# Chapter 12

(Some material taken from McGraw-Hill/College Physics/Giambattista, the majority from your Katz textbook)

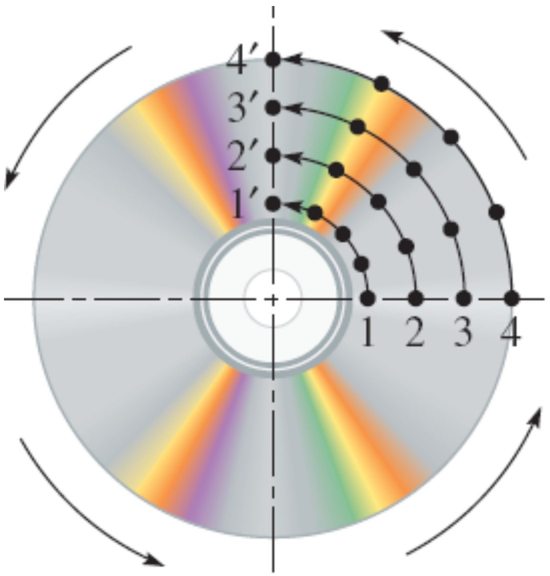
## Rotation of a Rigid Body

To describe circular motion, we **could** use the previous definitions of displacement, velocity, and acceleration from past chapters.

But much of the circular motion around us occurs in the rotation of a rigid object.

A **rigid body** is one for which the distance between any two points of the body remains the same when the body is translated or rotated (in other words, a misshapen brick is a rigid body, and so is putty as long as it does not get deformed)

**WE ALREADY SAW THIS, SO JUST A REVIEW…**




When a DVD spins inside a DVD player, different points on the DVD have different velocities and accelerations.

The velocity and acceleration of a given point keep changing direction as the DVD spins.

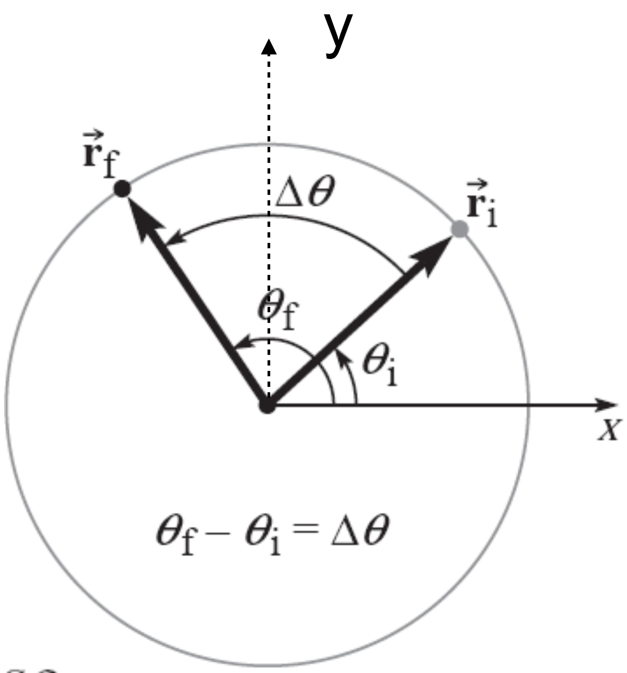
It is much simpler to say "the DVD spins at 210 rpm" than to say "a point 6.0 cm from the rotation axis of the DVD is moving at 1.3 m/s."

One point on the CD **does not move.** Which one?

## Angular Displacement and Angular Velocity

To simplify the description of circular motion, we concentrate on *angles* instead of distances.

Instead of displacement, we speak of **angular displacement $\Delta\theta$** , angle through which the DVD turns.



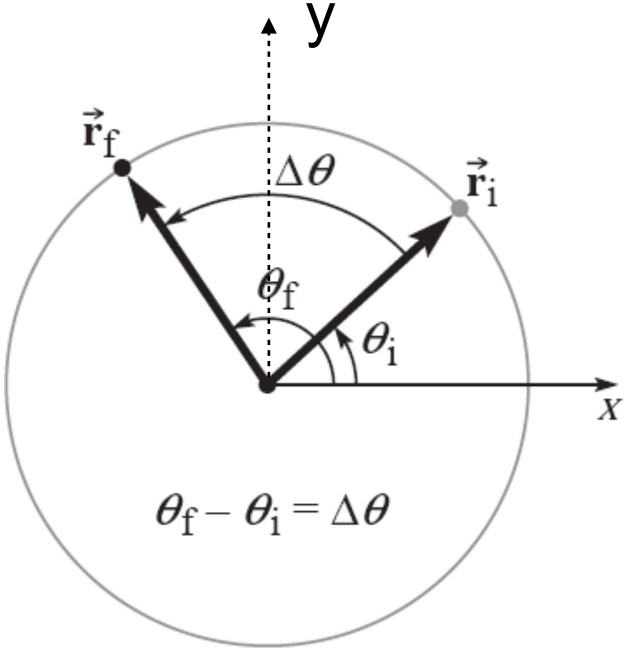

Not our normal usage (as in a clock), so be careful!
Also, what happens if we view the spinning disc from the other side?

+ Counterclockwise

– Clockwise

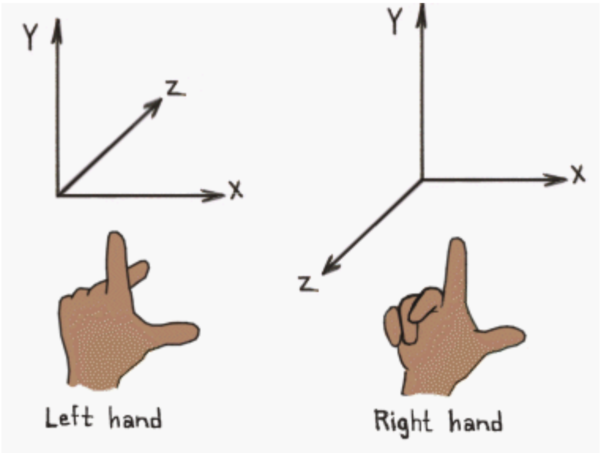
## Axis of rotation

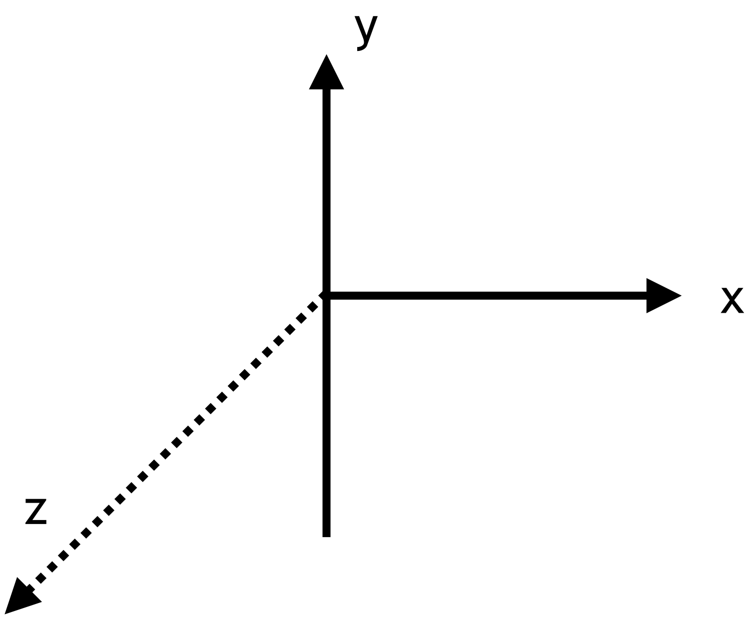
How do we define the rotation here? Along what axis? We choose an axis **perpendicular** to the direction of motion. Here we move in the x-y plane, so we can define the axis of rotation along the z axis. But in which direction? Into or out of the board?



