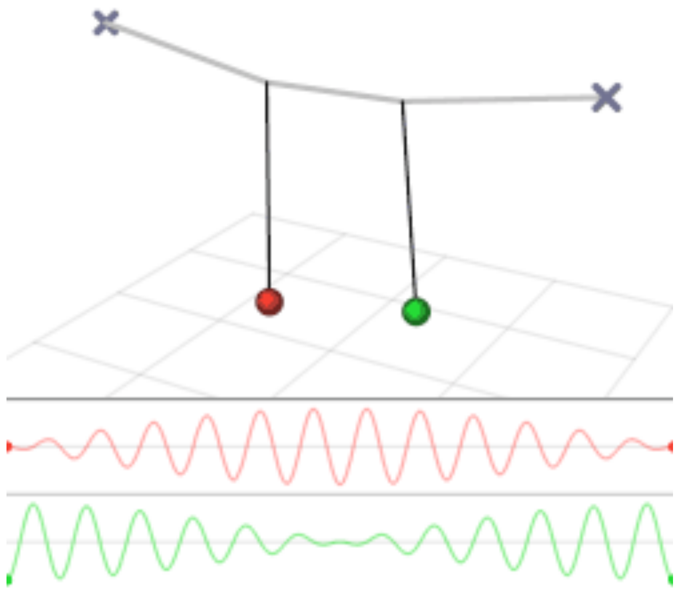
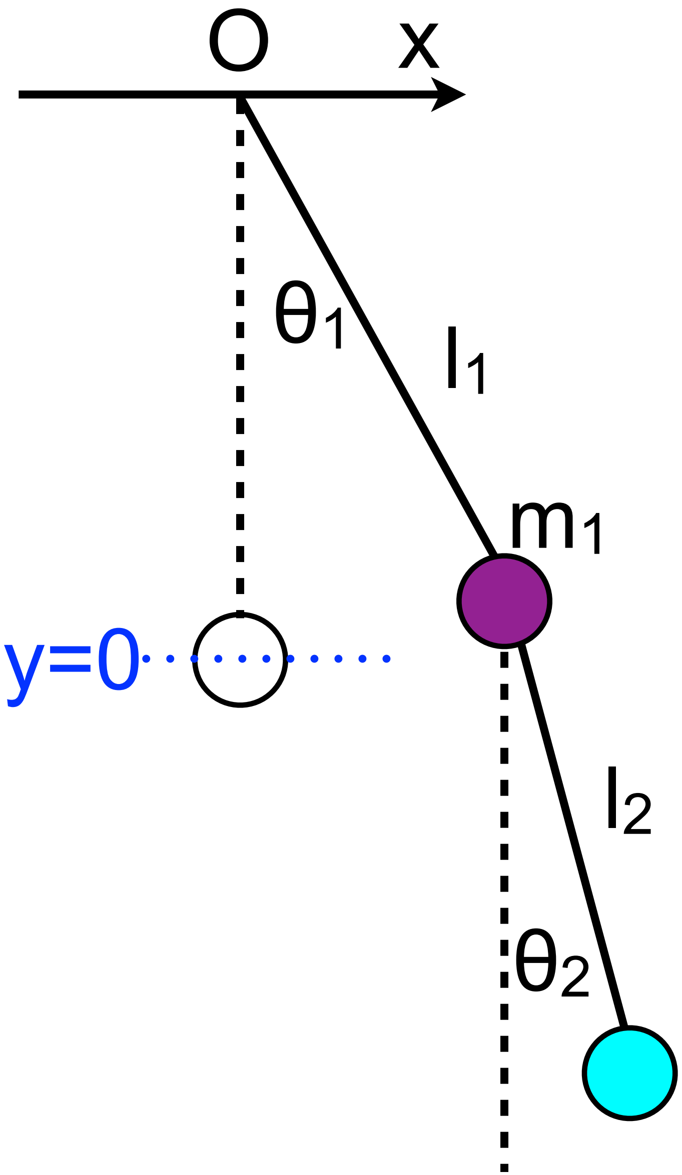
On to another subject

What happens if we return to oscillations, but put multiple oscillators in the system, and we couple/connect them together)

In other words, what if the behavior of one oscillator affects the behavior of another?



Recall this example...



$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}$$

$$-m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin\theta_1 =$$

$$\frac{d}{dt}\left[m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right]$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}$$

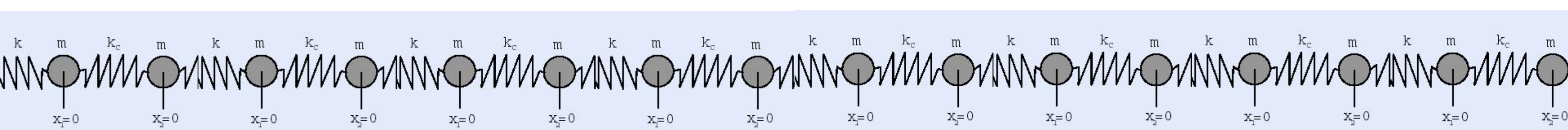
$$m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2 =$$

$$\frac{d}{dt}\left[m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)\right]$$

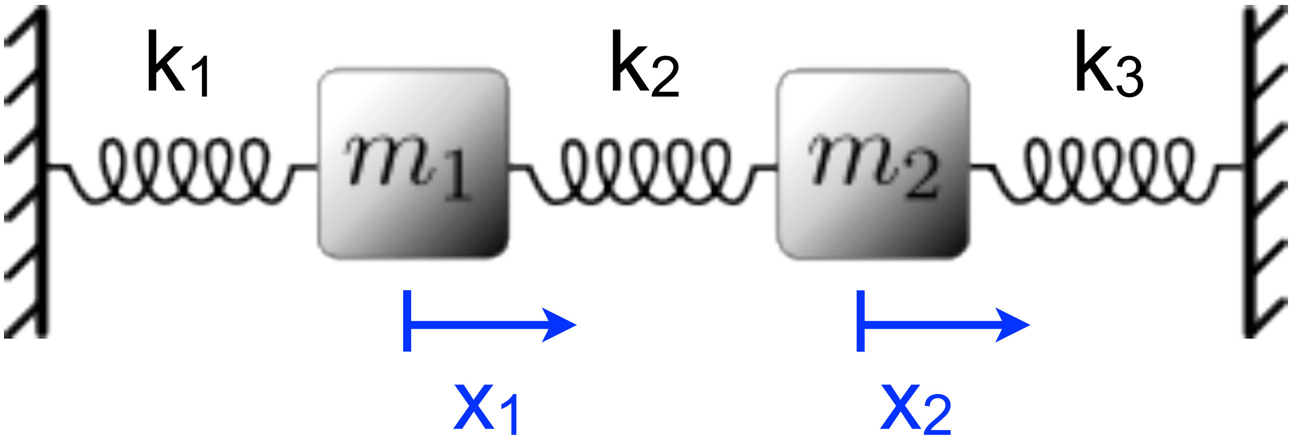
What we'll deal with

We solved systems with 1 oscillator, and will work on systems with 2 or 3 oscillators. If you go into certain areas of physics (such as condensed matter), you might have to deal with N objects in a coupled system, where N is very large (almost infinite!)

So, not this:



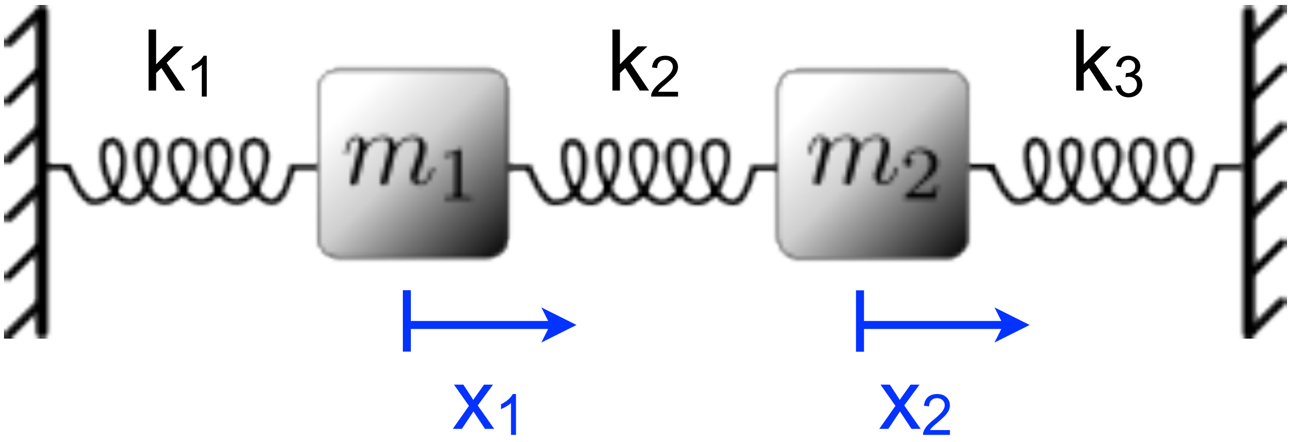
Start with a simple coupled case



If $k_2 = 0$ then springs are not coupled to each other, but with k_2 can see that if m_1 moves to the right, then this compresses k_2 and will push m_2 to the right (for example)

Start with a simple coupled case

x_i defined from equilibrium position



Force on $m_1 = -k_1x_1 - k_2(x_1 - x_2) = m_1a_1$

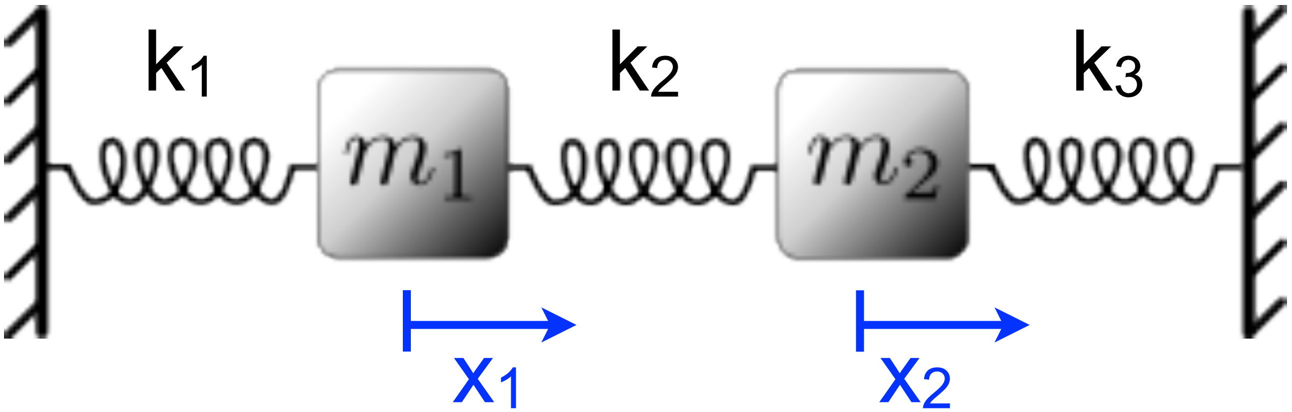
Check that this is in the right direction!

Force on $m_2 = -k_2(x_2 - x_1) - k_3(x_2) = m_2a_2$

Can again check the + and - signs

Rewriting

x_i defined from equilibrium position



$$m_1 \ddot{x}_1 = -(k_1 + k_2)x_1 + k_2 x_2$$

$$m_2 \ddot{x}_2 = -(k_2 + k_3)x_2 + k_2 x_1$$

Let's work out this matrix to make sure we see that it contains the same info

Matrix form of Hooke's Law

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

Guessing at the solution (as always)

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$z_1(t) = x_1(t) + iy_1(t) = a_1 e^{i\omega t}, a_1 = \alpha_1 e^{-i\delta_1}$$

$$z_2(t) = x_2(t) + iy_2(t) = a_2 e^{i\omega t}, a_2 = \alpha_2 e^{-i\delta_2}$$

$$\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} e^{i\omega t} = \mathbf{a} e^{i\omega t}$$

We are of course interested in **x**(t) = Re(**z**(t))

Plugging in our guess

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

$$\mathbf{x}(t) = \text{Re } \mathbf{z}(t) = \text{Re} \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} e^{i\omega t} = \text{Re } \mathbf{a}e^{i\omega t}$$

$$-\omega^2 \mathbf{M}\mathbf{a}e^{i\omega t} = -\mathbf{K}\mathbf{a}e^{i\omega t}$$

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$$

This again looks a lot like the equations we got one for a single oscillator. Can have **a** = 0, which is a trivial solution of no motion. Or else...

Solving such matrix equations (generically)

$$\mathbf{Q} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{Q} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow ax_1 + bx_2 = 0, cx_1 + dx_2 = 0$$

$$x_2 = -(c/d)x_1$$

$$ax_1 + b(-c/d)x_1 = 0$$

$$x_1 = 0 \text{ or } ad - bc = 0$$

$$\det\, Q = |Q| \text{ means } ad - bc$$

And for 3 dimensions? Gets a big uglier...

$$\mathbf{Q} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\mathbf{Q} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow ax_1 + bx_2 + cx_3 = 0,$$

$$dx_1 + ex_2 + fx_3 = 0$$

$$gx_1 + hx_2 + ix_3 = 0$$

$$(dc/f)x_1 + (ec/f)x_2 + cx_3 = 0$$

$$(di/f)x_1 + (ei/f)x_2 + ix_3 = 0$$

$$(a - dc/f)x_1 + (b - ec/f)x_2 = 0$$

$$(g - di/f)x_1 + (h - ei/f)x_2 = 0$$

$$(af - dc)x_1 + (bf - ec)x_2 = 0$$

$$(gf - di)x_1 + (hf - ei)x_2 = 0$$

$$x_1 = x_2(ec - bf)/(af - dc)$$

$$x_2(gf - di)(ec - bf)/(af - dc) + (hf - ei)x_2 = 0$$

$$(gf - di)(ec - bf) + (hf - ei)(af - dc) = 0$$

$$a(hf^2 - eif) + b(dif - gf^2) + c(gef - die) + d(eic - hfc) = 0$$

$$a(hf^2 - eif) + b(dif - gf^2) + c(gef - hfd) = 0$$

$$a(hf - ei) + b(di - gf) + c(ge - hd) = 0$$

$$a(ei - fh) - b(di - fh) + c(dh - ge) = 0$$

$$|\mathbf{Q}| = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Back to our guess

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

$$\mathbf{x}(t) = \text{Re } \mathbf{z}(t) = \text{Re} \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} e^{i\omega t} = \text{Re } \mathbf{a}e^{i\omega t}$$

$$-\omega^2 \mathbf{M}\mathbf{a}e^{i\omega t} = -\mathbf{K}\mathbf{a}e^{i\omega t}$$

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{a} = 0$$

Eigenvalue equation

$$|\mathbf{K} - \omega^2 \mathbf{M}| = 0$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

Let's simplify things

$$|\mathbf{K} - \omega^2 \mathbf{M}| = 0$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

As Taylor does, assume $m_1 = m_2$, $k_1 = k_2 = k_3$

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

Let’s simplify things

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

$$|\mathbf{K} - \omega^2 \mathbf{M}| = \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = 0$$

$$(2k - m\omega^2)^2 - k^2 = 0$$

$$m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$$

$$\omega^2 = \frac{4km \pm \sqrt{16k^2m^2 - 12m^2k^2}}{2m^2}$$

$$\omega^2 = (k/m)(2 \pm 1)$$

Normal Frequencies

Two purely sinuisoidal frequencies possible. One is standard sqrt(k/m), other is sqrt(3k/m)

But recall...

$$\mathbf{x}(t) = \text{Re}\ \mathbf{z}(t) = \text{Re}\begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} e^{i\omega t} = \text{Re}\ \mathbf{a}e^{i\omega t}$$

Let's plug in our first normal frequency, ω=sqrt(k/m), and find the first normal mode

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a}e^{i\omega t} = 0$$

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \rightarrow$$

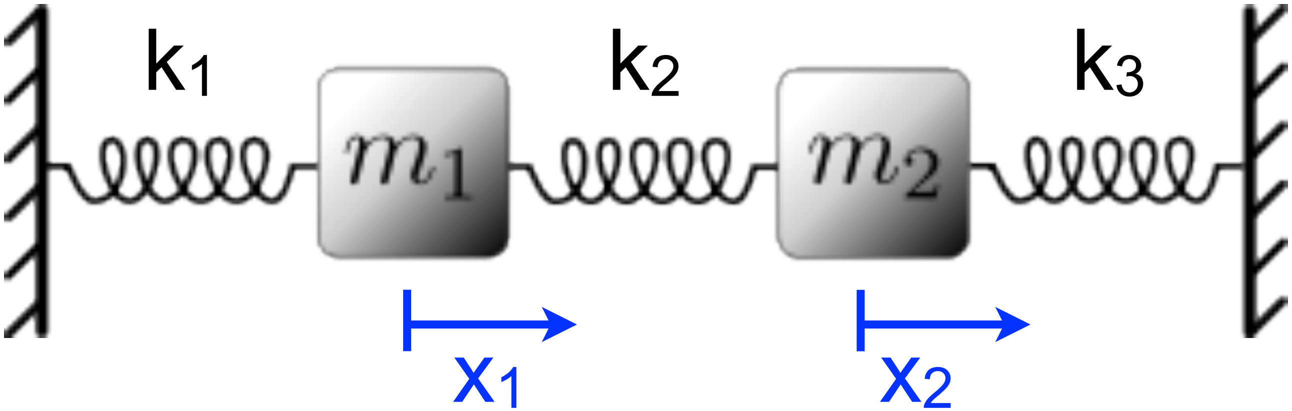
$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$a_1 - a_2 = 0, a_1 = a_2 = Ae^{-i\delta}$$

$$\mathbf{z}(t) = \begin{pmatrix} A \\ A \end{pmatrix} e^{i(\omega t - \delta)}$$

$$\mathbf{x}(t) = \text{Re}\ \mathbf{z}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A\cos(\omega t - \delta) \\ A\cos(\omega t - \delta) \end{pmatrix}$$

