadratic equation

Ignore negative R solution

Follow Taylor, too

Projectiles (finishing the algebra)

$$R = \frac{3v_{x0}\tau}{2}\left[\frac{-1}{2} + \sqrt{\frac{1}{4} + \frac{4v_{y0}}{3v_{\text{term}}}}\right]$$

Now assume drag force is small, so $v_{y0} \ll v_{\text{term}} \cdots$

$$R = \frac{3v_{x0}\tau}{2}\left[\frac{-1}{2} + \sqrt{\frac{1}{4}\left(1 + \frac{16v_{y0}}{3v_{\text{term}}}\right)}\right]$$

$$R = \frac{3v_{x0}\tau}{2}\left[\frac{-1}{2} + \frac{1}{2}\sqrt{1 + \frac{16v_{y0}}{3v_{\text{term}}}}\right]$$

$$(1+z)^n \sim 1 + nz + \frac{n(n-1)}{2}z^2 \text{ for small z}$$

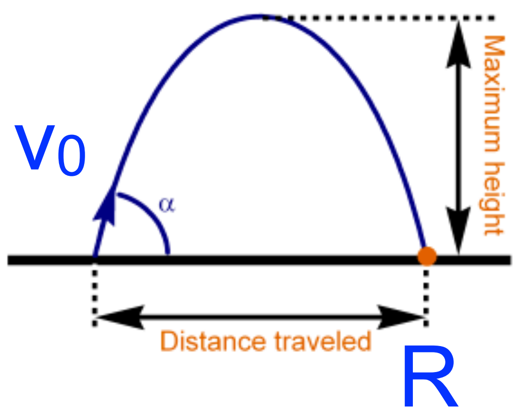
$$R = \frac{3v_{x0}\tau}{2}\left[\frac{-1}{2} + \frac{1}{2}\left(1 + \frac{8v_{y0}}{3v_{\text{term}}} - \frac{1}{8}\frac{16^2 v_{y0}^2}{3^2 v_{\text{term}}^2}\right)\right]$$

$$R = \frac{3v_{x0}\tau}{2}\left(\frac{4v_{y0}}{3v_{\text{term}}} - \frac{16v_{y0}^2}{9v_{\text{term}}^2}\right)$$