If we draw x and y axes in the normal way, which way does the z axis go?

**Right-hand rule to the rescue!**
**Let's take a close look**

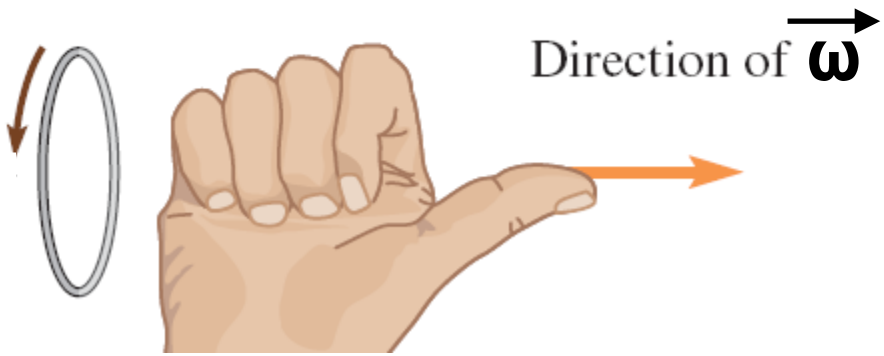


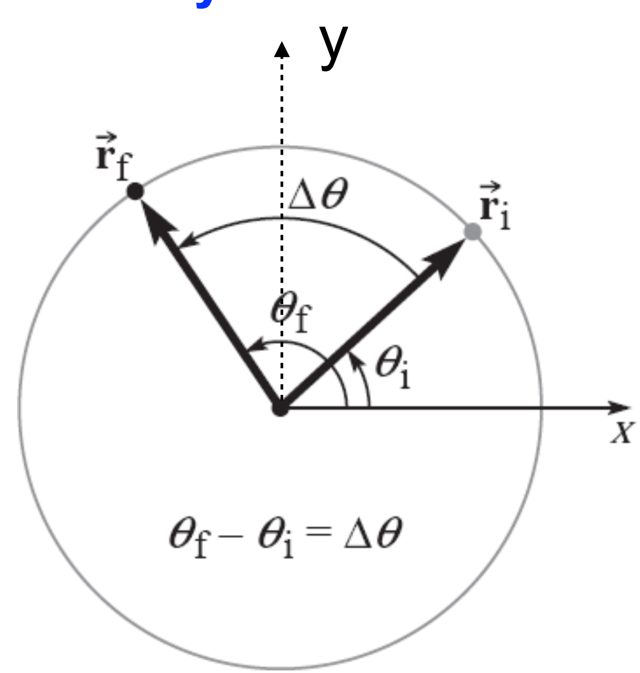

https://blenderartists.org/t/why-the-y-axis/640720/4

## Axis of rotation

So the +z axis is out of the board, towards you.




**So the axis of rotation is +z axis.**
**Do we see why?**




Curl your fingers in the direction of rotation with your **RIGHT HAND** and your thumb points along the rotation axis

Hopefully these look familiar to you in some way?

## Definition of angular displacement:

$$\Delta\theta = \theta_{\mathrm{f}} - \theta_{\mathrm{i}}$$

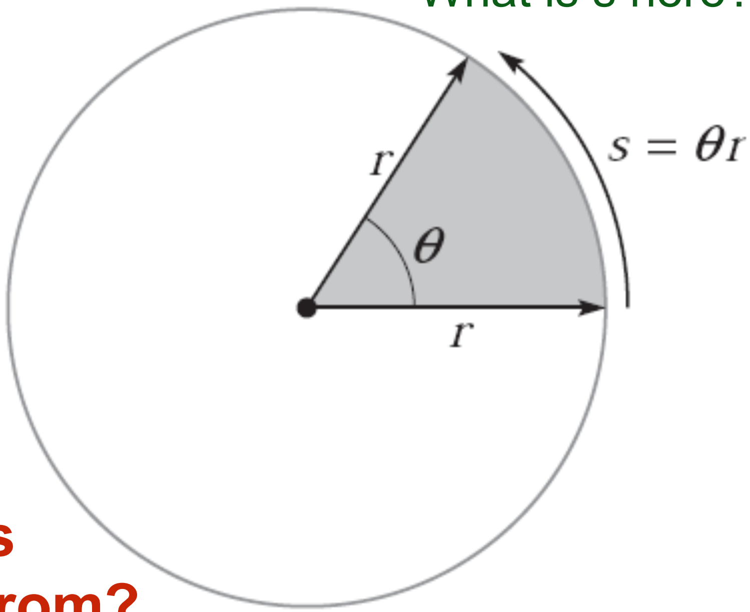
## Definition of average angular velocity:

$$\omega_{\mathrm{av}} = \frac{\Delta\theta}{\Delta t}$$

## Definition of instantaneous angular velocity:

$$\omega = \frac{d\theta}{dt}$$

What is s here?




## Radian Measure

In many situations the most convenient angle measure is the **radian**.

$$\theta \text{ (in radians)} = \frac{s}{r}$$

**Where does this come from?**

**For full circle:** $\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \text{ rad}$

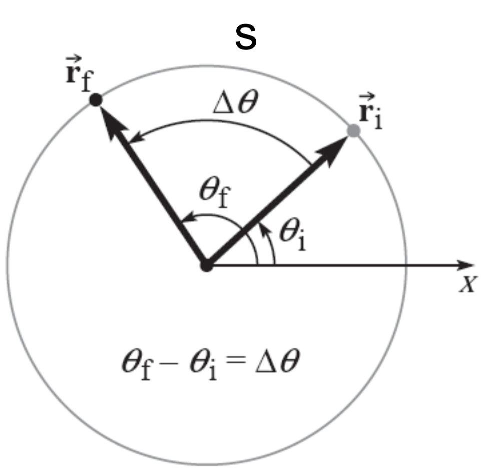
$$360° = 2\pi \text{ rad}$$

**Be careful about which angular units you're using! (Especially on a calculator)**

## Relation Between Linear and Angular Speed

For a point moving in a circular path of radius *r* , the linear distance traveled along the circular path during an angular displacement of Δθ (in radians) is the arc length *s* where

$$s = r|\Delta\theta| = r|\theta_f - \theta_i| \quad \text{(angles in radians)}$$



## Relation Between Linear and Angular Speed

For a point moving in a circular path of radius *r* ,

$$s = r|\Delta\theta| = r|\theta_f - \theta_i| \quad \text{(angles in radians)}$$

$$v_{av} = \frac{s}{\Delta t} = \frac{r|\Delta\theta|}{\Delta t} \quad (\Delta\theta \text{ in radians})$$

Taking the limit as $\Delta t$ approaches zero (ie the derivative),

$$v = r\frac{d\theta}{dt} = r\omega$$

## Period and Frequency

When the speed of a point moving in a circle is constant, its motion is called **uniform circular motion**.