The first normal mode



So in first normal mode both blocks move together, as if $k_2 = 0$, with frequency ω=sqrt(k/m)

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a}e^{i\omega t} = 0$$

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

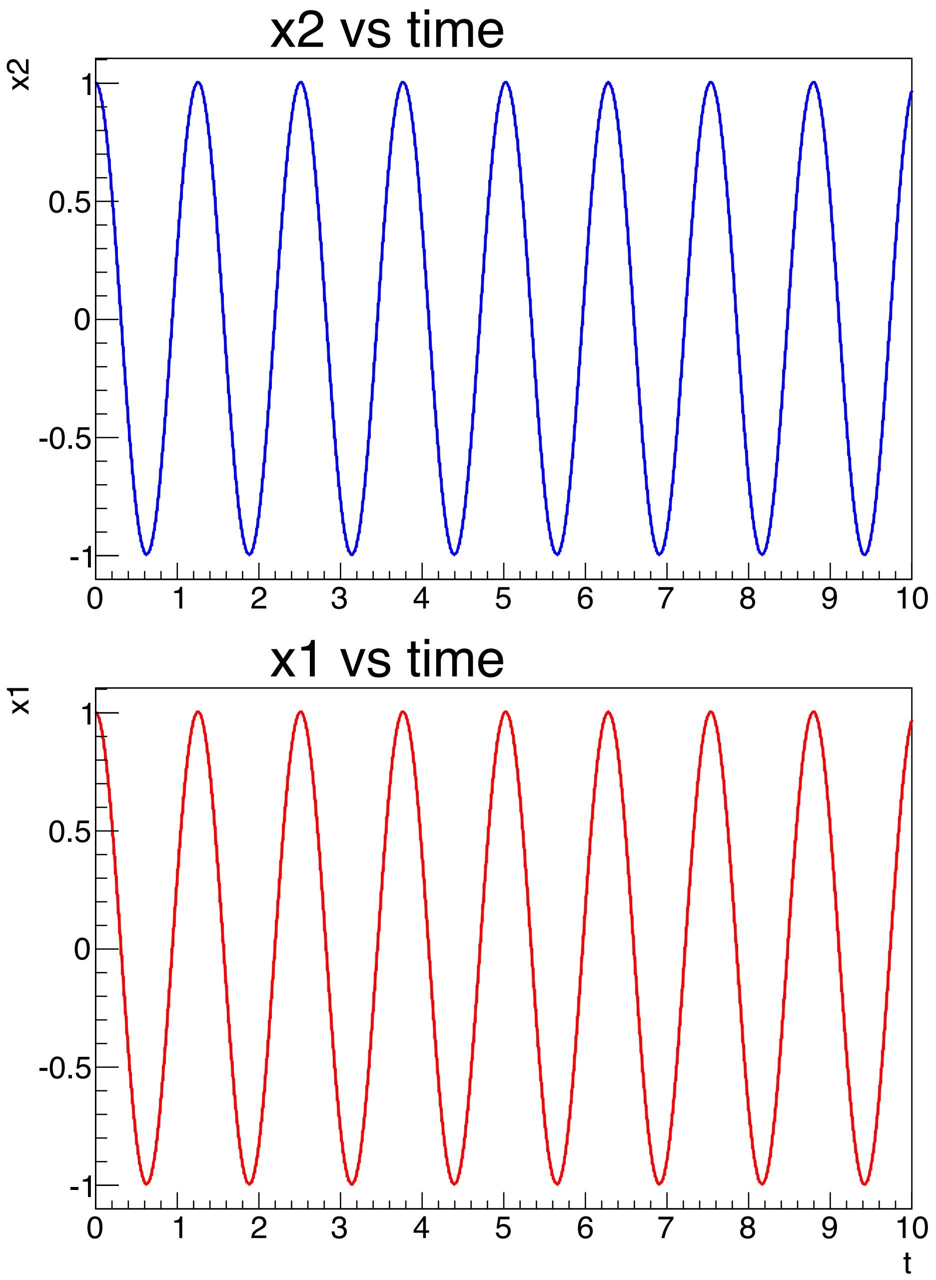
$$a_1 - a_2 = 0, a_1 = a_2 = Ae^{-i\delta}$$

$$\mathbf{z}(t) = \begin{pmatrix} A \\ A \end{pmatrix} e^{i(\omega t - \delta)}$$

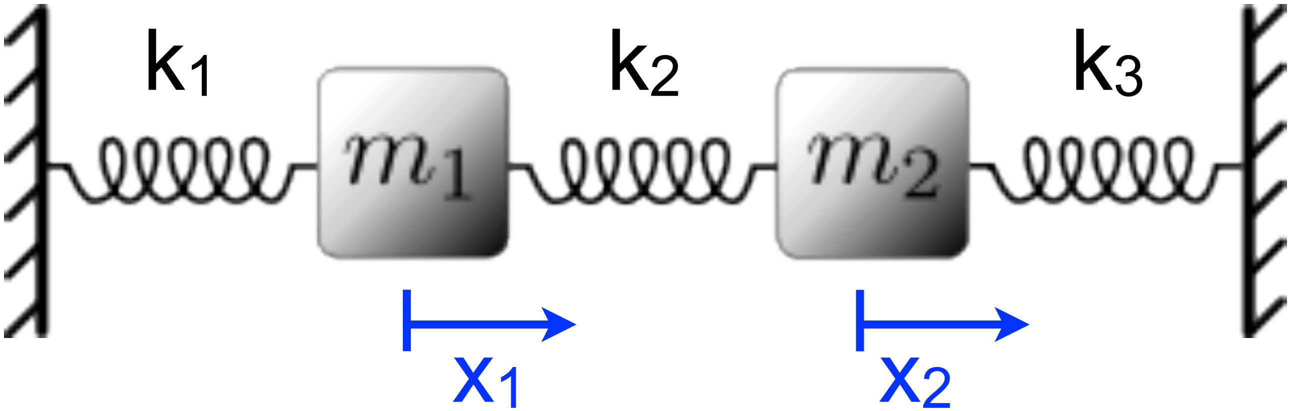
$$\mathbf{x}(t) = \text{Re } \mathbf{z}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A\cos(\omega t - \delta) \\ A\cos(\omega t - \delta) \end{pmatrix}$$

The first normal mode

In
phase!



The second normal mode



So in second normal mode both blocks move in opposite direction, with frequency ω=sqrt(3k/m), ie out of phase

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{a}e^{i\omega t} = 0$$

$$\mathbf{K} - \omega^2 \mathbf{M} = \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \to$$

$$\begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

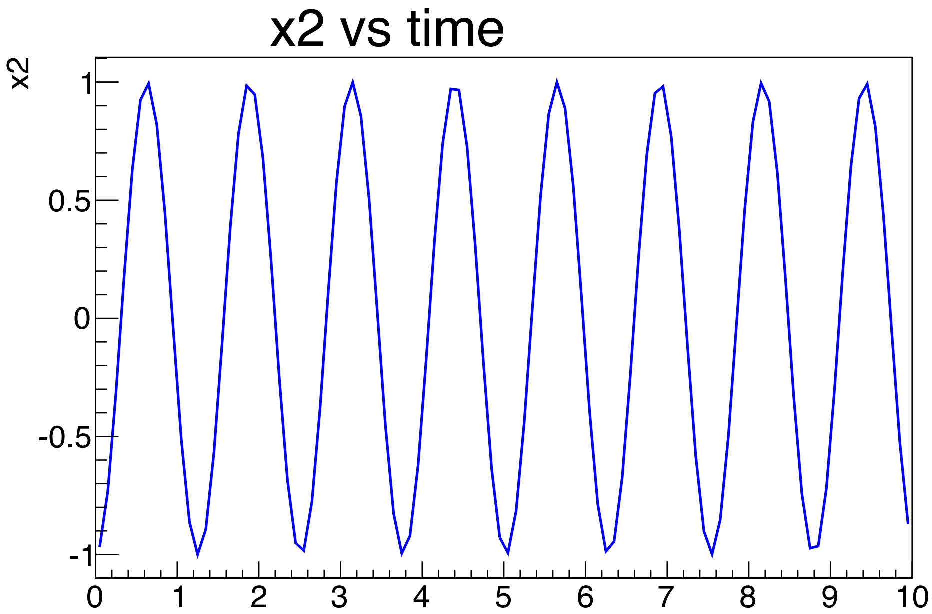
$$a_1 + a_2 = 0, a_1 = -a_2 = Ae^{-i\delta}$$

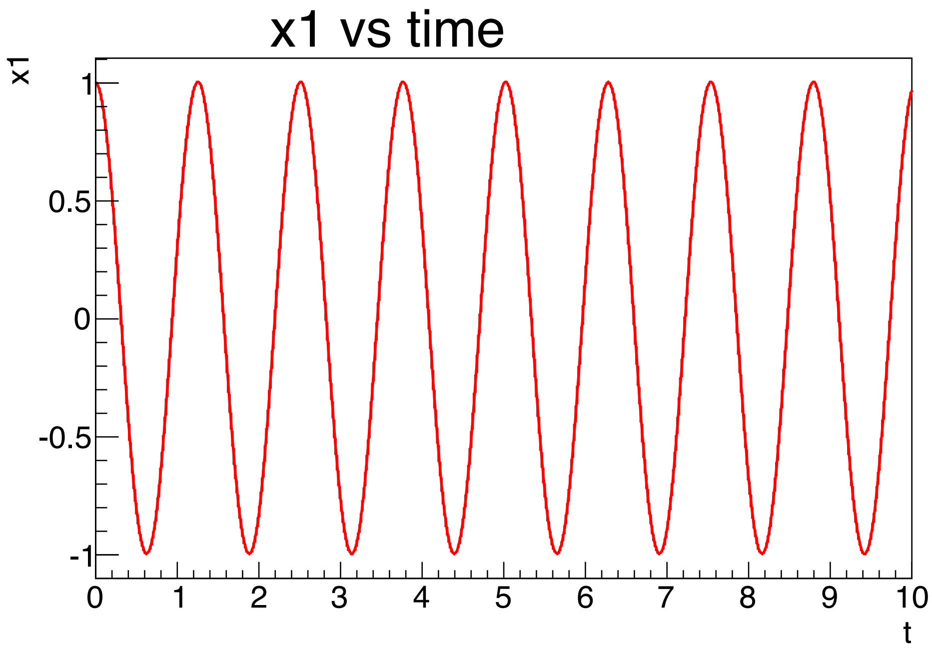
$$\mathbf{z}(t) = \begin{pmatrix} A \\ -A \end{pmatrix} e^{i(\omega t - \delta)}$$

$$\mathbf{x}(t) = \text{Re } \mathbf{z}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A\cos(\omega t - \delta) \\ -A\cos(\omega t - \delta) \end{pmatrix}$$

The second normal mode

Out of phase!





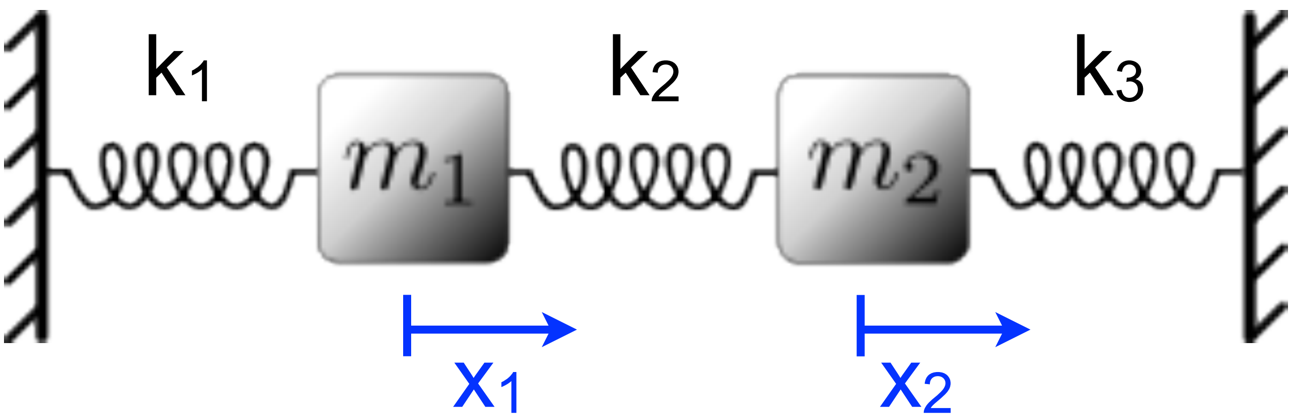
The generic solution

with $\omega_1=\text{sqrt}(k/m)$, $\omega_2=\text{sqrt}(3k/m)$

$$\mathbf{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1\cos(\omega_1 t - \delta_1) + A_2\cos(\omega_2 t - \delta_2) \\ -A_1\cos(\omega_1 t - \delta_1) - A_2\cos(\omega_2 t - \delta_2) \end{pmatrix}$$

Note: Two second order differential equations, so four unknown constants!

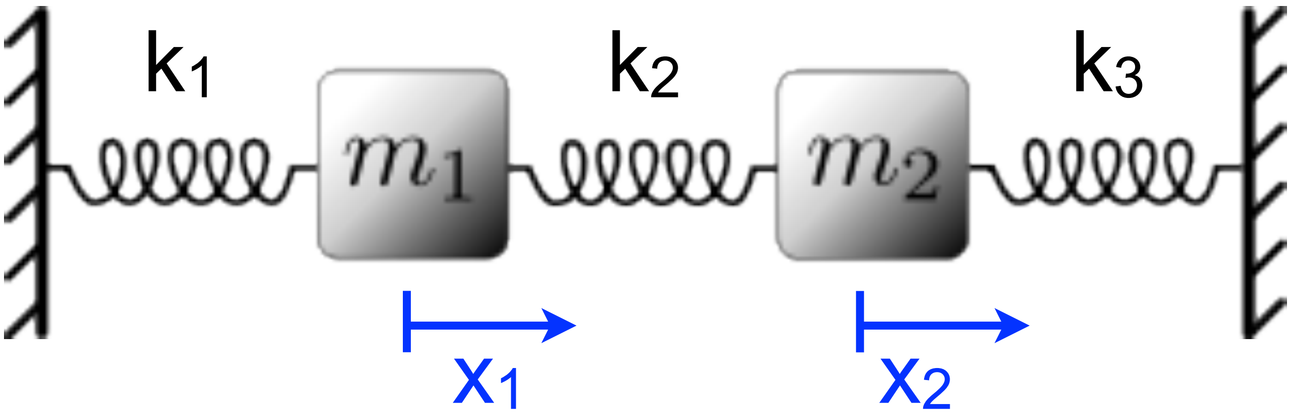
A not-so-nice feature of our coordinate system is that in both normal modes, x_1 and x_2 do not vary independently. Would be nice to have coordinates where that is true...