Remember that $\tau = m/b$ and $v_{\text{term}} = mg/b$

$$R = \frac{2v_{x0}v_{y0}}{g} - \frac{8v_{x0}v_{y0}^2}{3gv_{\text{term}}}$$

$$R = \frac{2v_{x0}v_{y0}}{g}\left(1 - \frac{4v_{y0}}{3v_{\text{term}}}\right)$$



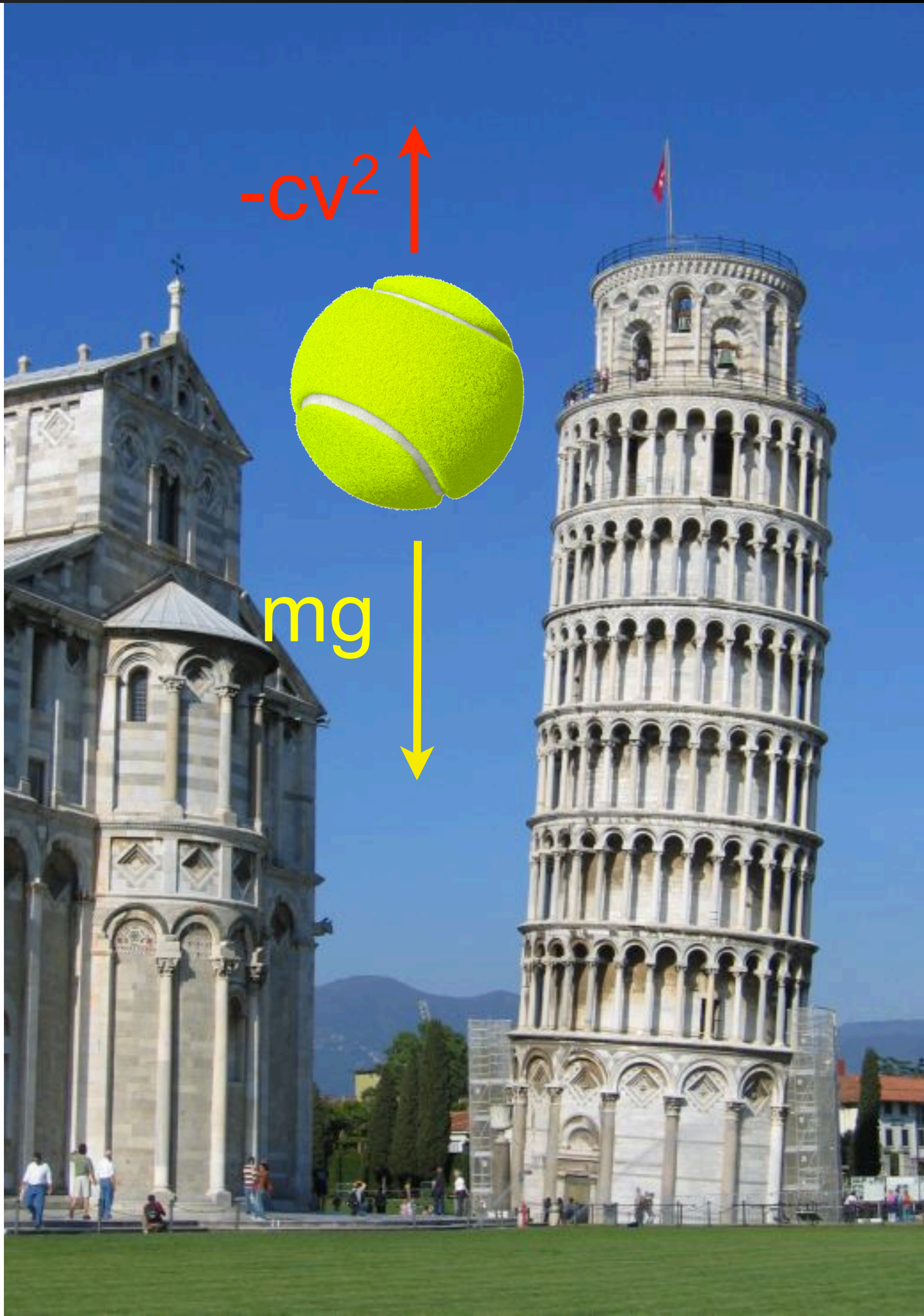
How does this compare with no-drag answer?

Phew

Quadratic air resistance / drag

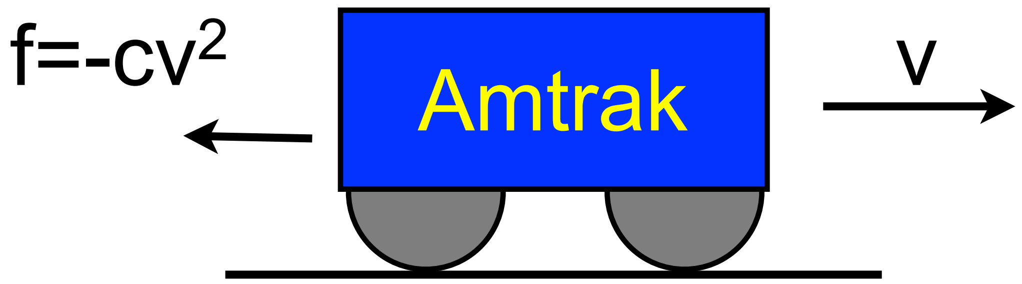
For larger objects, quadratic drag/air resistance is more the norm than linear version (which is easier to solve)

Make sure to read Taylor 2.1 discussion of this...