The time for the point to travel completely around the circle is called the **period** of the motion, *T*.

The **frequency (f)** of the motion, which is the number of revolutions per unit time, is defined as

$$f = \frac{1}{T}$$

## Period and Frequency

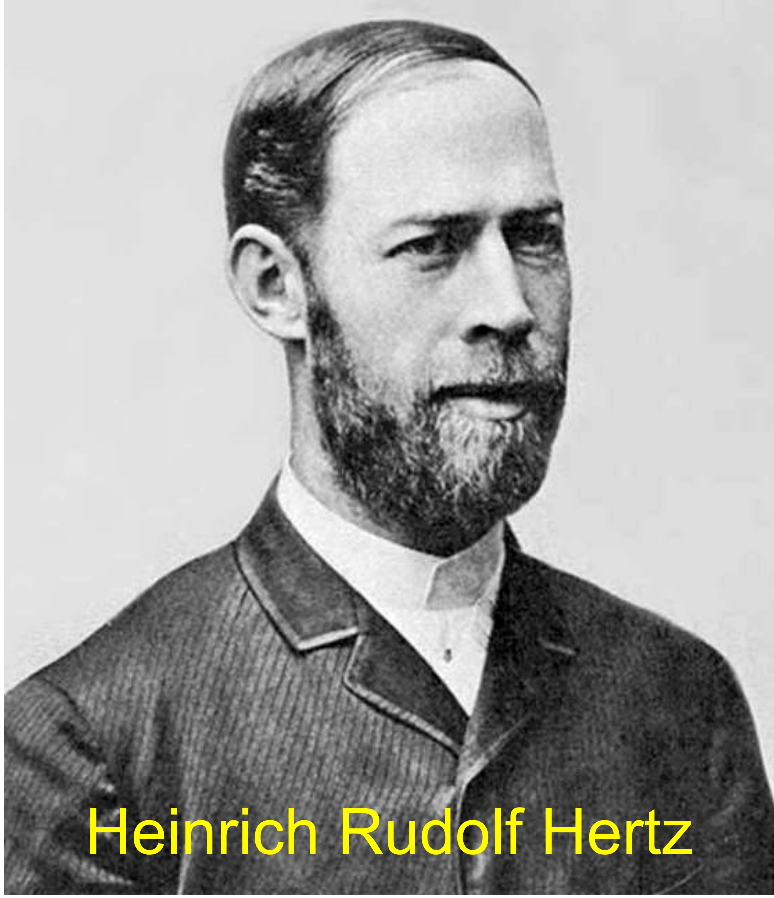
The speed is the total distance traveled divided by the time taken,

$$v = \frac{2\pi r}{T} = 2\pi r f$$

Then, for uniform circular motion

$$|\omega| = \frac{v}{r} = 2\pi f \quad (\omega \text{ in radians per unit time})$$

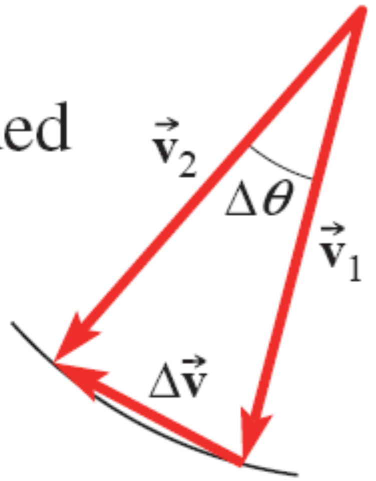
where, in SI units, angular velocity *w* is measured in rad/s and frequency *f* is measured in hertz (Hz).



## Magnitude of the Radial Acceleration

$$|\Delta\vec{\mathbf{v}}| = \text{arc length} = \text{radius of circle} \times \text{angle subtended}$$
$$= v\,|\Delta\theta| = v\,|\omega|\,\Delta t$$

**Here only talking about radial acceleration (moving in a circle)**




$$a_r = |\vec{a}| = \frac{|d\vec{v}|}{dt} = v|\omega|$$

$$v = |\omega| r$$

$$a_\text{r} = \frac{v^2}{r} \quad \text{or} \quad a_\text{r} = \omega^2 r \quad (\omega \text{ in radians per unit time})$$

## Average angular acceleration:

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

## Instantaneous angular acceleration:

$$\alpha = \frac{d\omega}{dt}$$

$$v_t = r\,|\omega|$$

Our normal tangential velocity vs angular speed relationship

**TANGENTIAL acceleration**

$$a_t = \frac{dv_t}{dt} = r\left|\frac{d\omega}{dt}\right|$$

$$a_t = r\,|\alpha|$$

Relate tangential acceleration to angular acceleration!

A centrifuge is spinning at 5400 rpm.

(a) Find the period (in s) and frequency (in Hz) of the motion.
(b) If the radius of the centrifuge is 14 cm, how fast (in m/s) is an object at the outer edge moving?

**Solution**

(a)

$$f = 5400\ \frac{\text{rev}}{\text{min}} \times \frac{1\ \text{min}}{60\ \text{s}} = 90\ \text{rev/s}$$

$$f = 90\ \text{Hz} = 90\ \text{s}^{-1}$$

$$T = 1/f = 0.011\ \text{s}$$

(b)

$$|\omega| = 90\ \frac{\text{rev}}{\text{s}} \times 2\pi\ \frac{\text{rad}}{\text{rev}} = 180\pi\ \text{rad/s}$$

$$|\omega| = 2\pi f = 180\ \pi\ \text{rad/s}$$

$$v = |\omega| r = 180\pi\, \text{s}^{-1} \times 0.14\ \text{m} = 79\ \text{m/s}$$

# Relationships Between $\theta$, $\omega$, and $\alpha$ for Constant Angular Acceleration

| **Constant Acceleration Along *x*-Axis** | **Constant Angular Acceleration** |
| --- | --- |
| $\Delta v_x = v_{fx} - v_{ix} = a_x \, \Delta t$ | $\Delta\omega = \omega_f - \omega_i = \alpha \, \Delta t$ |
| $\Delta x = \frac{1}{2}(v_{fx} + v_{ix}) \, \Delta t$ | $\Delta\theta = \frac{1}{2}(\omega_f + \omega_i) \, \Delta t$ |
| $\Delta x = v_{ix} \, \Delta t + \frac{1}{2} a_x (\Delta t)^2$ | $\Delta\theta = \omega_i \, \Delta t + \frac{1}{2}\alpha (\Delta t)^2$ |
| $v_{fx}^2 - v_{ix}^2 = 2a_x \, \Delta x$ | $\omega_f^2 - \omega_i^2 = 2\alpha \, \Delta\theta$ |

**Analogous equations for linear and angular motion!**
**Hopefully these look very familiar to you by now**

A potter's wheel rotates from rest to 210 rpm in a time of 0.75 s.