Introduce:

$$\zeta_1(t) = \frac{1}{2}x_1(t) + x_2(t)$$

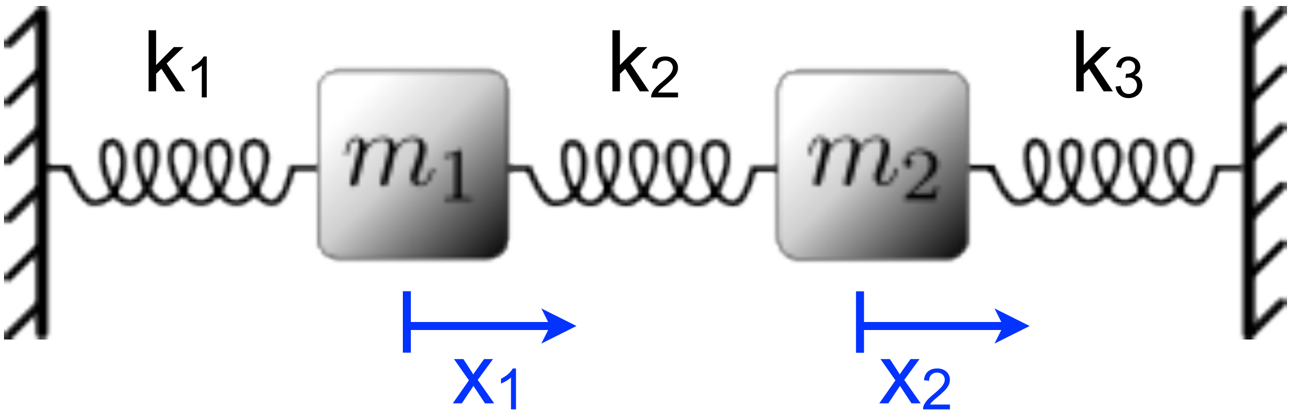
$$\zeta_2(t) = \frac{1}{2}x_1(t) - x_2(t)$$



We then have:

$$\zeta_1(t) = \frac{1}{2}(x_1(t) + x_2(t)) = A_1 \cos(\omega_1 t - \delta_1)$$

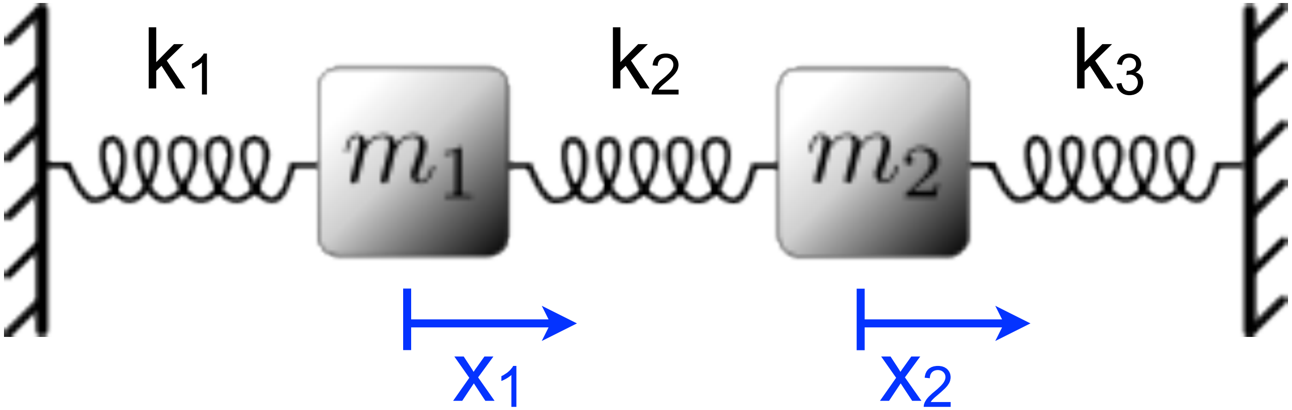
$$\zeta_2(t) = \frac{1}{2}(x_1(t) - x_2(t)) = 0$$



We get:

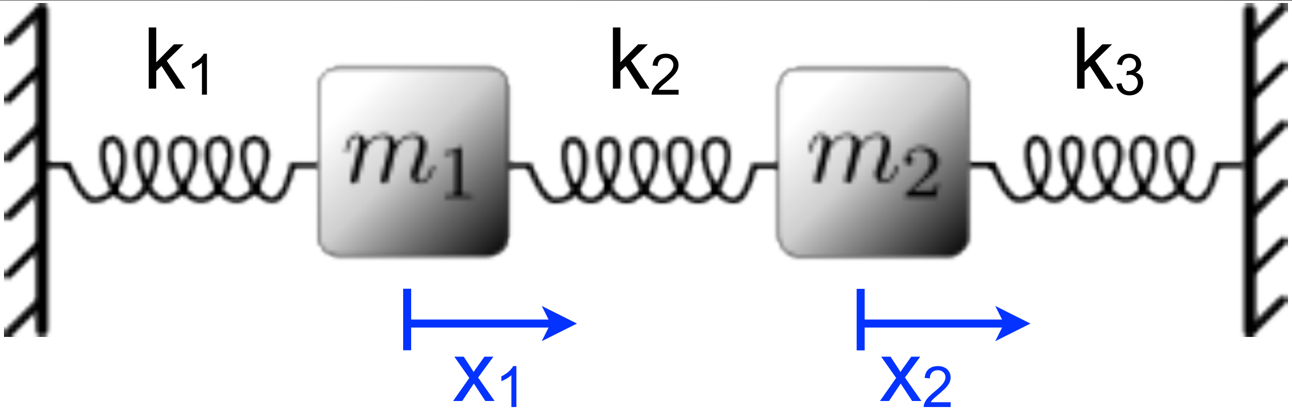
$$\zeta_1(t) = \frac{1}{2}(x_1(t) + x_2(t)) = 0$$

$$\zeta_2(t) = \frac{1}{2}(x_1(t) - x_2(t)) = A_2 \cos(\omega_2 t - \delta_2)$$



Taylor 11.2, 11.4

Two weakly coupled oscillators



$m_1 = m_2$,

$k_1 = k_3 = k$,

$k_2 << k$

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \mathbf{K} = \begin{bmatrix} k + k_2 & -k_2 \\ -k_2 & k + k_2 \end{bmatrix}$$

$$\mathbf{K} - \mathbf{M}\omega^2 = \begin{bmatrix} k + k_2 - m\omega^2 & -k_2 \\ -k_2 & k + k_2 - m\omega^2 \end{bmatrix}$$

$$|\mathbf{K} - \mathbf{M}\omega^2| = 0 \rightarrow (k + k_2 - m\omega^2)^2 - k_2^2 = 0$$

$$m^2\omega^4 - 2(k + k_2)m\omega^2 + k^2 + k + 2^2 + 2kk_2 - k_2^2 = 0$$

$$m^2\omega^4 - 2(k + k_2)m\omega^2 + k^2 + 2kk_2 = 0$$

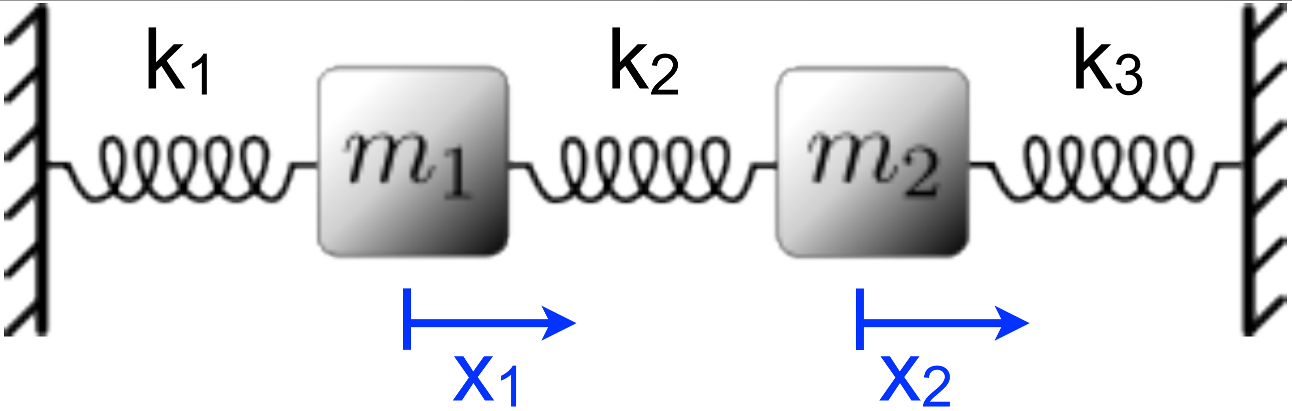
$$\omega^2 = \frac{2m(k + k_2) \pm \sqrt{4m^2(k + k_2)^2 - 4m^2(k^2 + 2kk_2)}}{2m^2}$$

$$\omega^2 = \frac{k + k_2}{m} \pm \frac{\sqrt{4m^2k_2^2 + 8m^2kk_2 - 8m^2kk_2}}{2m^2}$$

Check that we get right answer if $k_2 = k$

$$\omega^2 = \frac{k + k_2}{m} \pm \frac{k_2}{m}$$

Now let's use the weakly coupled assumption



$k_1 = k_3 = k$, $k_2 << k$ so
$\omega_1 = \mathrm{sqrt}(k/m) \sim \omega_1 \sim \omega_2$

$$\omega_1 = \sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{k + 2k_2}{m}}$$

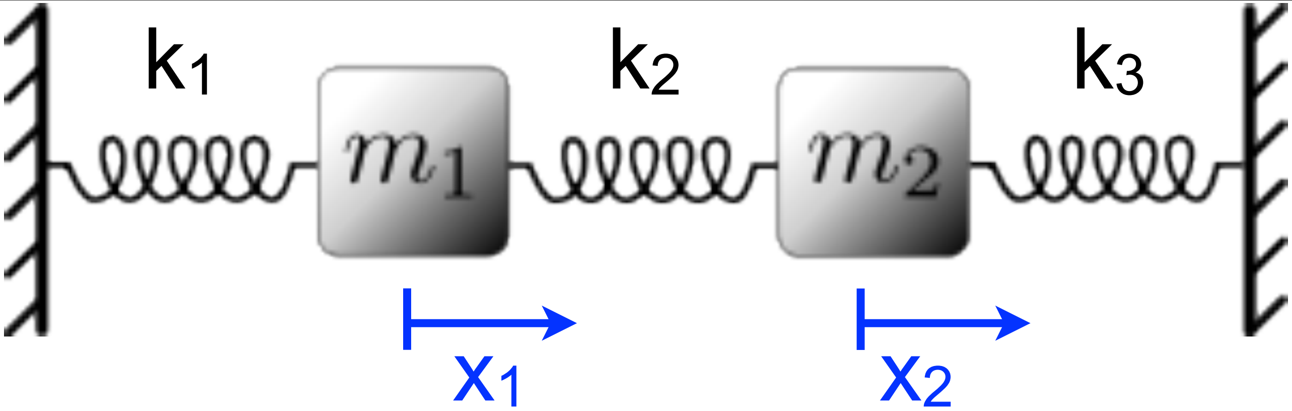
$$\omega_0 = \frac{\omega_1 + \omega_2}{2}$$

$$\omega_1 = 2\omega_0 - \omega_2 = \omega_0 - \epsilon$$

$$\omega_2 = 2\omega_0 - \omega_1 = \omega_0 + \epsilon$$

$$\omega_2 - \omega_1 = 2\epsilon$$

Now the normal modes

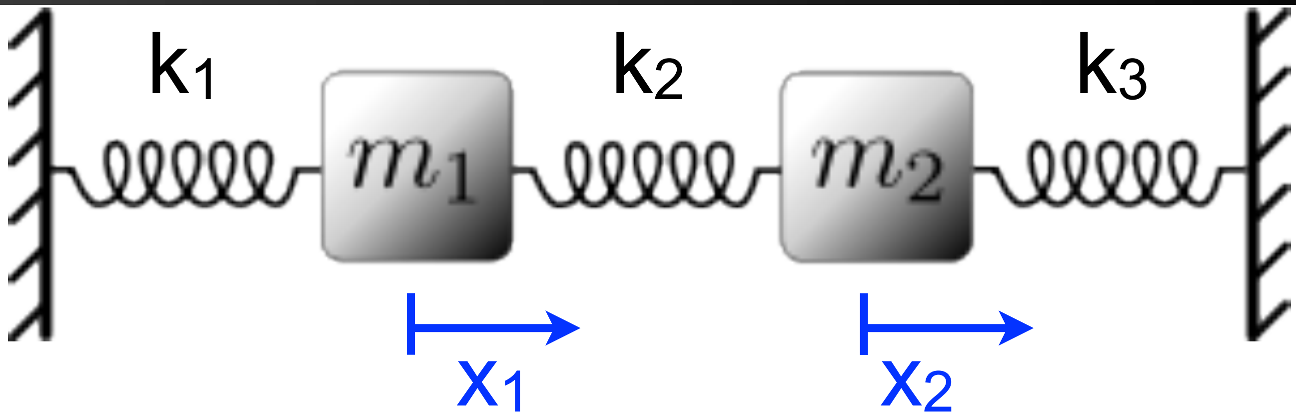


$$\mathbf{z}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_0 - \epsilon)t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_0 + \epsilon)t}$$

$$\mathbf{z}(t) = \left\{ C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\epsilon t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\epsilon t} \right\} e^{i\omega_0 t}$$

Term in {} varies very very slowly with time since ε is small. Only changes **z**(t) over long periods of time

Now the normal modes



$$\mathbf{z}(t) = \left\{ C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\epsilon t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\epsilon t} \right\} e^{i\omega_0 t}$$

Just one choice, but an interesting one

$$C_1 = C_2 = A/2$$

$$\mathbf{z}(t) = \left\{ A/2 \begin{bmatrix} e^{-i\epsilon t} + e^{i\epsilon t} \\ e^{-i\epsilon t} - e^{i\epsilon t} \end{bmatrix} \right\} e^{i\omega_0 t}$$

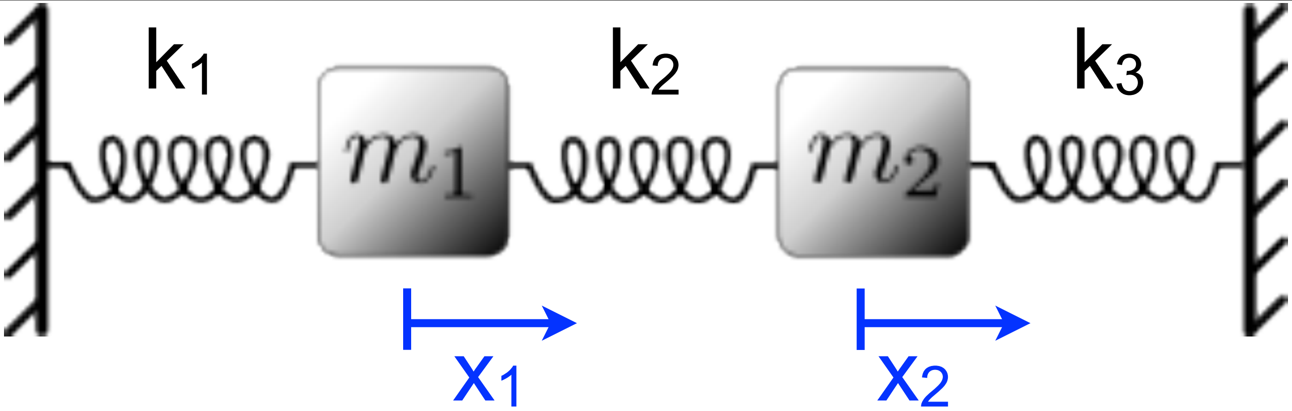
Is this clear/ obvious to all?

$$\mathbf{z}(t) = \left\{ A \begin{bmatrix} \cos \epsilon t \\ -i \sin \epsilon t \end{bmatrix} \right\} e^{i\omega_0 t}$$

$$x_1(t) = \text{Re } \mathbf{z}_1(t) = \text{Re } (A \cos \epsilon t \cos \omega_0 t + Ai \cos \epsilon t \sin \omega_0 t) = A \cos \epsilon t \cos \omega_0 t$$

$$x_2(t) = \text{Re } \mathbf{z}_2(t) = \text{Re } (-Ai \sin \epsilon t \cos \omega_0 t - Ai^2 \sin \epsilon t \sin \omega_0 t) = A \sin \epsilon t \sin \omega_0 t$$

Examining the solution



$$x_1(t) = A\cos\epsilon t\cos\omega_0 t$$

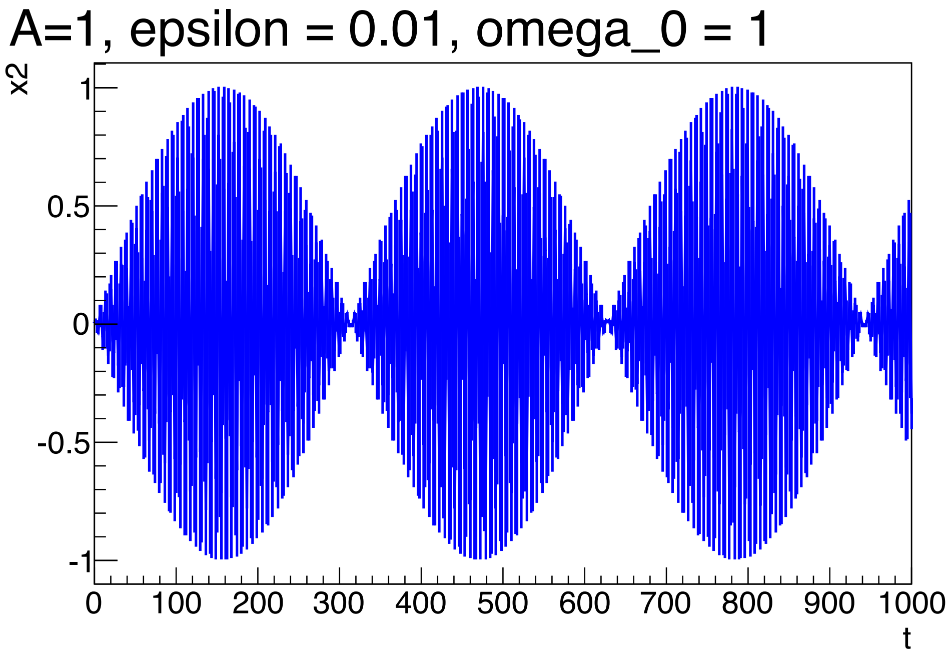
$$x_2(t) = A\sin\epsilon t\sin\omega_0 t$$

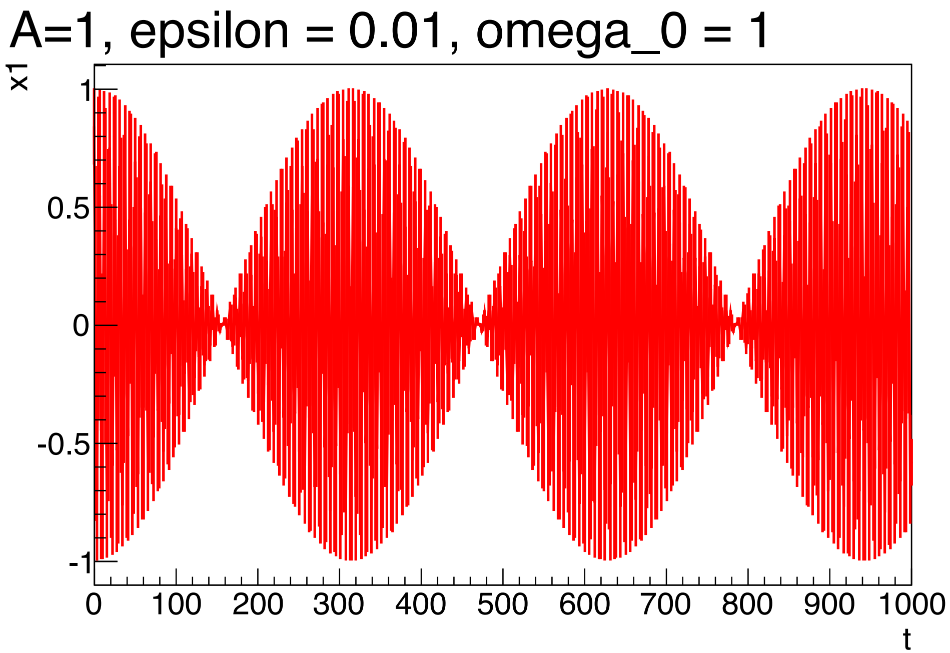
$$\dot{x}_1(t) = -A\epsilon\sin\epsilon t\cos\omega_0 t - A\omega_0\cos\epsilon t\sin\omega_0 t$$

$$\dot{x}_2(t) = A\epsilon\cos\epsilon t\sin\omega_0 t + A\omega_0\sin\epsilon t\cos\omega_0 t$$

What does our solution look like at t=0?