Starting off in easier dimension (no gravity or other F_{ext})

$$m\dot{v}_x = -cv_x^2$$



$$m\frac{dv}{dt} = -cv^2$$

$$m\frac{dv}{v^2} = -cdt$$

$$\int_{v_0}^{v} m\frac{dv'}{v'^2} = \int_0^t -cdt'$$

$$m\int_{v_0}^{v} \frac{dv'}{v'^2} = -ct$$

$$-m[1/v']_{v_0}^{v} = -ct$$

$$m(\frac{1}{v} - \frac{1}{v_0}) = ct$$

Separation of variables is another trick we will use many times this course

Finishing off the algebra

$$m\dot{v}_x = -cv_x^2$$

$$m\left(\frac{1}{v} - \frac{1}{v_0}\right) = ct$$

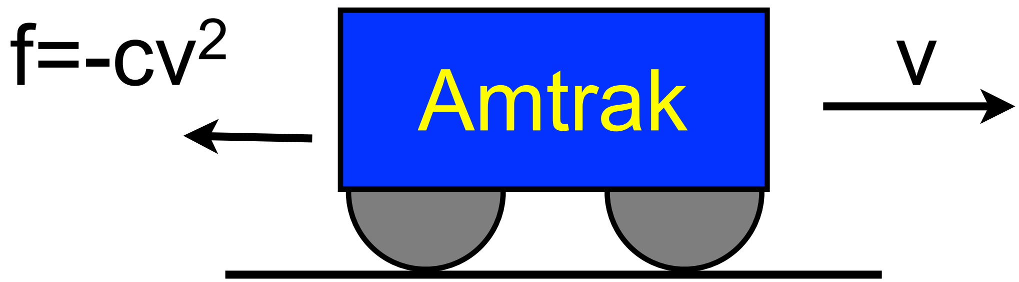
$$\frac{1}{v} = \frac{ct}{m} + \frac{1}{v_0}$$

$$\frac{1}{v} = \frac{m + v_0 ct}{mv_0}$$

$$v(t) = \frac{mv_0}{m + v_0 ct}$$

$$v(t) = \frac{v_0}{1 + v_0 ct/m}$$

$$v(t) = \frac{v_0}{1 + t/\tau}, \tau = \frac{m}{cv_0}$$



Finishing off the algebra

$$v(t) = dx/dt = \frac{v_0}{1 + t/\tau}$$

$$dx = dt \frac{v_0}{1 + t/\tau}$$

$$\int_{x_0}^{x} dx' = \int_0^t dt' \frac{v_0}{1 + t'/\tau}$$

$$x - x_0 = v_0 \int_0^t \frac{dt'}{1 + t'/\tau}$$

$$z = 1 + t'/\tau, dt' = \tau dz$$

$$x - x_0 = \tau v_0 \int_1^{1+t/\tau} \frac{dz}{z}$$

$$x - x_0 = \tau v_0 [\log z]_1^{1+t/\tau}$$

$$x = x_0 + \tau v_0 \log(1 + t/\tau) - \log(1) = \tau v_0 \log(1 + t/\tau)$$

$$x = x_0 + v_0 \tau \log(1 + t/\tau)$$

To ponder... what does this approach as drag force gets small?