(a) What is the angular acceleration of the wheel during this time, assuming constant angular acceleration?

(b) How many revolutions does the wheel make during this time interval?

(c) Find the tangential and radial components of the acceleration of a point 12 cm from the rotation axis when the wheel is spinning at 180 rpm.

# Solution

(a)

$$\omega_i = 0 \text{ rad/s}$$

$$\omega_f = 210 \frac{\text{rev}}{\text{min}} \times \frac{1}{60} \frac{\text{min}}{\text{s}} \times 2\pi \frac{\text{rad}}{\text{rev}} = 7.0\pi \text{ rad/s}$$

$$\alpha = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{7.0\pi \text{ rad/s} - 0}{0.75 \text{ s} - 0} = \frac{7.0\pi \text{ rad/s}}{0.75 \text{ s}} = 29 \text{ rad/s}^2$$

# Solution

(b)

$$\Delta\theta = \tfrac{1}{2}(\omega_f + \omega_i)\,\Delta t = \tfrac{1}{2}(7.0\pi \text{ rad/s} + 0)(0.75 \text{ s}) = 8.25 \text{ rad}$$

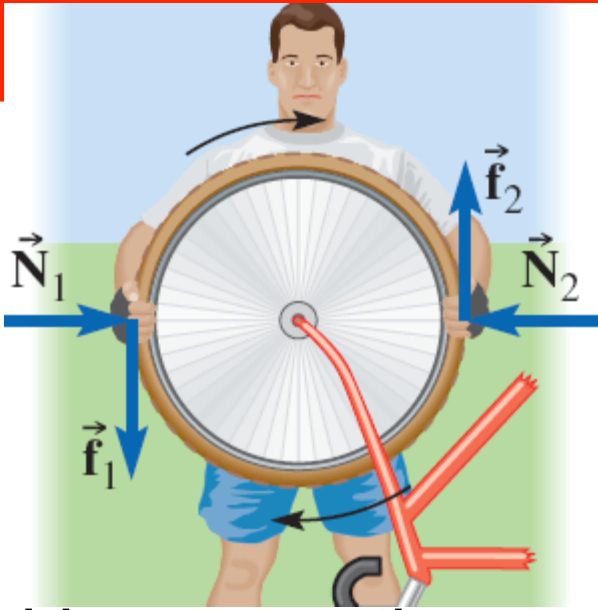
$$\frac{8.25 \text{ rad}}{2\pi \text{ rad/rev}} = 1.3 \text{ rev}$$

(c)

$$\omega = 180\,\frac{\text{rev}}{\text{min}} \times \frac{1}{60}\,\frac{\text{min}}{\text{s}} \times 2\pi\,\frac{\text{rad}}{\text{rev}} = 6.0\pi \text{ rad/s}$$

$$a_r = \omega^2 r = (6.0\pi \text{ rad/s})^2 \times 0.12 \text{ m} = 43 \text{ m/s}^2$$

$$a_t = \alpha r = 29 \text{ rad/s}^2 \times 0.12 \text{ m} = 3.5 \text{ m/s}^2$$

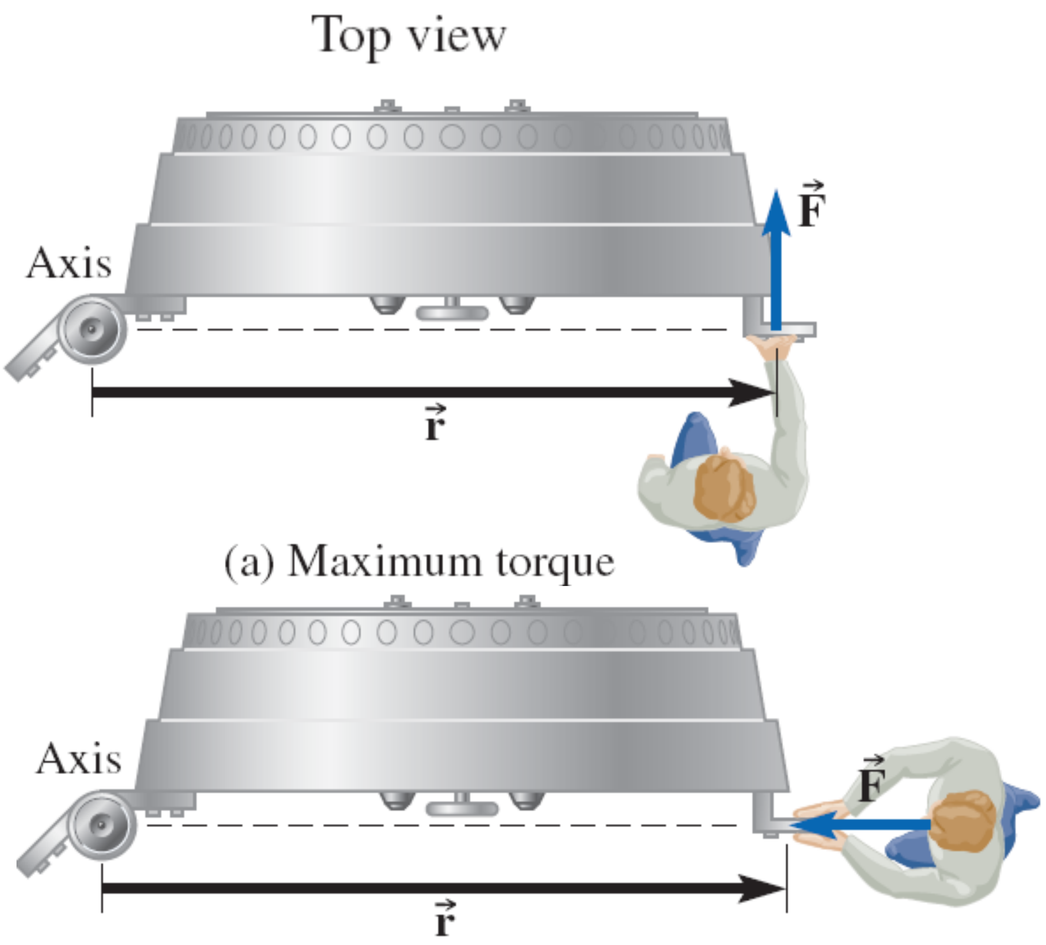



A quantity related to force, called **torque**, plays the role in rotation that force itself plays in translation.

A torque is not separate from a force; it is impossible to exert a torque without exerting a force.

Torque is a measure of how effective a given force is at twisting or turning something. For something rotating about a fixed axis such as the bicycle wheel, a torque can *change* the rotational motion either by making it rotate faster or by slowing it down.