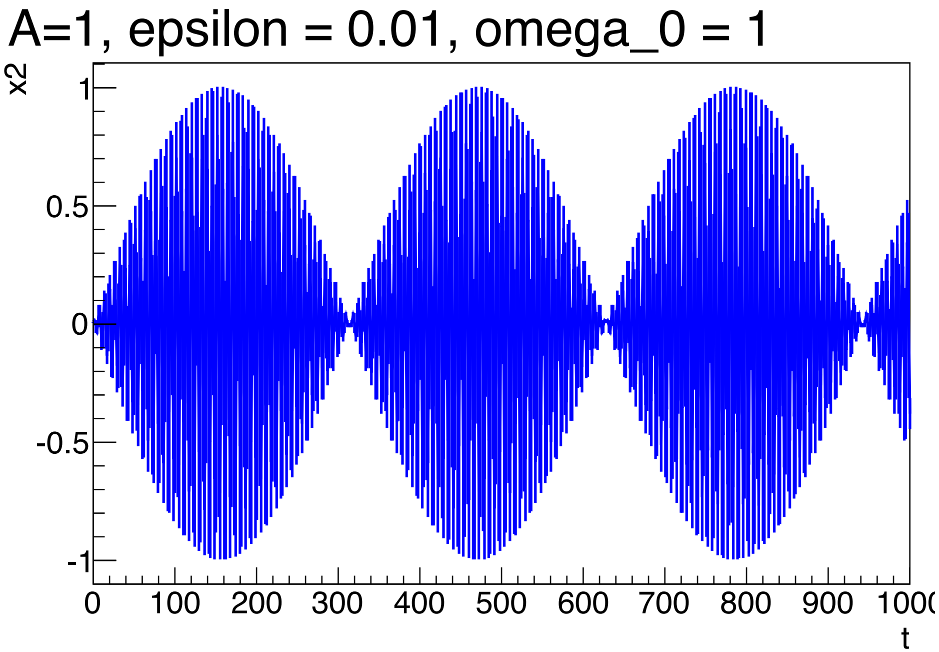
Examining the solution

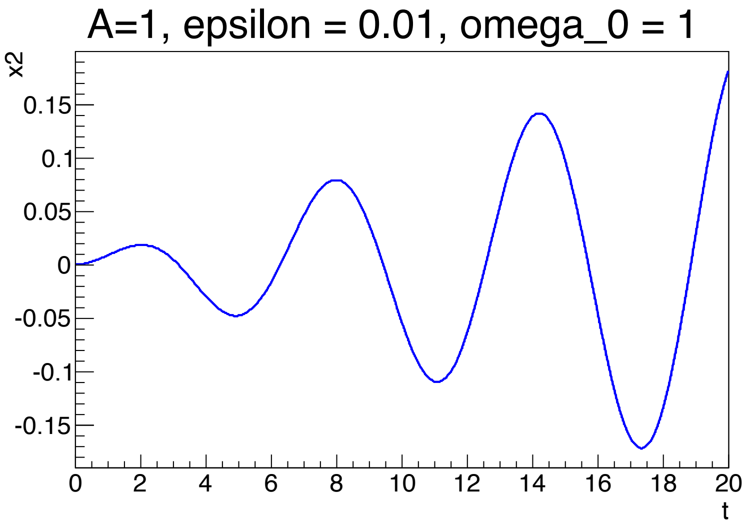


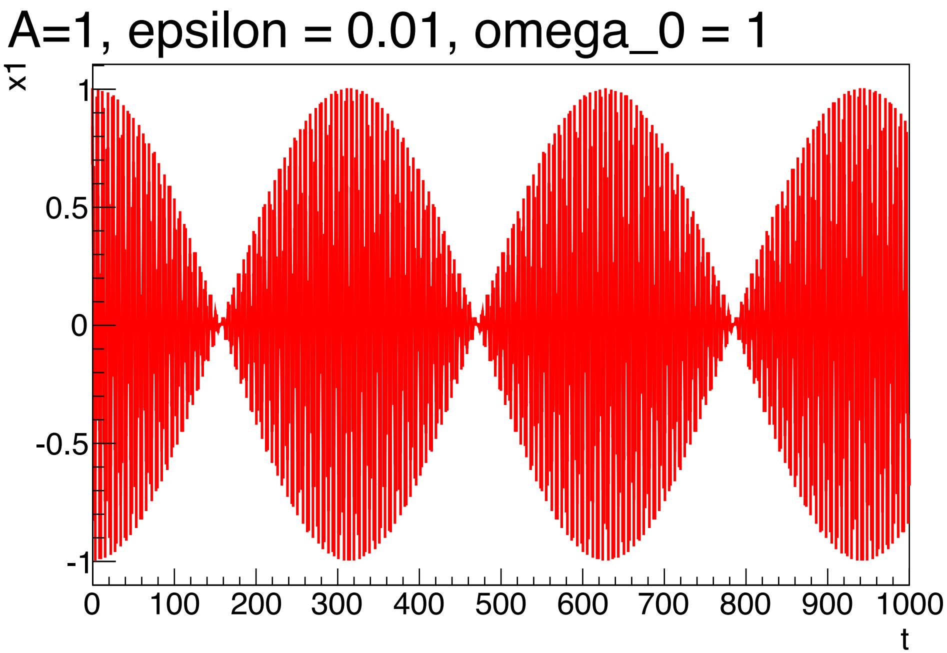


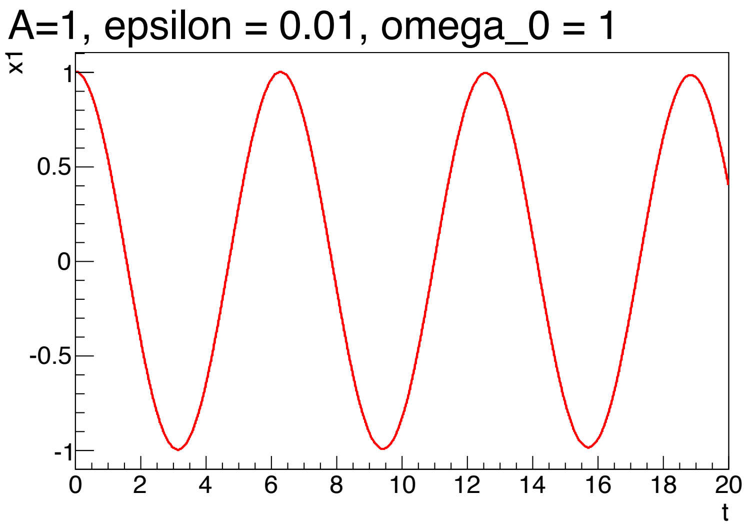
Beat patterns due to interference (constructive and destructive) between the modes! At any given time, one interference is constructive, the other destructive

Examining the solution









Let's rewrite our solutions again

$$x_1(t) = A\cos\epsilon t\cos\omega_0 t$$

$$x_2(t) = A\sin\epsilon t\sin\omega_0 t$$

$$\zeta_1(t) = \frac{x_1(t) + x_2(t)}{2} = A/2\left(\cos\epsilon t\cos\omega_0 t + \sin\epsilon t\sin\omega_0 t\right)$$

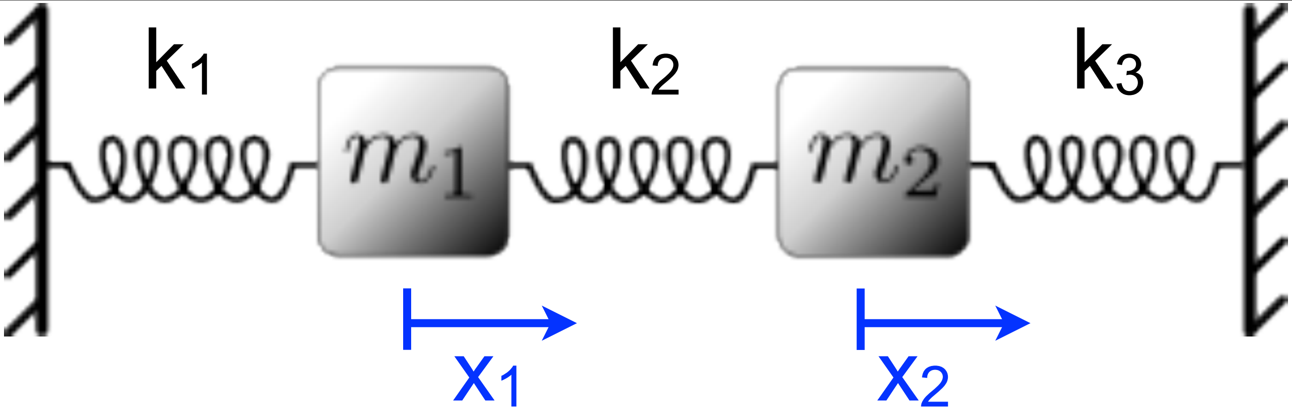
$$\zeta_2(t) = \frac{x_1(t) - x_2(t)}{2} = A/2\left(\cos\epsilon t\cos\omega_0 t - \sin\epsilon t\sin\omega_0 t\right)$$

$$\zeta_1(t) = \frac{A}{2}\cos(\omega_0 t - \epsilon t) = \frac{A}{2}\cos\omega_1 t$$

$$\zeta_2(t) = \frac{A}{2}\cos(\omega_0 t + \epsilon t) = \frac{A}{2}\cos\omega_2 t$$

It's the normal modes that oscillate with equal amplitude / simpler motion!

To the Lagrangian formalism



$$T = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2$$

$$U = \frac{k_1}{2}x_1^2 + \frac{k_2}{2}(x_2 - x_1)^2 + \frac{k_3}{2}x_2^2$$

$$\mathcal{L} = T - U = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 - \frac{k_1}{2}x_1^2 - \frac{k_2}{2}(x_2 - x_1)^2 - \frac{k_3}{2}x_2^2$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -k_1 x_1 + k_2(x_2 - x_1)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -k_3 x_2 - k_2(x_2 - x_1)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1 \dot{x}_1$$

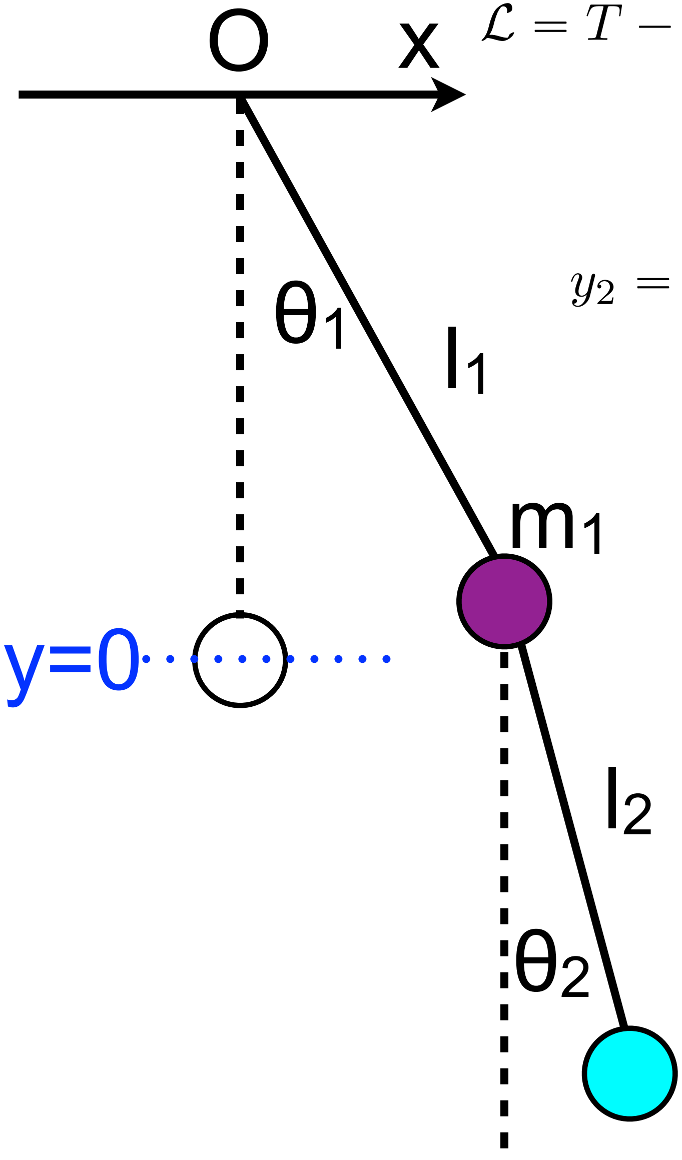
$$\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_2 \dot{x}_2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1)$$

No clear advantage here but for other problems this is much easier

Finally moving back to this (a recap)



$$\mathcal{L} = T - U = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - m_1 g y_1 - m_2 g y_2$$

$$x_1 = l_1 \sin\theta_1, y_1 = l_1(1 - \cos\theta_1)$$

$$x_2 = x_1 + l_2 \sin\theta_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

$$y_2 = y_1 + l_2(1 - \cos\theta_2) = l_1(1 - \cos\theta_1) + l_2(1 - \cos\theta_2)$$

$$\dot{x_1} = l_1 \cos\theta_1 \dot{\theta}_1$$

$$\dot{y_1} = l_1 \sin\theta_1 \dot{\theta}_1$$

$$\dot{x_1}^2 + \dot{y_1}^2 = l_1^2 \dot{\theta}_1^2$$

$$\dot{x_2} = l_1 \cos\theta_1 \dot{\theta_1} + l_2 \cos\theta_2 \dot{\theta_2}$$

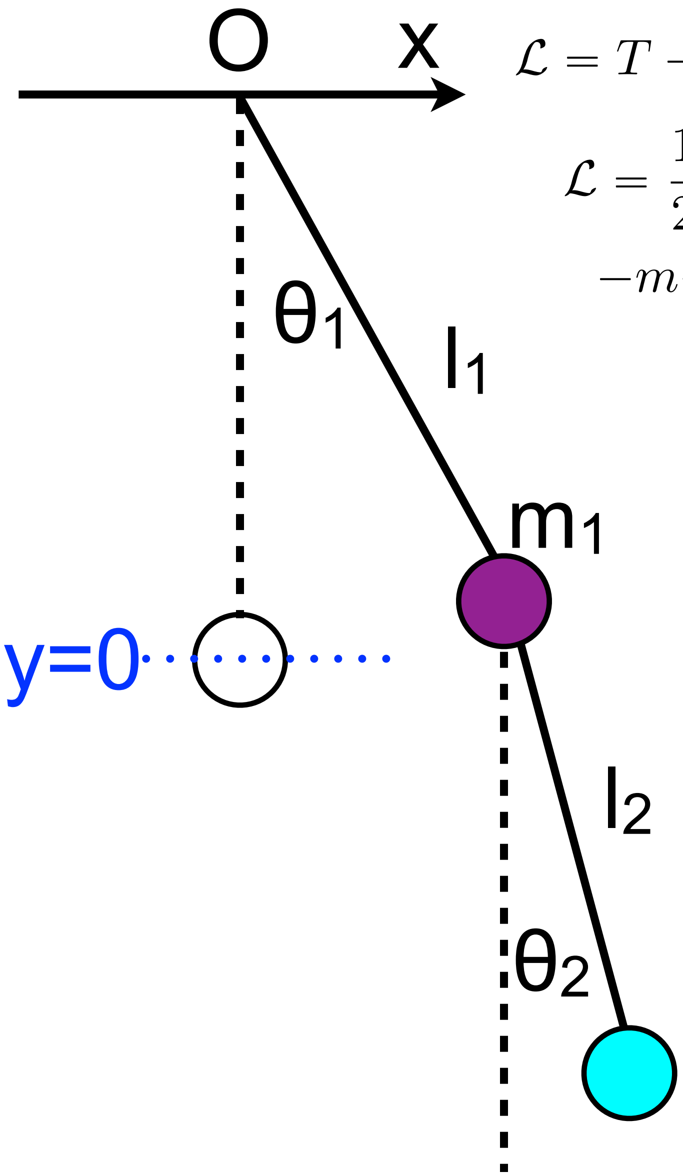
$$\dot{y_2} = l_1 \sin\theta_1 \dot{\theta_1} + l_2 \sin\theta_2 \dot{\theta_2}$$

$$\dot{x_2}^2 + \dot{y_2}^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)$$

$$\dot{x_2}^2 + \dot{y_2}^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

Makes sense that second mass velocity has factor related to angular difference

This does not have a nice, analytical solution



$$\mathcal{L} = T - U = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - m_1 g y_1 - m_2 g y_2$$