Now back to this

$$m\dot{v} = mg - cv^2$$

$$\dot{v} = g - (c/m)v^2$$

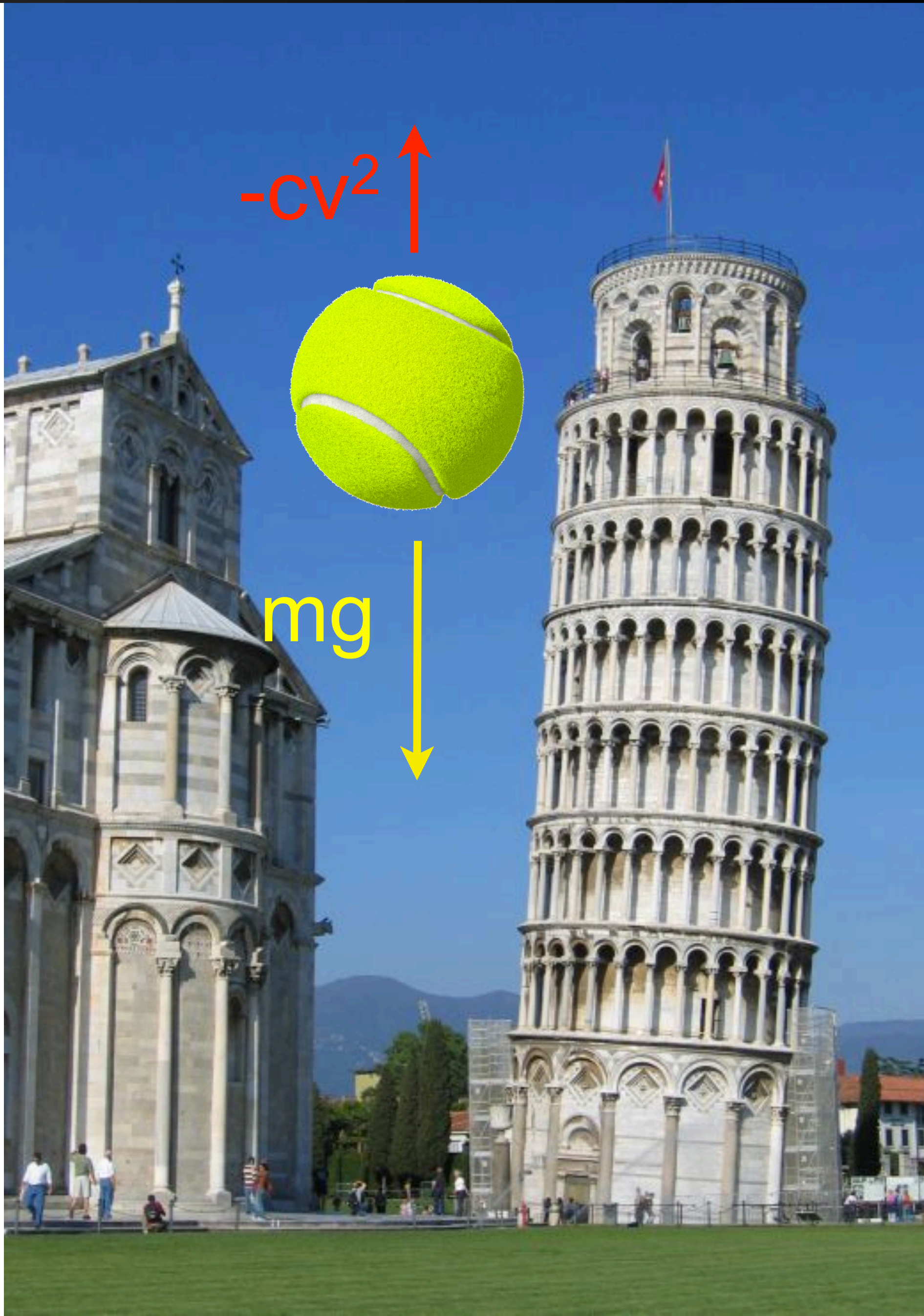
$$\dot{v} = 0 \rightarrow v_{\text{ter}} = \sqrt{\frac{mg}{c}}$$