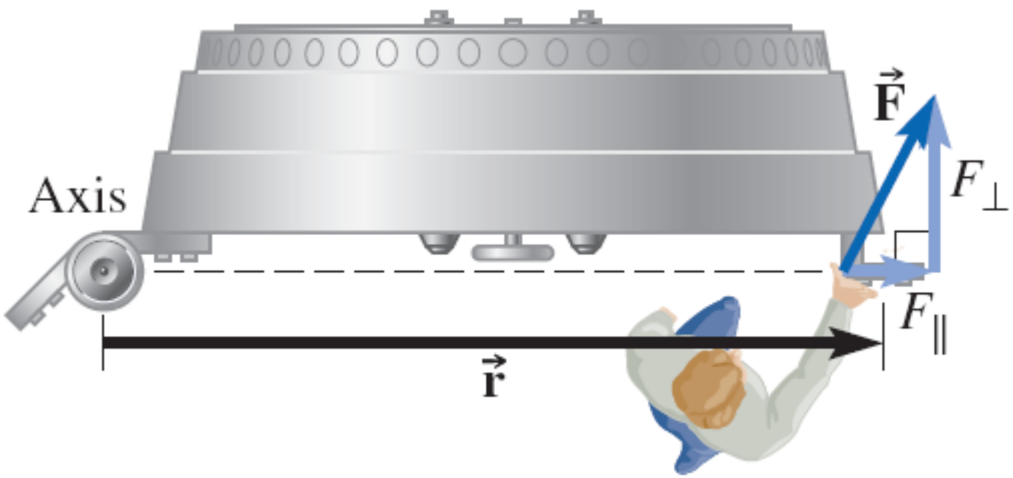
# Relationship Between Force and Torque




(a) Maximum torque

(b) Zero torque

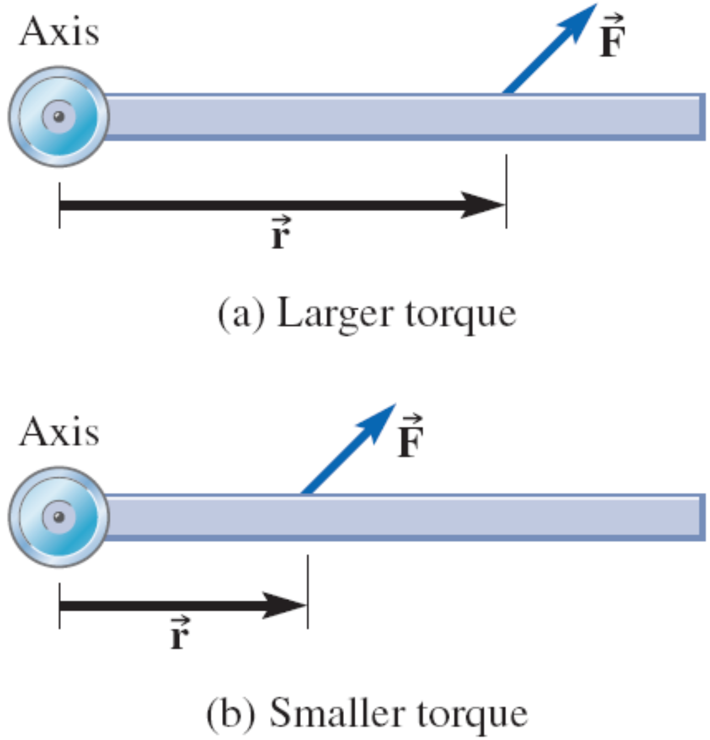
# Relationship Between Force and Torque




(c) Less torque

Torque depends on the direction of the applied force.

# Relationship Between Force and Torque




Torque is proportional to the distance between the rotation axis and the **point of application** of the force (the point at which the force is applied)

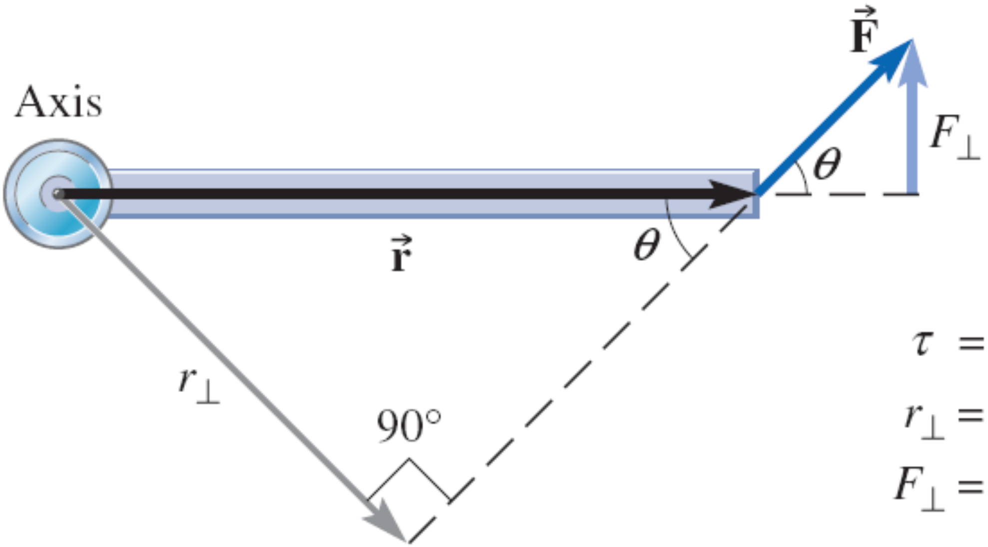
# Definition of torque:

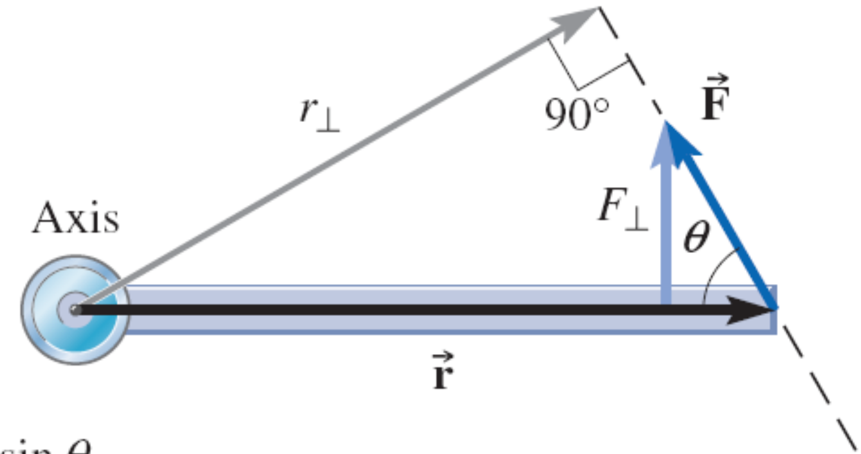
$$\tau = \pm r F_{\perp}$$

The sign of the torque is *not* determined by the sign of the angular velocity (in other words, whether the wheel is spinning counter-clockwise or clockwise); rather, it is determined by the sign of the angular ***acceleration*** the torque would cause if acting alone.