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)$$
$$-m_1 g l_1(1-\cos\theta_1) - m_2 g\left(l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)\right)$$

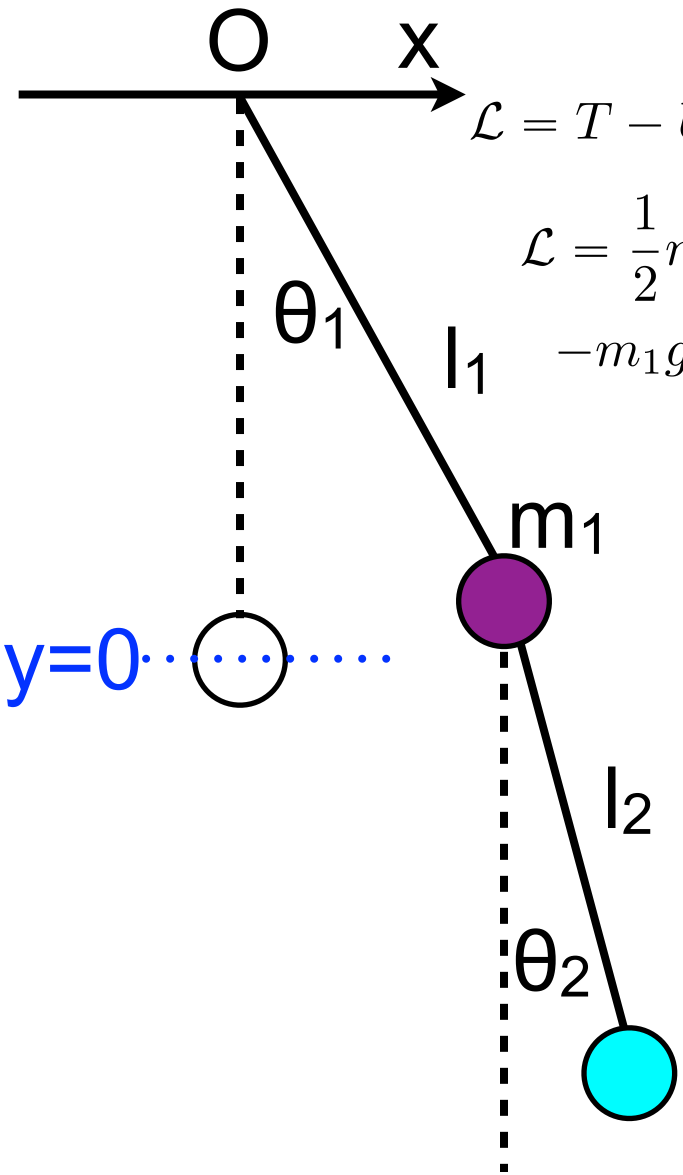
$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}$$

$$-m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin\theta_1 =$$
$$\frac{d}{dt}\left[m_1 l_1^2\dot{\theta}_1 + m_2 l_1^2\dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2\cos(\theta_1 - \theta_2)\right]$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}$$

$$m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2 =$$
$$\frac{d}{dt}\left[m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos(\theta_1 - \theta_2)\right]$$

Let's see if small angles have an analytical solution



$$\mathcal{L} = T - U = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - m_1 g y_1 - m_1 g y_2$$

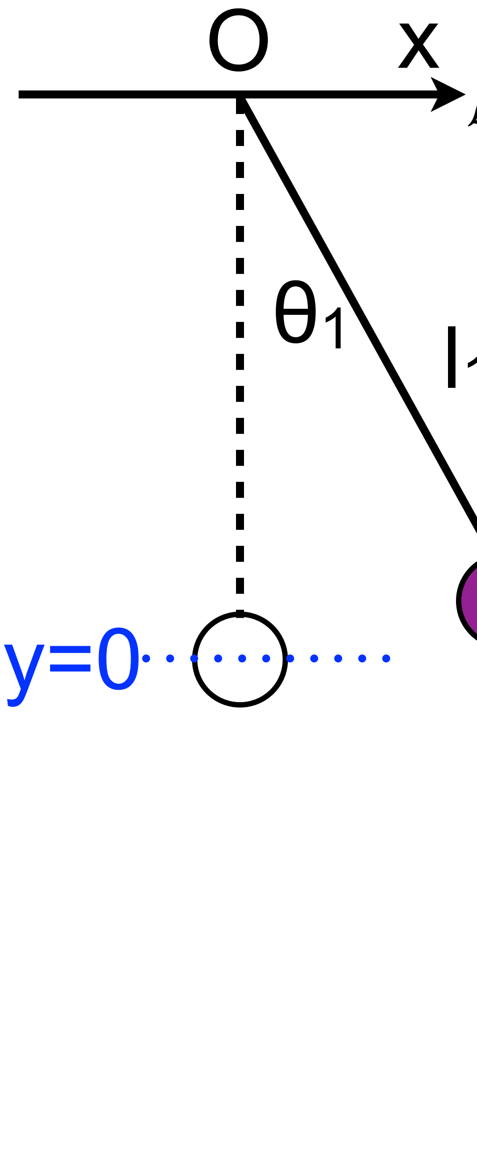
$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)$$

$$-m_1 g l_1(1 - \cos\theta_1) - m_2 g\left(l_1(1 - \cos\theta_1) + l_2(1 - \cos\theta_2)\right)$$

$$\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2$$

Two small numbers multiplied by a number close to 1. Approximate the cosine by 1

Let's see if small angles have an analytical solution



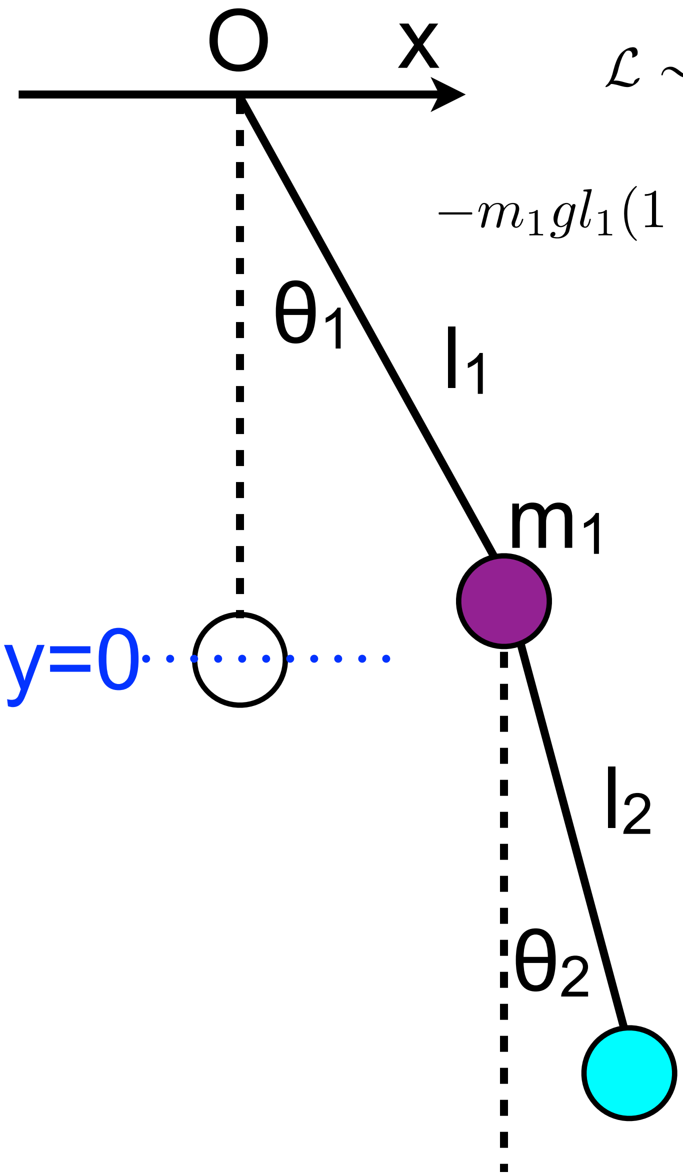
$$\mathcal{L} = T - U = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - m_1 g y_1 - m_1 g y_2$$

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)$$
$$-m_1 g l_1(1-\cos\theta_1) - m_2 g\left(l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)\right)$$

$$\cos(\theta) \sim 1 - \theta^2/2$$

Approximate with this Taylor expansion (not multiplied by other small numbers so we shouldn't use just "1")

Our new Lagrangian

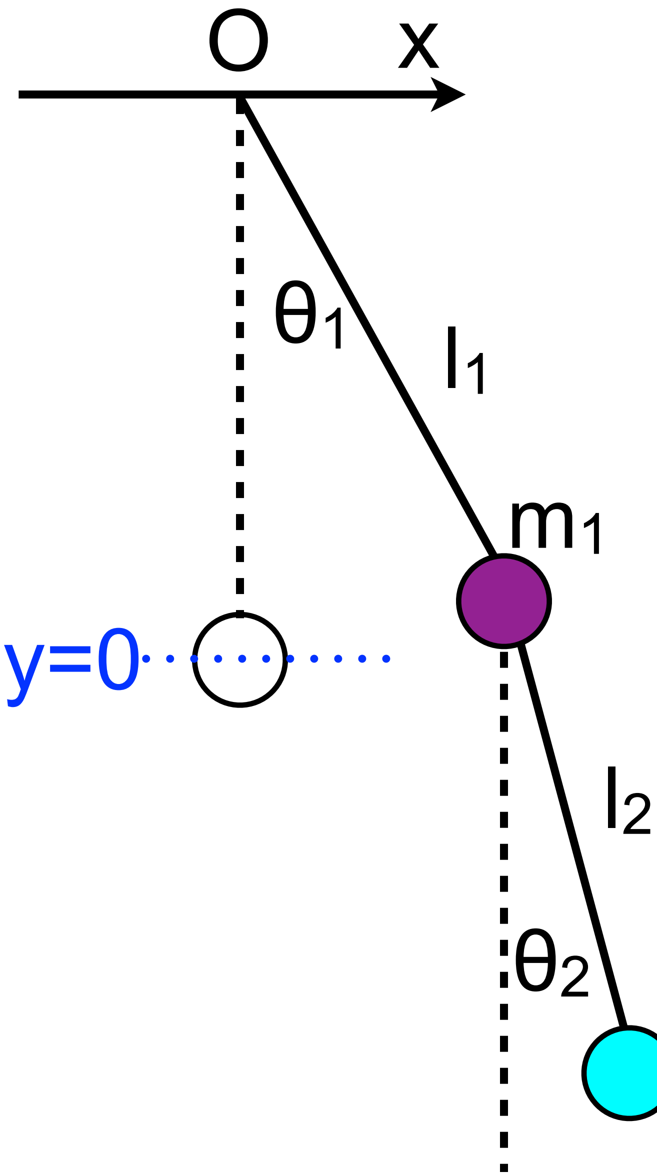


$$\mathcal{L} \sim \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\right)$$
$$-m_1 g l_1(1 - 1 + \frac{\theta_1^2}{2}) - m_2 g\left(l_1(1 - 1 + \frac{\theta_1^2}{2}) + l_2(1 - 1 + \frac{\theta_2^2}{2})\right)$$

Can drop constants from Lagrangian ...

$$\mathcal{L} \sim \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\right)$$
$$-m_1 g l_1\frac{\theta_1^2}{2} - m_2 g\left(l_1\frac{\theta_1^2}{2} + l_2\frac{\theta_2^2}{2}\right)$$

Equations with simplified Lagrangian



$$\mathcal{L} \sim \frac{1}{2} m_1 l_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \right)$$

$$- m_1 g l_1 \frac{\theta_1^2}{2} - m_2 g \left(l_1 \frac{\theta_1^2}{2} + l_2 \frac{\theta_2^2}{2} \right)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -(m_1 + m_2) g l_1 \theta_1$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = -m_2 g l_2 \theta_2$$

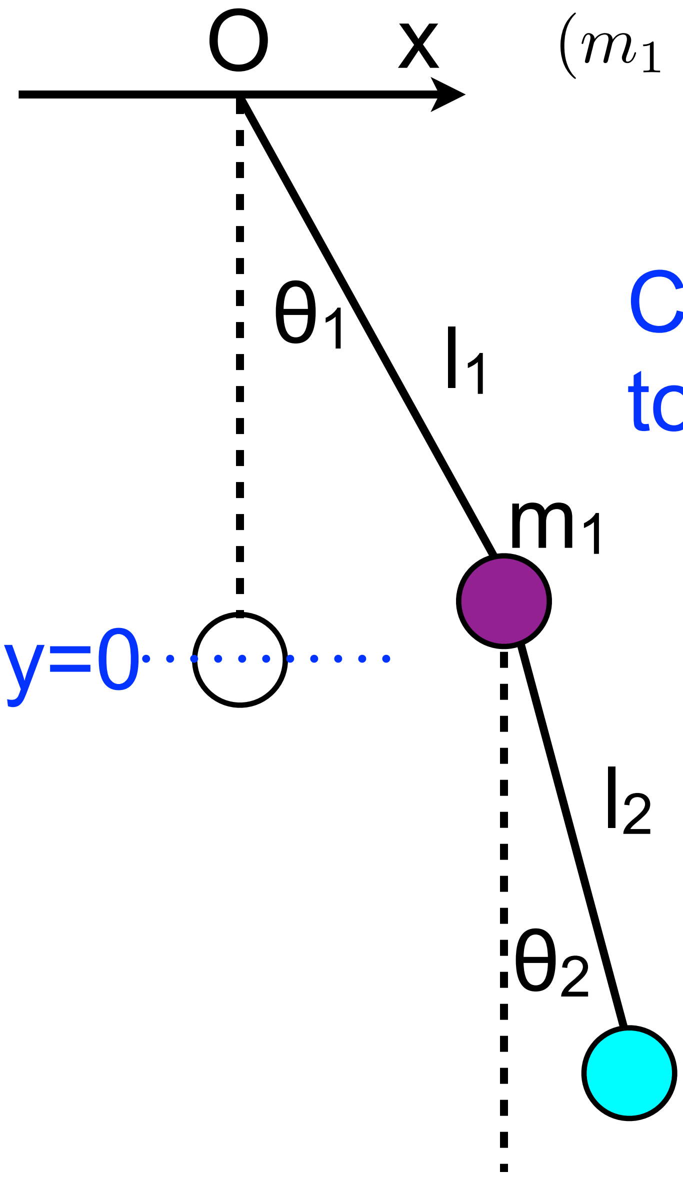
$$\frac{\partial \mathcal{L}}{\partial \dot\theta_1} = (m_1 + m_2) l_1^2 \dot\theta_1 + m_2 l_1 l_2 \dot\theta_2$$

$$\frac{\partial \mathcal{L}}{\partial \dot\theta_2} = m_2 l_2^2 \dot\theta_2 + m_2 l_1 l_2 \dot\theta_1$$

$$(m_1 + m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 = -(m_1 + m_2) g l_1 \theta_1$$

$$m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 = -m_2 g l_2 \theta_2$$

Our coupled equations



$$(m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 = -(m_1+m_2) g l_1 \theta_1$$
$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 = -m_2 g l_2 \theta_2$$

Can write in new form analogous to coupled springs:

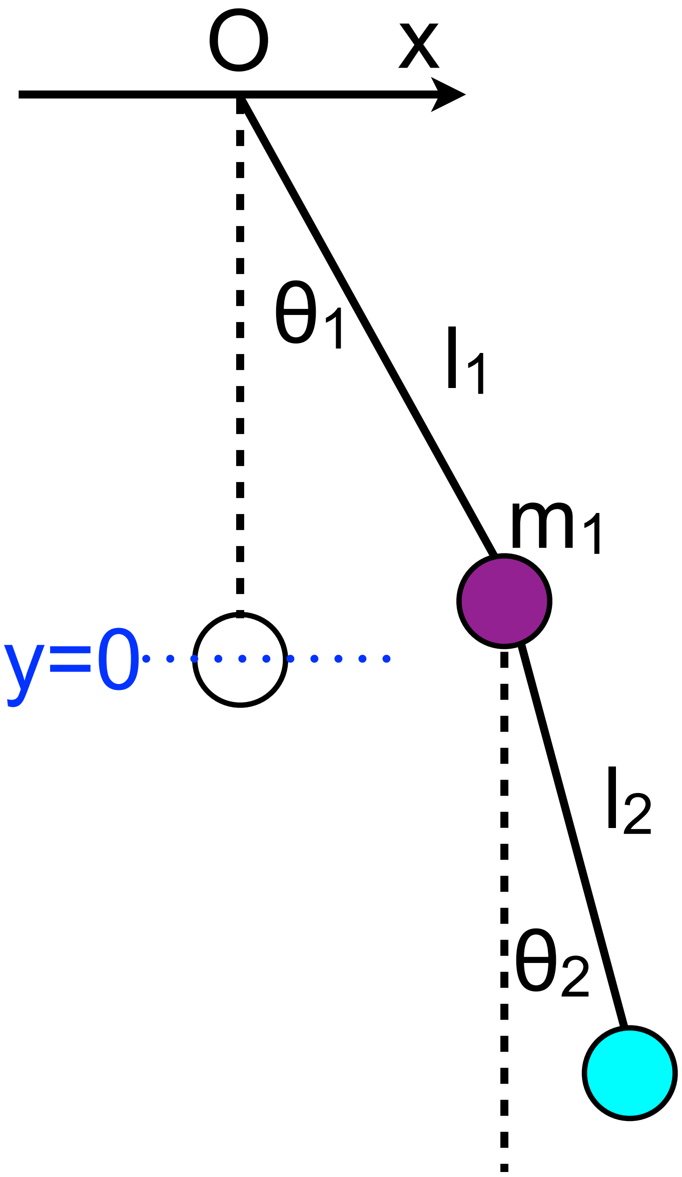
$$\mathbf{M}\ddot{\theta} = -\mathbf{K}\theta$$