$$\dot{v} = g[1 - (\frac{v}{v_{\text{ter}}})^2]$$

$$\frac{dv}{dt} = g[1 - (\frac{v}{v_{\text{ter}}})^2]$$

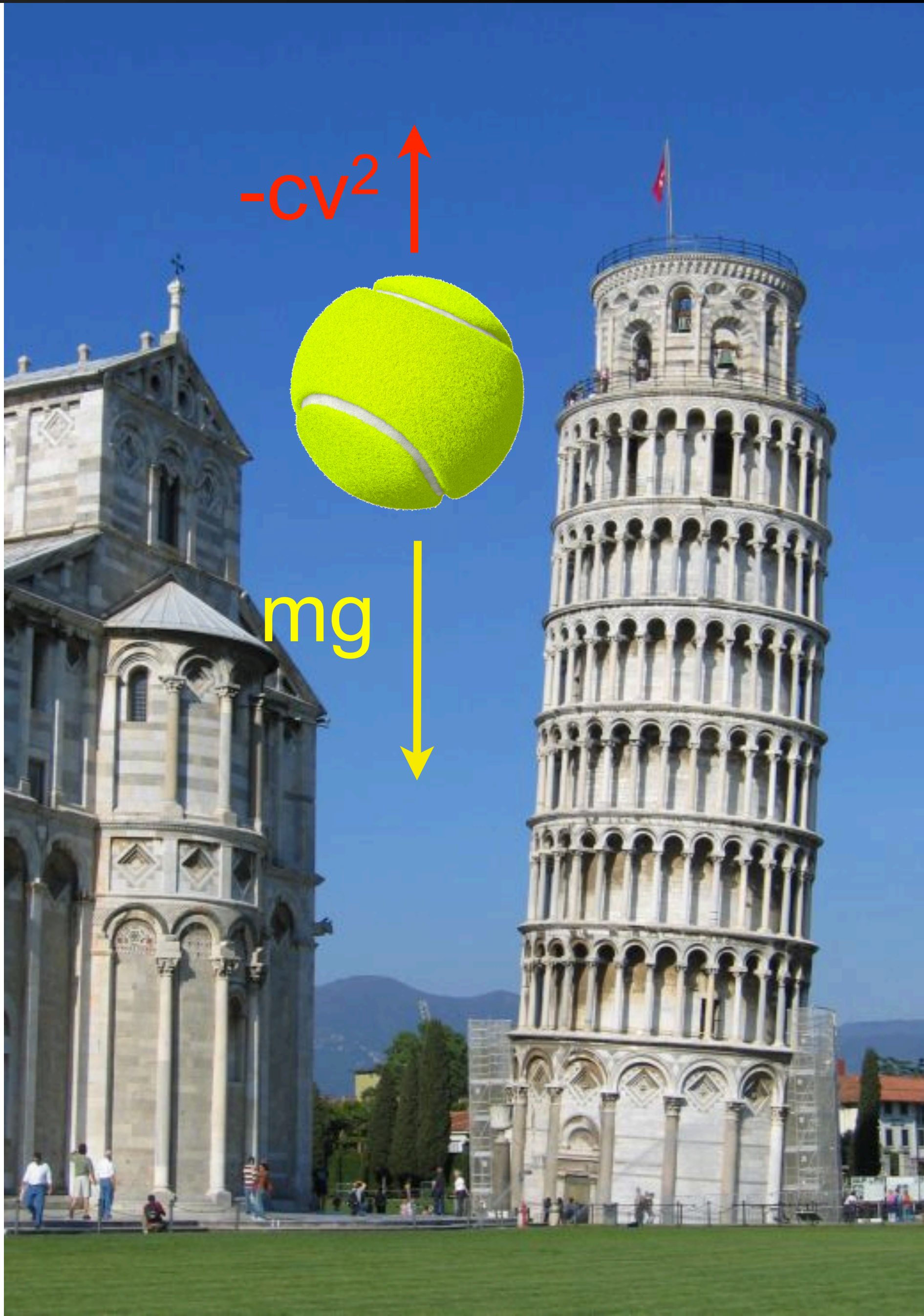
$$\frac{dv}{[1 - (\frac{v}{v_{\text{ter}}})^2]} = gdt$$

$$\int_{v_0}^{v} \frac{dv'}{[1 - (\frac{v'}{v_{\text{ter}}})^2]} = \int_0^t gdt'$$



Prepared again?

Just want to sketch results here - we won't go
over them in detail now (doesn't teach you much)



The easy version of the math

$$\int_{v_0}^{v} \frac{dv'}{[1 - (\frac{v'}{v_{\text{ter}}})^2]} = \int_0^t g dt'$$

Taylor's cover gives the answer to that integral = arctanh. Or....

The integral is some function, arctanh of the velocity = time, so need to invert it to get velocity as a function of time