To determine the sign of a torque, imagine which way the torque would make the object begin to spin if it is initially not rotating.

## Lever Arms




$$\tau = rF \sin\theta$$

$$r_\perp = r \sin\theta$$

$$F_\perp = F \sin\theta$$

$$\tau = \pm r F_\perp = \pm r(F \sin\theta)$$

$$\tau = \pm(r \sin\theta)F$$

$$\tau = \pm r_\perp F$$

The distance $r_\perp$ is called the **lever arm** (or **moment arm**).

## Center of Gravity

We have seen that the torque produced by a force depends on the point of application of the force. What about gravity?

When we need to find the total torque due to the forces of gravity acting on an object, the total force of gravity can be considered to act at a single point. This point is called the **center of gravity**.

If the gravitational field is uniform in magnitude and direction, then the center of gravity of an object is located at the object's **center of mass**, which we just learned how to calculate a few chapters ago

The concepts of torque and rotational inertia can be used to formulate a "Newton's second law for rotation" —a law that fills the role of $\sum\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ for rotation about a fixed axis.

# Rotational form of Newton's second law:

$$\sum \tau = I\alpha$$

**Let's go over each of the individual pieces above**

The sum of the torques due to internal forces acting on a rigid object is always zero. **WHY?!**
Therefore, only *external* torques need be included in the net torque.

**Compare with:**

$$\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

I is the **rotational inertia**, a measure of an object's resistance to have angular acceleration (just like mass m is a measure of an object's resistance to be linearly accelerated)

# Challenge that we have:

$$\sum \tau = I\alpha$$

**Angular acceleration is a vector, so torque must be too!**

$$\tau = \pm rF_{\perp} = \pm r(F \sin \theta)$$

$$\tau = \pm(r \sin \theta)F$$

$$\tau = \pm r_{\perp} F$$

Torque is made up of two vectors (**r** and **F**).
How to combine to get a third vector?

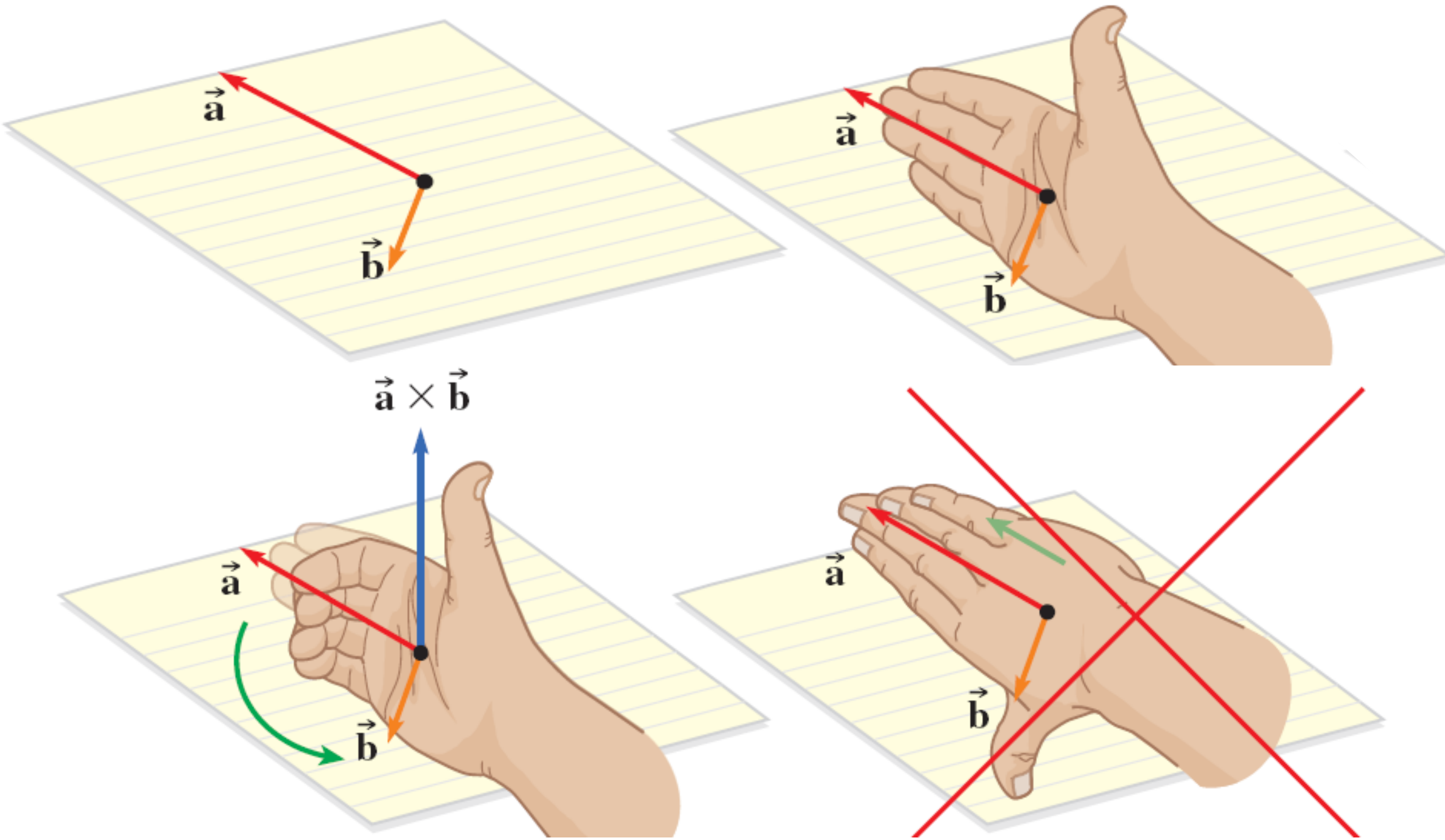
$$\vec{\tau} = \vec{r} \times \vec{F}$$

"Torque equals **r** cross **F**" (the cross product of **r** and **F**)

The cross product of two vectors, **C** = **A** x **B** is another vector
**|C| = |A|**|**B**|sin θ
What is the direction of **C**? We use a right-hand rule again

You will use cross products a lot next semester!