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} (m_1+m_2)l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} (m_1+m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix}$$

Yet again, we guess a solution



$$\mathbf{M}\ddot{\theta} = -\mathbf{K}\theta$$

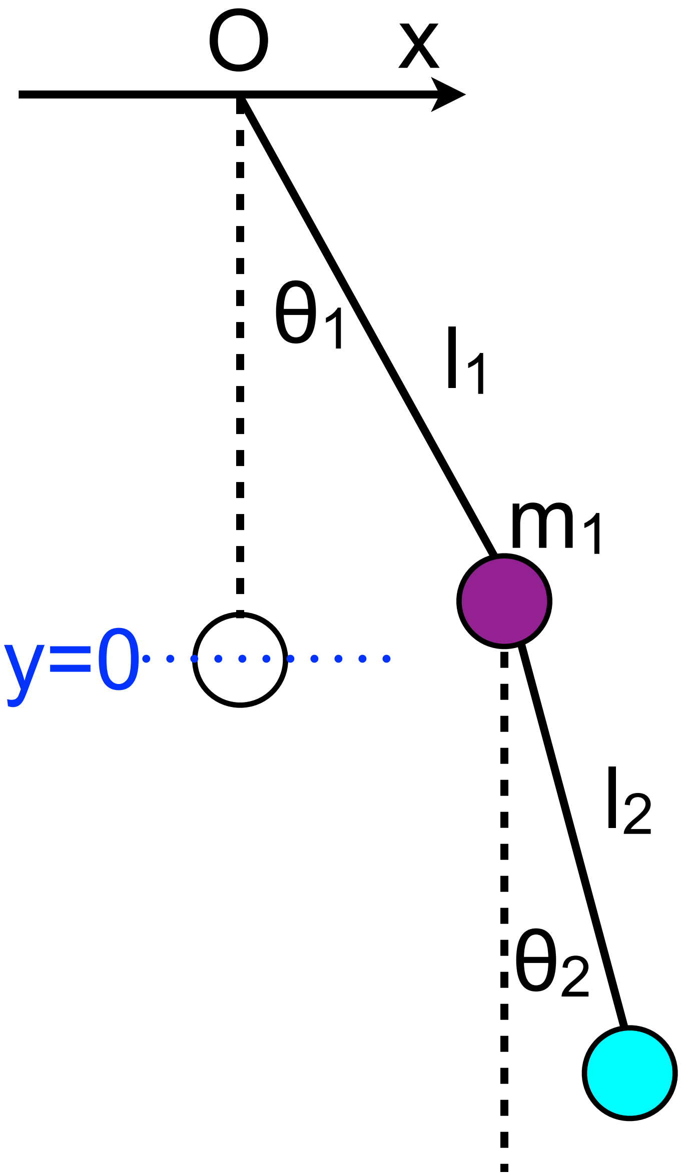
$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} (m_1 + m_2)l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} (m_1 + m_2)g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix}$$

$$\theta = \mathrm{Re}\ \mathbf{z}(t) = \mathrm{Re}\ \mathbf{a}e^{i\omega t} = \mathrm{Re} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t}$$

So we once again look to solve



$$|\mathbf{K} - \omega^2 \mathbf{M}| = 0$$

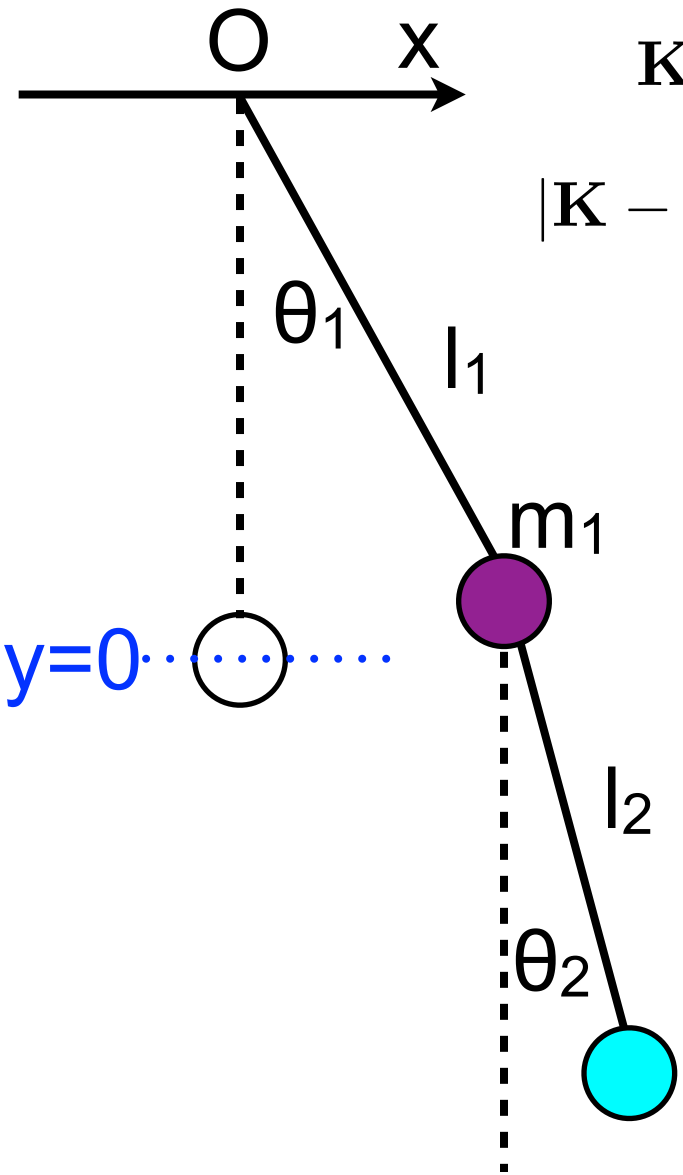
Following Taylor, let's simplify things to make life easier, and assume $m = m_1 = m_2$, $l = l_1 = l_2$

$$\omega_0^2 = \sqrt{g/L}$$

$$\mathbf{M} = \begin{bmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{bmatrix} = ml^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 2mgl & 0 \\ 0 & mgl \end{bmatrix} = ml^2 \begin{bmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{bmatrix}$$

Plugging in the determinant



$$\mathbf{K} - \omega^2\mathbf{M} = ml^2 \begin{bmatrix} 2\omega_0^2 - 2\omega^2 & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{bmatrix}$$

$$|\mathbf{K} - \omega^2\mathbf{M}| = 0 = (2\omega_0^2 - 2\omega^2)(\omega_0^2 - \omega^2) - \omega^4 = 0$$

$$2\omega_0^4 - 2\omega_0^2\omega^2 - 2\omega_0^2\omega^2 + 2\omega^4 - \omega^4 = 0$$

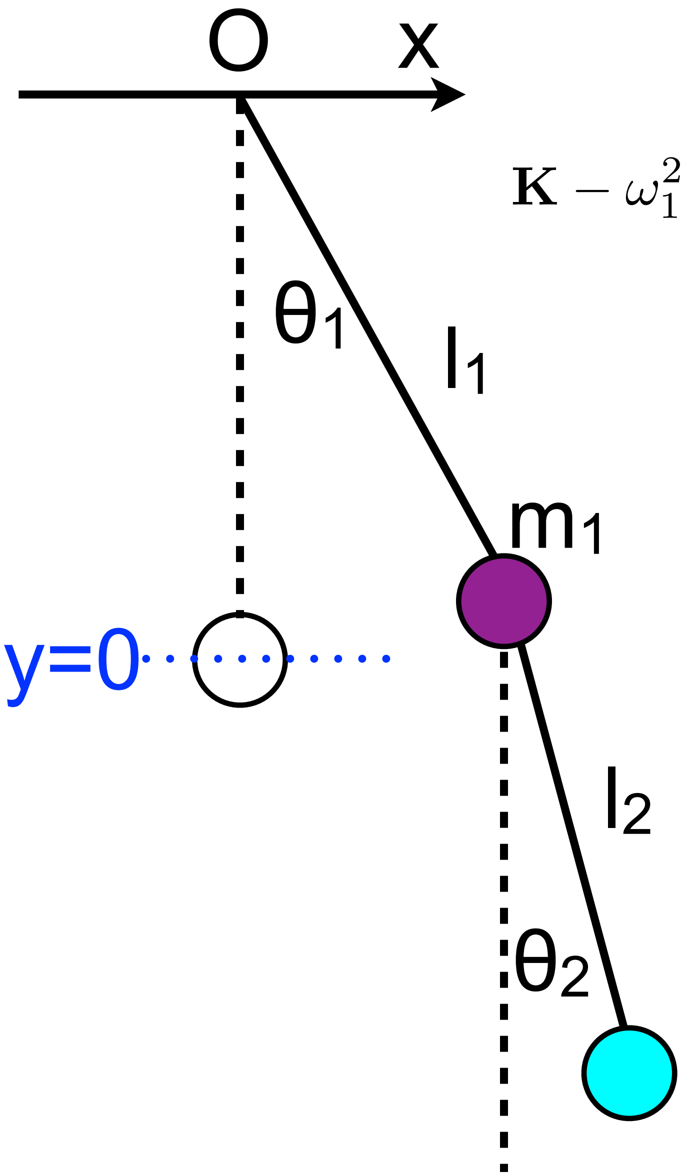
$$\omega^4 + \omega^2(-4\omega_0^2) + 2\omega_0^4 = 0$$

$$\omega^2 = \frac{4\omega_0^2 \pm \sqrt{16\omega_0^4 - 4(2)(\omega_0^4)}}{2}$$

$$\omega^2 = 2\omega_0^2 \pm \sqrt{2}\omega_0^2 = \omega_0^2(2 \pm \sqrt{2})$$

$$\omega_1^2 = \omega_0^2(2 - \sqrt{2}), \omega_2^2 = \omega_0^2(2 + \sqrt{2})$$

Looking for the first normal mode



$$(\mathbf{K} - \omega_1^2 \mathbf{M})\mathbf{a}e^{i\omega t} = 0$$

$$\mathbf{K} - \omega_1^2 \mathbf{M} = ml^2 \begin{bmatrix} 2\omega_0^2 - 2(2-\sqrt{2})\omega_0^2 & -(2-\sqrt{2})\omega_0^2 \\ -(2-\sqrt{2})\omega_0^2 & \omega_0^2 - (2-\sqrt{2})\omega_0^2 \end{bmatrix}$$

$$\mathbf{K} - \omega_1^2 \mathbf{M} = ml^2\omega_0^2 \begin{bmatrix} 2\sqrt{2} - 2 & -2+\sqrt{2} \\ -2+\sqrt{2} & \sqrt{2}-1 \end{bmatrix}$$

$$(\mathbf{K} - \omega_1^2 \mathbf{M}) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$(2\sqrt{2}-2)a_1 + a_2(-2+\sqrt{2}) = 0$$

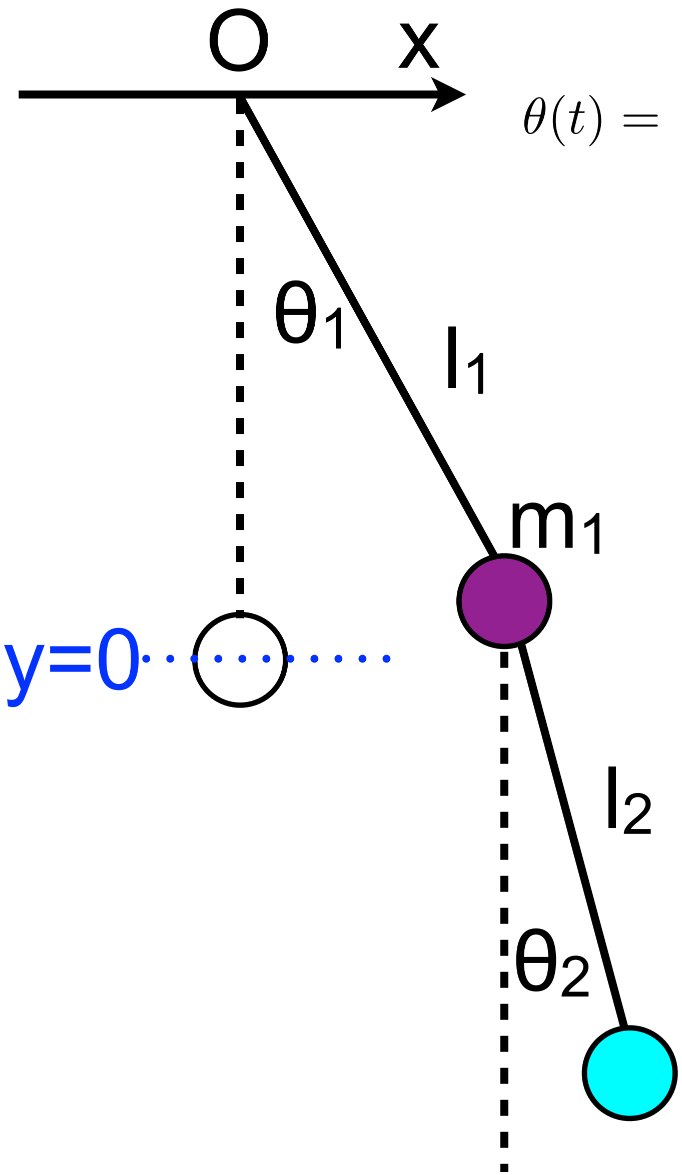
$$a_2 = \frac{2\sqrt{2}-2}{2-\sqrt{2}} a_1$$

$$a_2 = \frac{(2\sqrt{2}-2)(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})} a_1$$

$$a_2 = \frac{4\sqrt{2}+4-4-2\sqrt{2}}{2} a_1$$

$$a_2 = \sqrt{2}a_1$$

Looking for the first normal mode



$$\theta(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix} = \text{Re } \mathbf{z}(t) = \text{Re } \mathbf{a}e^{i\omega_1 t} = \text{Re} \begin{bmatrix} a_1 \\ \sqrt{2}a_1 \end{bmatrix} e^{i\omega_1 t}$$

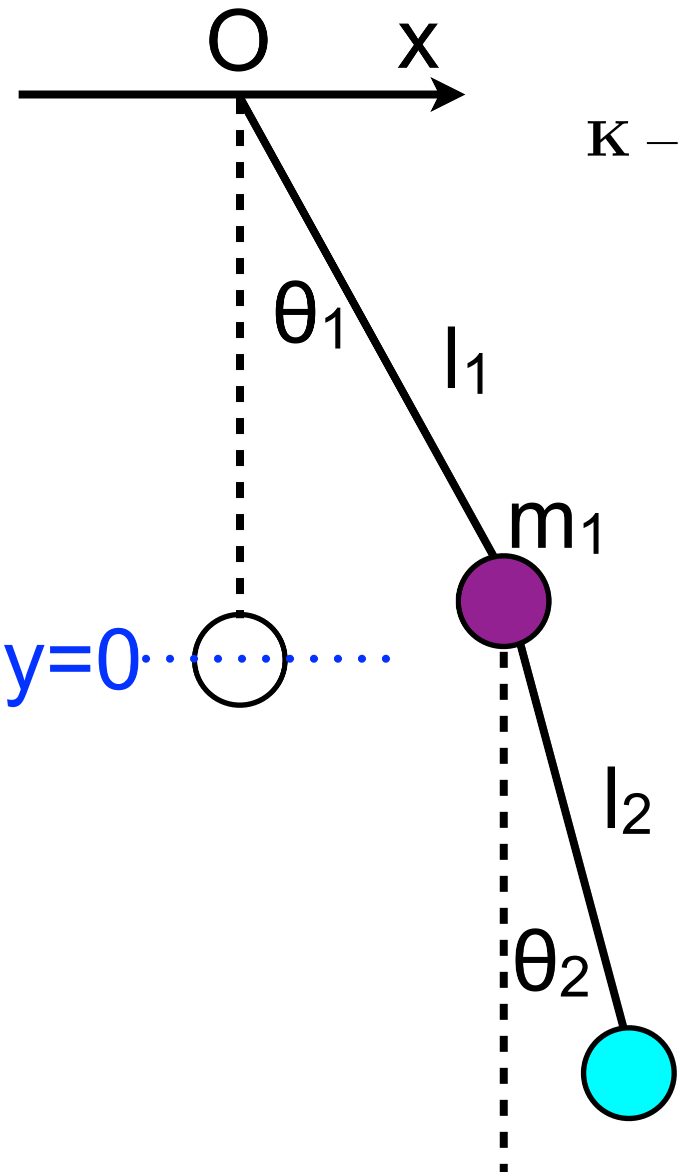
$$a_1(t) = A_1 e^{-i\delta_1}$$

$$\theta(t) = A_1 \text{ Re } e^{-i\delta_1} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{i\omega_1 t}$$

$$\theta(t) = A_1 \cos(\omega_1 t - \delta_1) \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