The less easy version of the math

$$\int_{v_0}^{v} \frac{dv'}{[1-(\frac{v'}{v_{\rm ter}})^2]} = \int_0^t g dt'$$

$$\frac{1}{2}\int_{v_0}^{v} \frac{dv'}{1+v'/v_t} + \frac{dv'}{1-v'/v_t} = gt$$

$$z = 1 + v'/v_t, dv' = v_t dz$$

$$q = 1 - v'/v_t, dv' = -v_t dq$$

$$\frac{v_t}{2}\left[\int_{1+v_0/v_t}^{1+v/v_t} \frac{dz}{z} - \int_{1-v_0/v_t}^{1-v/v_t} \frac{dq}{q}\right] = gt$$

$$\frac{v_t}{2}\left[[\ln(z)]_{1+v_0/v_t}^{1+v/v_t} - [\ln(q)]_{1-v_0/v_t}^{1+v/v_t}\right] = gt$$

$$\frac{v_t}{2}[\ln(1+v/v_t) - \ln(1+v_0/v_t) + \ln(1-v_0/v_t) - \ln(1-v/v_t)] = gt$$

$$\ln\left(\frac{1+v/v_t}{1-v/v_t}\right) + \ln\left(\frac{1-v_0/v_t}{1+v_0/v_t}\right) = 2gt/v_t$$

$$\ln\left(\frac{1+v/v_t}{1-v/v_t}\right) = 2gt/v_t + \ln\left(\frac{1+v_0/v_t}{1-v_0/v_t}\right)$$

$$(1+v/v_t)/(1-v/v_t) = (1+v_0/v_t)/(1-v_0/v_t)e^{2gt/v_t}$$

$$\frac{2}{1-v/v_t} - 1 = \left(\frac{2}{1-v_0/v_t} - 1\right) e^{2gt/v_t}$$

$$v = v_t \frac{\frac{1+v_0/v_t}{1-v_0/v_t} e^{2gt/v_t} - 1}{\frac{1+v_0/v_t}{1-v_0/v_t} e^{2gt/v_t} + 1}$$

$$v = v_t \left[\frac{v_0(e^{2gt/v_t}+1) + v_t(e^{2gt/v_t}-1)}{v_0(e^{2gt/v_t}-1) + v_t(e^{2gt/v_t}+1)}\right]$$

Note that answer in Taylor starts with $v_0 = 0$ (in which case answer simplifies quite a bit!) Recommend you make sure when this is posted that you follow all of this

What about the position?

$$v = v_t \left[\frac{v_0(e^{2gt/v_t} + 1) + v_t(e^{2gt/v_t} - 1)}{v_0(e^{2gt/v_t} - 1) + v_t(e^{2gt/v_t} + 1)} \right]$$

Let's assume that $v_0 = 0$ (makes life a lot easier)

$$\frac{dy}{dt} = v(t) = \frac{v_t(e^{2gt/v_t} - 1)}{(e^{2gt/v_t} + 1)} = v_t \tanh(gt/v_t)$$

$$\int dy = \int v_t \tanh(gt'/v_t)dt'$$

$$y = \frac{v_t^2}{g} \ln \left[\cosh\left(\frac{gt}{v_t} \right) \right]$$

After changing variables, one of those integrals you look up (online, in mathematical physics book... or Taylor cover)

Aside on hyperbolic functions

Taken shamelessly from Wikipedia, these are useful to know (derivatives and integrals are in the front cover of your favorite book)

The hyperbolic functions are:

- Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}}$$

- Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}}$$

- Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

And now

Work on problem 2.8 in small groups or on your own (and then we'll go over it together)

Let's go over ...

Example 2.5 and then
Problem 2.23

How about projectiles with quadratic drag?

$$m\ddot{\mathbf{r}} = m\mathbf{g} - cv^2\hat{\mathbf{v}}$$

Newton's second law. And drag is always along velocity vector (in opposite direction)

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{v}$$

Definition of unit vector

$$m\ddot{\mathbf{r}} = m\mathbf{g} - cv\mathbf{v}$$

$$v = \sqrt{v_x^2 + v_y^2}$$

$$m\dot{\mathbf{v}} = m\mathbf{g} - c\mathbf{v}\sqrt{v_x^2 + v_y^2}$$

$$m\dot{v_x} = -cv_x\sqrt{v_x^2 + v_y^2}$$

$$m\dot{v_y} = -mg - cv_y\sqrt{v_x^2 + v_y^2}$$

Note the coupled equations. This cannot be solved analytically!

How about projectiles with quadratic drag?

We'll learn how to use computers to solve such problems in your HW assignment (which we'll open now) based on Taylor 2.43