# The Right-Hand Rule

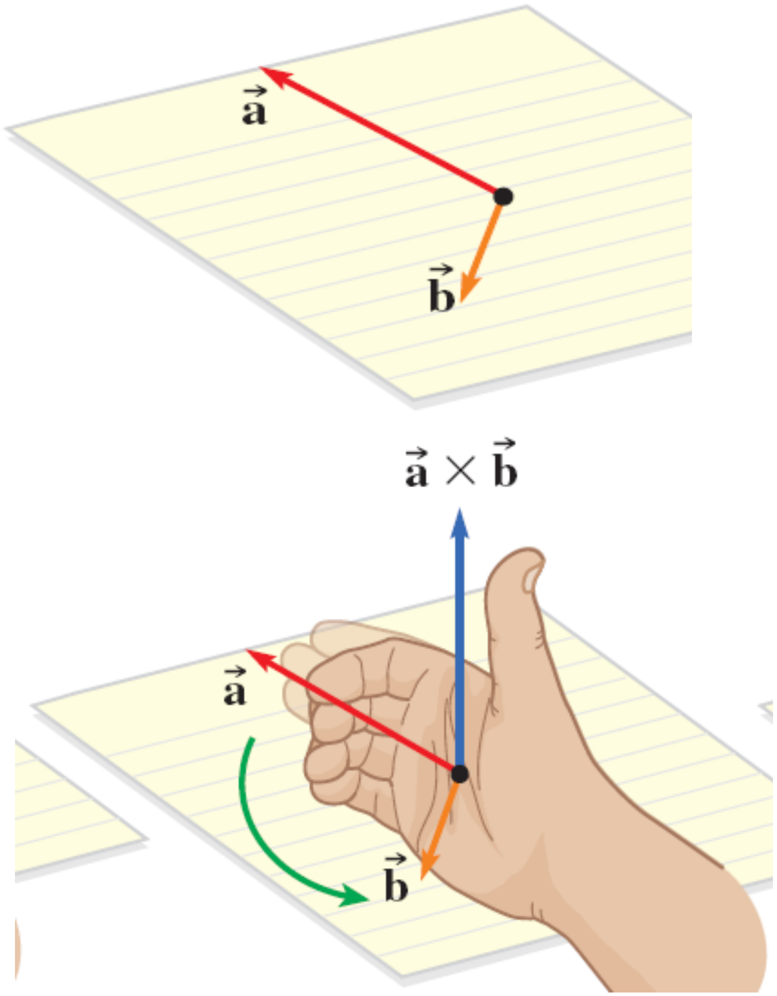



**Let's look carefully (as always, you have to use the right-hand rule, even if you are left-handed!)**

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The cross product of two vectors, **C** = **A** x **B** is another vector
What about **B** x **A**?

Let's check with the right-hand rule…

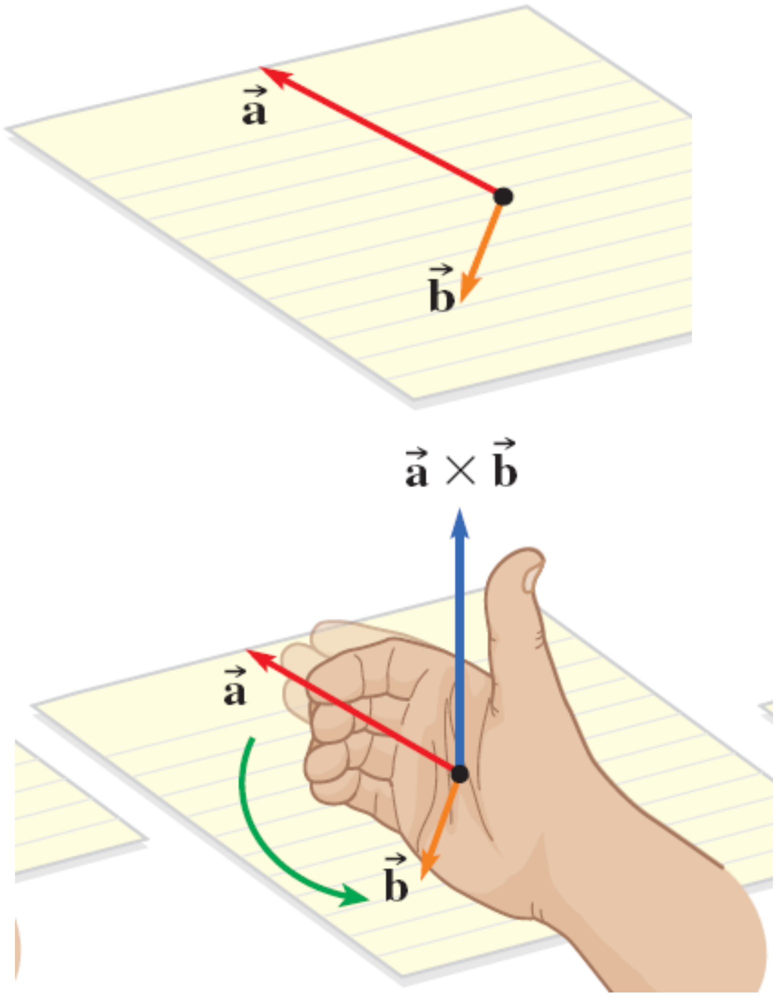


$$\vec{\tau} = \vec{r} \times \vec{F}$$

The cross product of two vectors, **C** = **A** x **B** is another vector

|**C**| = |**A**||**B**|sin θ

What if **A** and **B** are parallel? Anti-parallel? Perpendicular to one another?



A grinding wheel is a solid, uniform disk of mass 2.50 kg and radius 9.00 cm.

Starting from rest, what constant torque must a motor supply so that the wheel attains a rotational speed of 126 rev/s in a time of 6.00 s?

# Solution

**Look this up:** $I = \frac{1}{2}mr^2$

**I:** $\frac{1}{2} \times 2.50\ \text{kg} \times (0.0900\ \text{m})^2 = 0.010125\ \text{kg}\cdot\text{m}^2$

$$\omega = 126\ \frac{\text{rev}}{\text{s}} \times 2\pi \frac{\text{rad}}{\text{rev}}$$

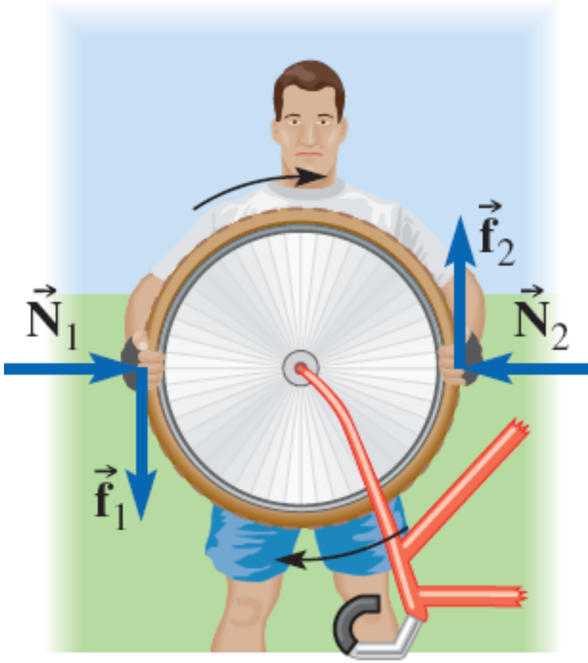
$$\alpha = \frac{\Delta\omega}{\Delta t}$$

$$\sum \tau = I\alpha = I\frac{\Delta\omega}{\Delta t}$$

$$= 0.010125\ \text{kg}\cdot\text{m}^2 \times \frac{126\ \text{rev/s} \times 2\pi\ \text{rad/rev}}{6.00\ \text{s}}$$