In first normal mode, both pendulums oscillate in phase, with second pendulum having amplitude sqrt(2) of the first one

Looking for the second normal mode



$$(\mathbf{K} - \omega_2^2\mathbf{M})\mathbf{a}e^{i\omega t} = 0$$

$$\mathbf{K} - \omega_2^2\mathbf{M} = ml^2 \begin{bmatrix} 2\omega_0^2 - 2(2+\sqrt{2})\omega_0^2 & -(2+\sqrt{2})\omega_0^2 \\ -(2+\sqrt{2})\omega_0^2 & \omega_0^2 - (2+\sqrt{2})\omega_0^2 \end{bmatrix}$$

$$\mathbf{K} - \omega_2^2\mathbf{M} = ml^2\omega_0^2 \begin{bmatrix} 2-4-2\sqrt{2} & -2-\sqrt{2} \\ -2-\sqrt{2} & 1-2-\sqrt{2} \end{bmatrix}$$

$$\mathbf{K} - \omega_2^2\mathbf{M} = ml^2\omega_0^2 \begin{bmatrix} -2-2\sqrt{2} & -2-\sqrt{2} \\ -2-\sqrt{2} & -1-\sqrt{2} \end{bmatrix}$$

$$\mathbf{K} - \omega_2^2\mathbf{M} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega_2 t} = 0 \rightarrow$$

$$(-2-2\sqrt{2})a_1 - (2+\sqrt{2})a_2 = 0$$

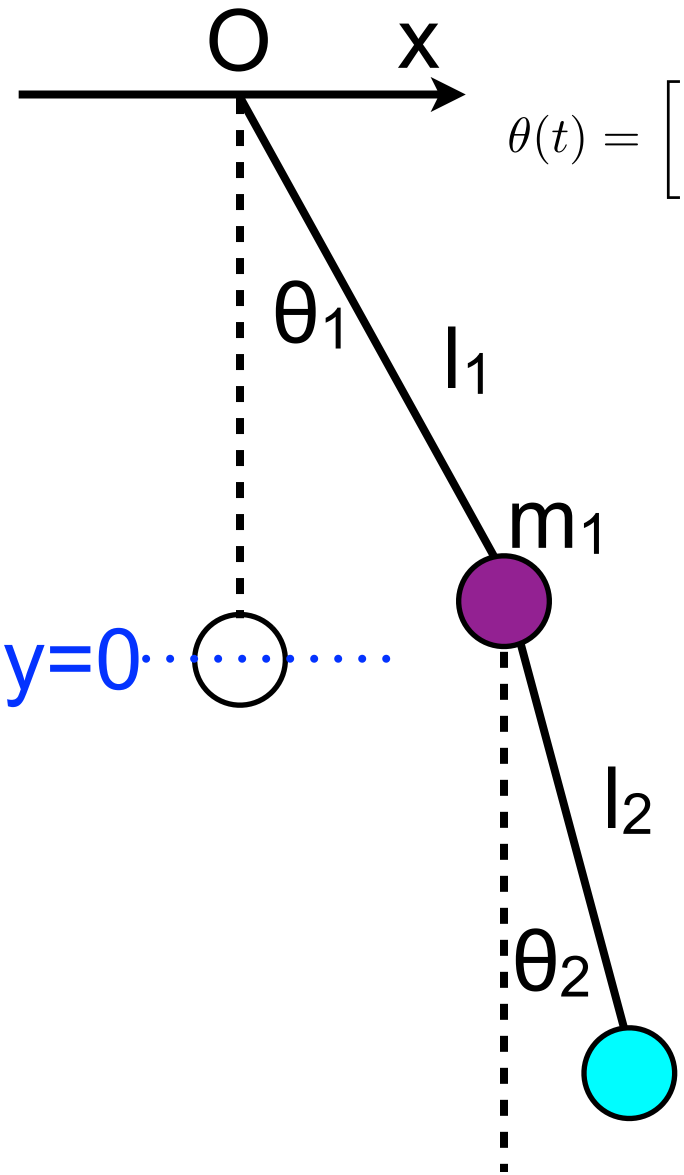
$$a_2 = \frac{-2-2\sqrt{2}}{2+\sqrt{2}}a_1$$

$$a_2 = \frac{(-2-2\sqrt{2})(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}a_1$$

$$a_2 = \frac{-4+2\sqrt{2}-4\sqrt{2}+4}{2}a_1$$

$$a_2 = -\sqrt{2}a_1$$

Looking for the second normal mode



$$\theta(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix} = \text{Re } \mathbf{z}(t) = \text{Re } \mathbf{a} e^{i\omega_2 t} = \text{Re} \begin{bmatrix} a_1 \\ -\sqrt{2} a_1 \end{bmatrix} e^{i\omega_2 t}$$

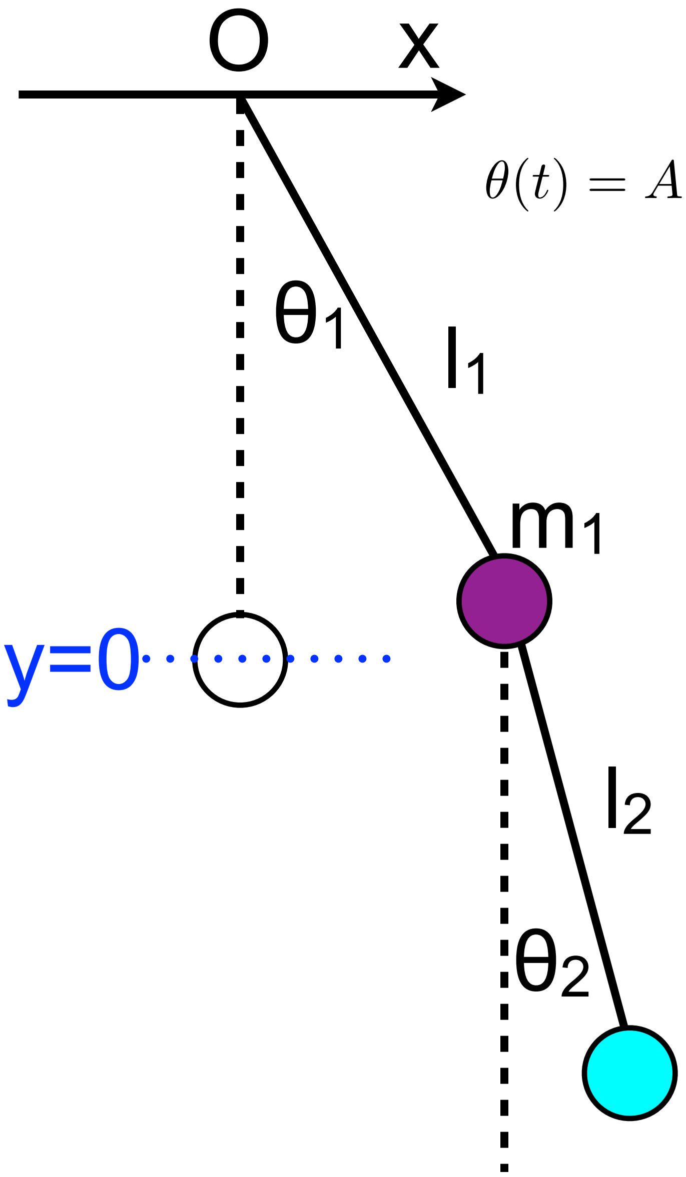
$$a_1(t) = A_2 e^{-i\delta_2}$$

$$\theta(t) = A_2 \text{ Re } e^{-i\delta_2} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} e^{i\omega_2 t}$$

$$\theta(t) = A_2 \cos(\omega_2 t - \delta_2) \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$$

In second normal mode, both pendulums oscillate out of phase, with second pendulum having amplitude sqrt(2) of the first one

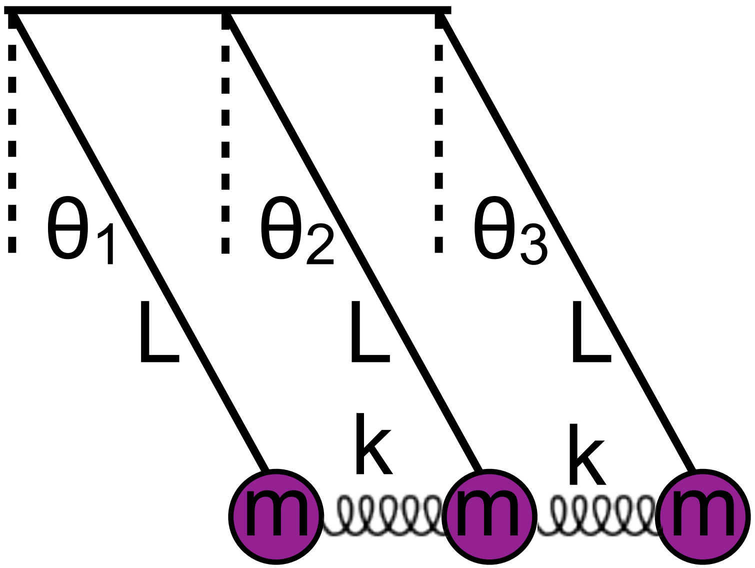
And as before



$$\theta(t) = A_1 \cos(\omega_1 t - \delta_1) \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} + A_2 \cos(\omega_2 t - \delta_2) \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$$

Generic solution is any arbitrary linear combination of the two solutions

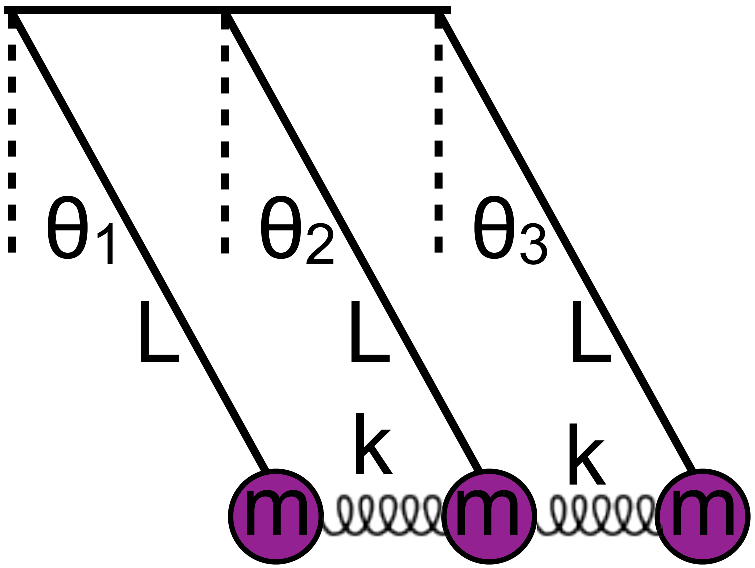
On to another system



Three equal mass attached via pendulums of equal length, connected via two springs of the same spring constant. What is the motion?

We will begin already knowing that we have to make small-angle approximations to find a solution

Three coupled pendulums



As before, expand cos(x) small ~1-x²/2

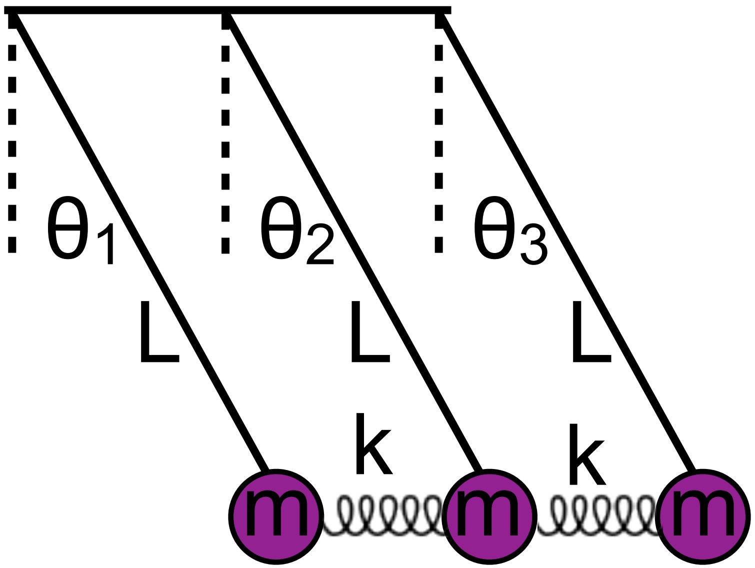
$$\mathcal{L} = T - U$$

$$T = \frac{mL^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2)$$

$$U_{grav} = mgL\left[(1 - \cos\theta_1) + (1 - \cos\theta_2) + (1 - \cos\theta_3)\right]$$

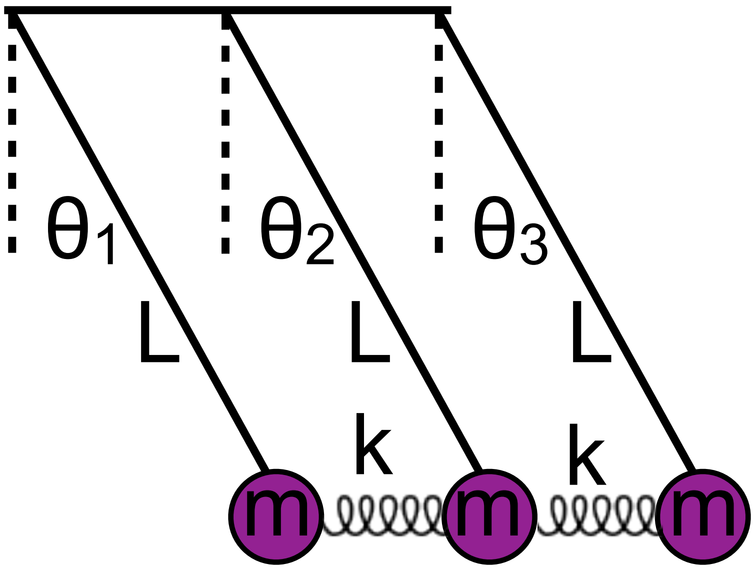
$$U_{grav} \sim \frac{mgL}{2}(\theta_1^2 + \theta_2^2 + \theta_3^2)$$

Three coupled pendulums



What about the spring PEs?

Three coupled pendulums



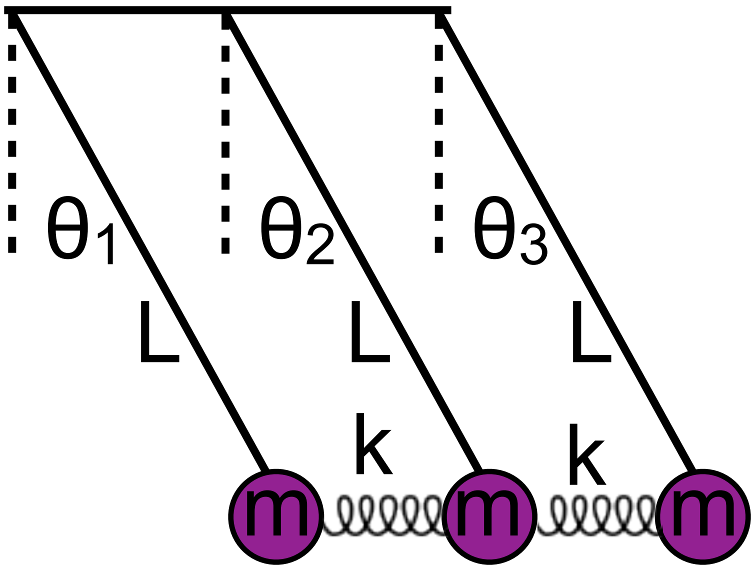
To first order, pendulums don't move in y direction, only x direction, and change is l sin(θ) ~ lθ. So ...

$$U_{spring} \sim \frac{k}{2}((x_2 - x_1)^2 + (x_3 - x_2)^2)$$

$$U_{spring} \sim \frac{k}{2}((L\theta_2 - L\theta_1)^2 + (L\theta_3 - L\theta_2)^2)$$

$$U_{spring} \sim \frac{kL^2}{2}(\theta_1^2 + 2\theta_2^2 + \theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3)$$

Putting it together



$$\mathcal{L} = T - U_{grav} - U_{spring} =$$

$$\mathcal{L} = \frac{mL^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) - \frac{kL^2}{2}(\theta_1^2 + 2\theta_2^2 + \theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3) - \frac{mgL}{2}(\theta_1^2 + \theta_2^2 + \theta_3^2)$$

The equations

$$\mathcal{L} = \frac{mL^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) - \frac{kL^2}{2}(\theta_1^2 + 2\theta_2^2 + \theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3) - \frac{mgL}{2}(\theta_1^2 + \theta_2^2 + \theta_3^2)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -kL^2\theta_1 + kL^2\theta_2 - mgl\theta_1$$

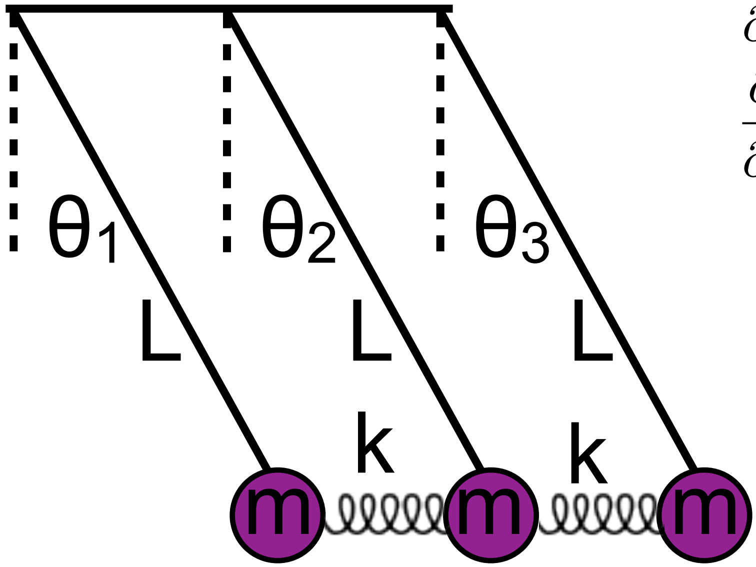
$$\frac{\partial \mathcal{L}}{\partial \theta_2} = -2kL^2\theta_2 + kL^2\theta_1 + kL^2\theta_3 - mgl\theta_2$$

$$\frac{\partial \mathcal{L}}{\partial \theta_3} = -kL^2\theta_3 + kL^2\theta_2 - mgl\theta_3$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = mL^2\dot{\theta}_1$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = mL^2\dot{\theta}_2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_3} = mL^2\dot{\theta}_3$$



The equations

$$mL^2\ddot{\theta}_1 = -kL^2\theta_1 + kL^2\theta_2 - mgl\theta_1$$

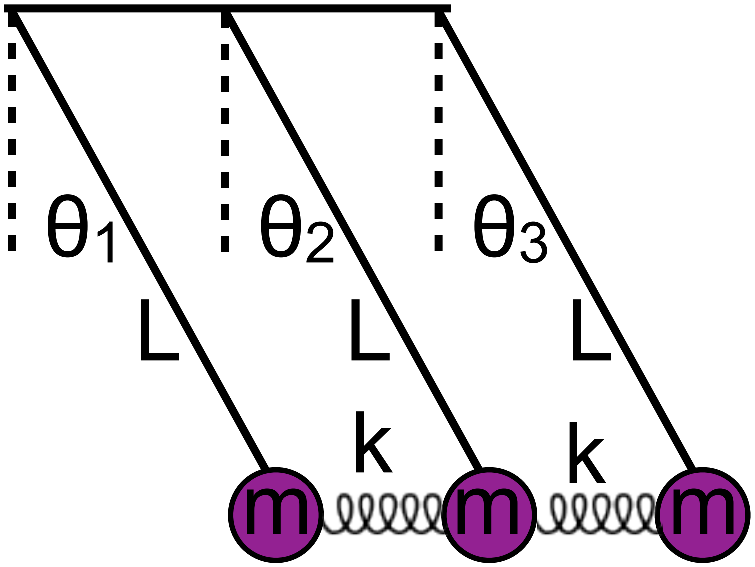
$$mL^2\ddot{\theta}_2 = -2kL^2\theta_2 + kL^2\theta_1 + kL^2\theta_3 - mgl\theta_2$$

$$mL^2\ddot{\theta}_3 = -kL^2\theta_3 + kL^2\theta_2 - mgl\theta_3$$

$$\mathbf{M}\ddot{\theta} = -\mathbf{K}\theta$$

$$\mathbf{M} = mL^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} kL^2 + mgL & -kL^2 & 0 \\ -kL^2 & 2kL^2 + mgL & -kL^2 \\ 0 & -kL^2 & kL^2 + mgL \end{bmatrix}$$

$$\mathbf{K} = L^2\begin{bmatrix} k + m\omega_0^2 & -k & 0 \\ -k & 2k + m\omega_0^2 & -k \\ 0 & -k & k + m\omega_0^2 \end{bmatrix}, \omega_0^2 = g/L$$



As usual, we look for the same oscillatory motion, and thus the same sort of solution to the eigenvalue equation

Determinant of 3x3 matrix

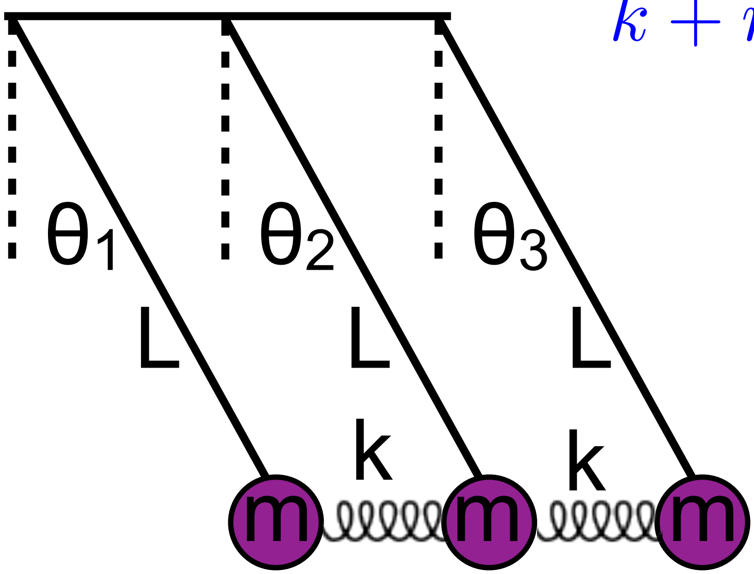
$$\left|\mathbf{K} - \omega^2\mathbf{M}\right| = L^2 \begin{vmatrix} k + m\omega_0^2 - m\omega^2 & -k & 0 \\ -k & 2k + m\omega_0^2 - m\omega^2 & -k \\ 0 & -k & k + m\omega_0^2 - m\omega^2 \end{vmatrix} = 0$$

$$(k + m\omega_0^2 - m\omega^2)\left((2k + m\omega_0^2 - m\omega^2)(k + m\omega_0^2 - m\omega^2) - k^2\right) - (-k)(-k(k + m\omega_0^2 - m\omega^2)) = 0$$

$$(k + m\omega_0^2 - m\omega^2)\left((2k + m\omega_0^2 - m\omega^2)(k + m\omega_0^2 - m\omega^2) - k^2 - k^2\right) = 0$$

One solution:

$$k + m\omega_0^2 - m\omega^2 = 0 \rightarrow \omega^2 = \omega_0^2 + k/m$$



Finding other eigenvalues

$$(2k + m\omega_0^2 - m\omega^2)(k + m\omega_0^2 - m\omega^2) - 2k^2 = 0$$

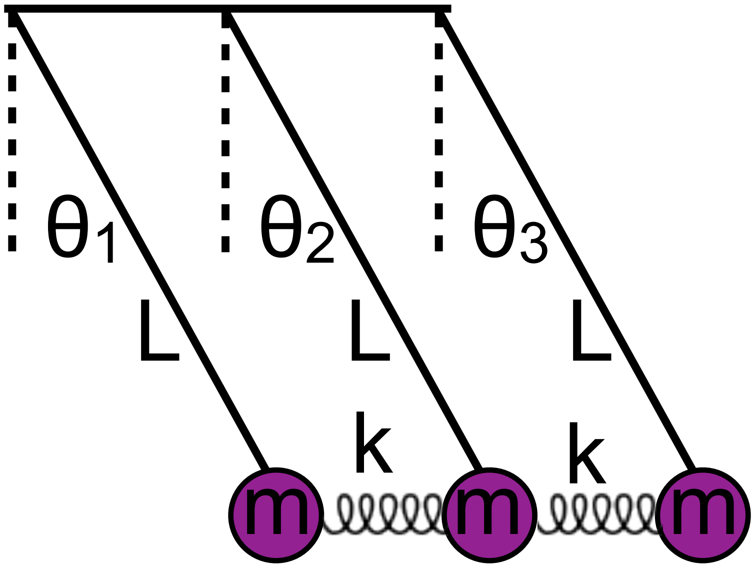
$$\omega^4(m^2) + \omega^2(-mk - m^2\omega_0^2 - 2km - m^2\omega_0^2) + (2km\omega_0^2 + km\omega_0^2 + m^2\omega_0^4) = 0$$

$$m^2\omega^4 + \omega^2(-3km - 2m^2\omega_0^2) + (3km\omega_0^2 + m^2\omega_0^4) = 0$$

$$\omega^2 = \frac{3km + 2m^2\omega_0^2 \pm \sqrt{(-3km - 2m^2\omega_0^2)^2 - 4m^2(3km\omega_0^2 + m^2\omega_0^4)}}{2m^2}$$

$$\omega^2 = \frac{3km + 2m^2\omega_0^2 \pm \sqrt{9k^2m^2 + 4m^2\omega_0^4 + 12km^3\omega_0^2 - 12km^3\omega_0^2 - 4m^4\omega_0^4}}{2m^2}$$

$$\omega^2 = \omega_0^2 + \frac{3k \pm 3k}{2m}$$



$$\omega_1^2 = \omega_0^2$$

$$\omega_2^2 = \omega_0^2 + \frac{k}{m}$$

From before

$$\omega_3^2 = \omega_0^2 + \frac{3k}{m}$$

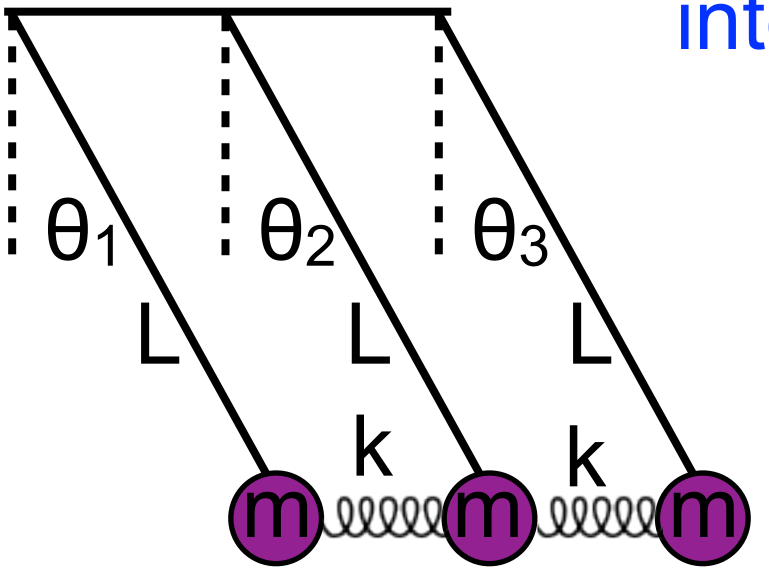
Finding the eigenvectors (first)

$$\left[\mathbf{K}-\omega_1^2\mathbf{M}\right]\mathbf{a} = L^2 \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

$$ka_1 - ka_2 = 0 \rightarrow a_1 = a_2$$

$$-ka_2 + ka_3 = 0 \rightarrow a_3 = a_2 = a_1$$

First eigenvector: all pendulums oscillate in unison. Physical interpretation?



$$\omega_1^2 = \omega_0^2$$

$$\omega_2^2 = \omega_0^2 + \frac{k}{m}$$

$$\omega_3^2 = \omega_0^2 + \frac{3k}{m}$$

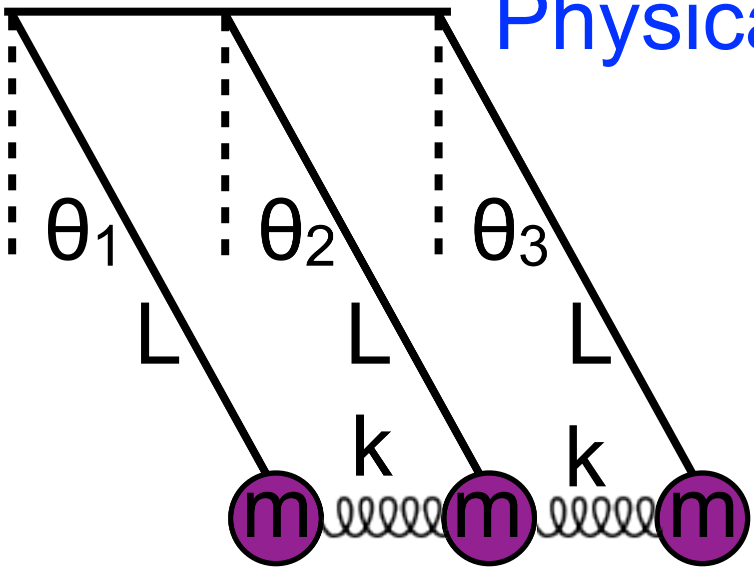
Finding the eigenvectors (second)

$$\left[\mathbf{K} - \omega_2^2 \mathbf{M}\right] \mathbf{a} = L^2 \begin{bmatrix} 0 & -k & 0 \\ -k & k & -k \\ 0 & -k & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

$$-ka_2 = 0 \rightarrow a_2 = 0$$

$$-ka_1 + ka_2 - ka_3 = 0 \rightarrow a_1 = -a_3$$

Second eigenvector: middle pendulum at rest, outer ones oscillate out of phase. Physical interpretation?



$$\omega_1^2 = \omega_0^2$$

$$\omega_2^2 = \omega_0^2 + \frac{k}{m}$$

$$\omega_3^2 = \omega_0^2 + \frac{3k}{m}$$

Finding the eigenvectors (third)

$$\left[\mathbf{K} - \omega_2^2 \mathbf{M}\right] \mathbf{a} = L^2 \begin{bmatrix} -2k & -k & 0 \\ -k & -k & -k \\ 0 & -k & -2k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

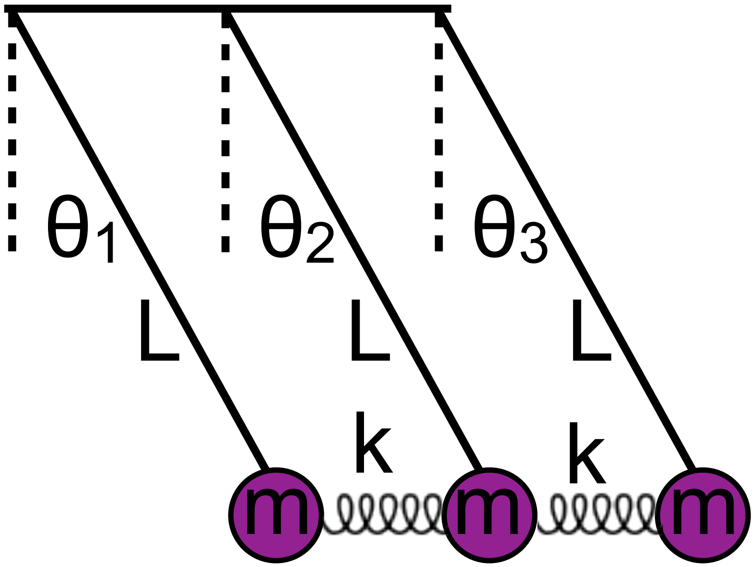
$$-2ka_1 - ka_2 = 0 \rightarrow a_1 = -a_2/2$$

$$-ka_1 - ka_2 - ka_3 = 0$$

$$-ka_2 - 2ka_3 = 0 \rightarrow a_3 = -a_2/2$$

$$\rightarrow a_1 = a_3 = -a_2/2$$

Third eigenvector: middle pendulum out of phase with outer ones, with twice the amplitude



$$\omega_1^2 = \omega_0^2$$

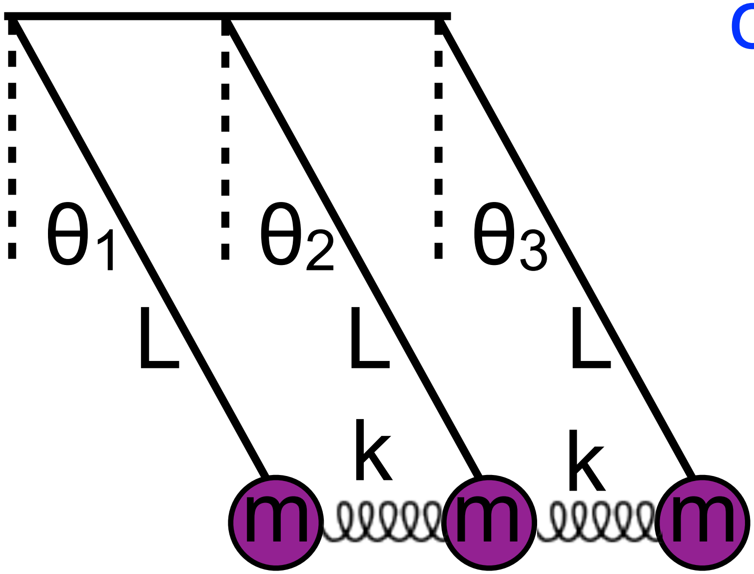
$$\omega_2^2 = \omega_0^2 + \frac{k}{m}$$

$$\omega_3^2 = \omega_0^2 + \frac{3k}{m}$$

And as always

$$\theta(t) = A_1 \cos(\omega_1 t - \delta_1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + A_2 \cos(\omega_2 t - \delta_2) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + A_3 \cos(\omega_3 t - \delta_3) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Generic solution is any arbitrary linear combination of the three solutions



$$\omega_1^2 = \omega_0^2$$

$$\omega_2^2 = \omega_0^2 + \frac{k}{m}$$

$$\omega_3^2 = \omega_0^2 + \frac{3k}{m}$$

Let's do one more problem together if there's time

Taylor 11.19

Homework, due in 1 week, as usual

Another computational problem - let's look at it now