**Check units:** $= 1.34\ \text{N}\cdot\text{m}$

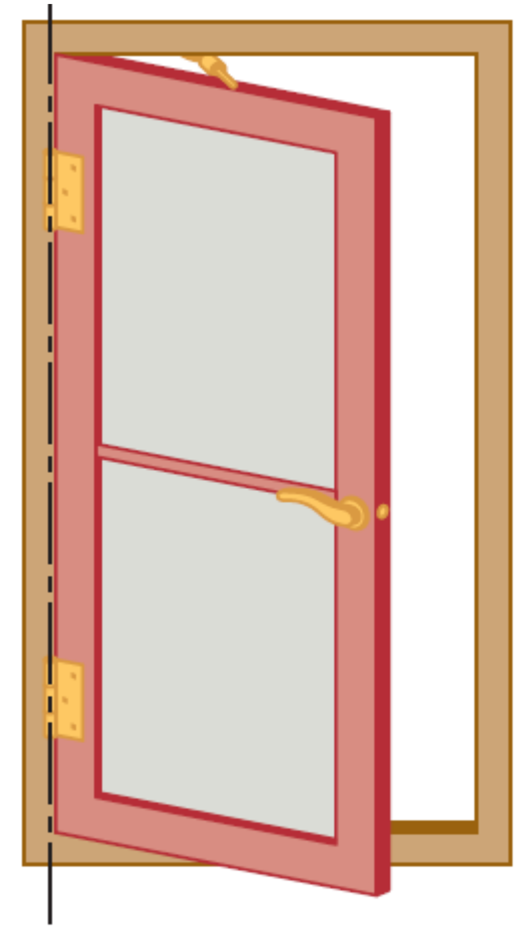
Group work!
https://forms.gle/p2rkzEJ58tPqEWrX7

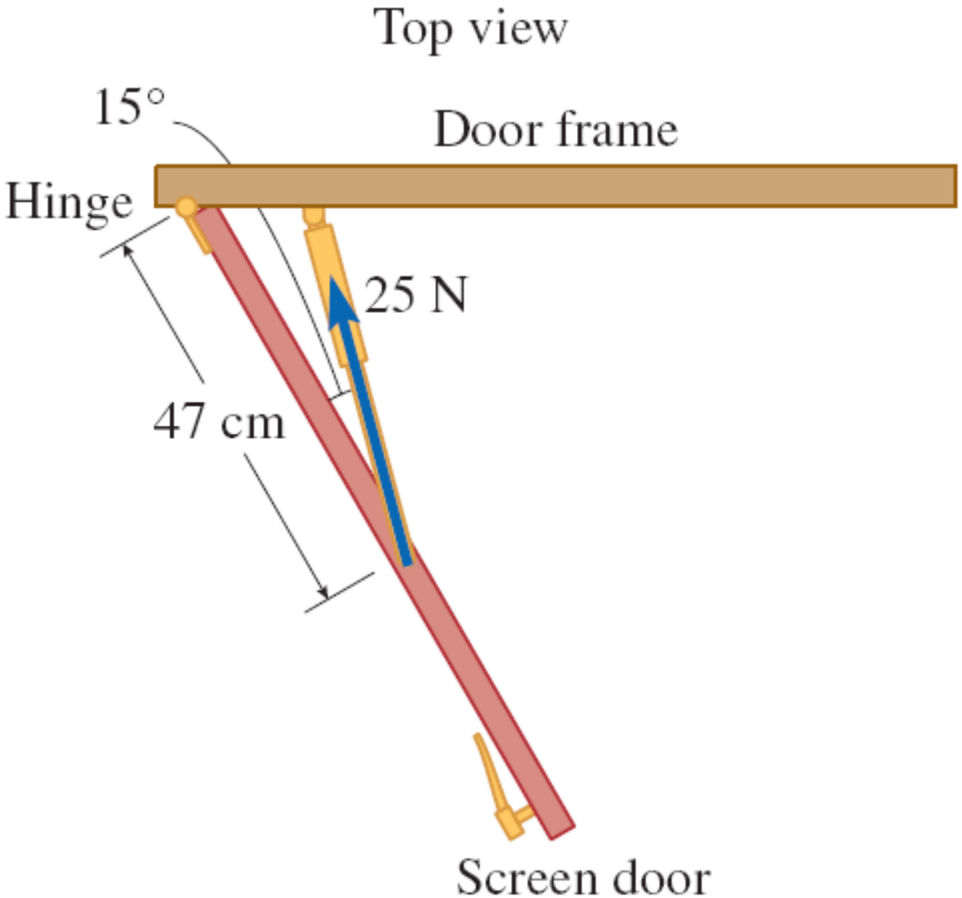



To stop a spinning bicycle wheel, suppose you push radially inward on opposite sides of the wheel with equal forces of magnitude 10.0 N.

The radius of the wheel is 32 cm and the coefficient of kinetic friction between the tire and your hand is 0.75. The wheel is spinning in the clockwise sense. What is the net torque on the wheel?

An automatic screen door closer attaches to a door 47 cm away from the hinges and pulls on the door with a force of 25 N, making an angle of 15° with the door.



An automatic screen door closer attaches to a door 47 cm away from the hinges and pulls on the door with a force of 25 N, making an angle of 15° with the door (as in the figure on the last slide.)

Find the magnitude of the torque exerted on the door due to this force about the rotation axis through the hinges using

(a) the perpendicular component of the force and
(b) the lever arm
(c) What is the sign of this torque as viewed from above?

# PHYS253 Chapter 12

Your hard drive is spinning at 7200 RPM. The computer senses that it needs to shut down and decelerates the disk at a constant value of 1000 radians/$s^2$. How long does it take for the disk to come to rest?

Your hard drive is spinning at 7200 RPM. At time t=0, the computer starts constant deceleration at 1000 radians/$s^2$, how far has the disk spun in 0.1 seconds?

During normal operation, a computer's hard disk spins at 7200 rpm. If it takes the hard disk 4.0 s to reach this angular velocity starting from rest, what is the average angular acceleration of the hard disk in rad/$s^2$?

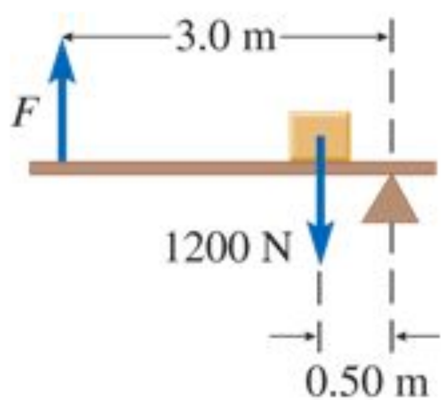
a) Find the average angular speed of the second hand of an analog clock.

b) What is its angular displacement during 5.0s?

Why is it easier to push open a swinging door from near the edge away from the hinges rather than in the middle of the door?

A child of mass 40.0 kg is sitting on a horizontal seesaw at a distance of 2.0 m from the supporting axis. What is the magnitude of the torque about the axis due to the weight of the child?

A weight of 1200 N rests on a level at a point 0.50 m from a support. On the same side of the support, at a distance of 3.0 m from it, an upward force with magnitude F is applied. Ignore the weight of the board itself. If the system is in equilibrium, what is F?



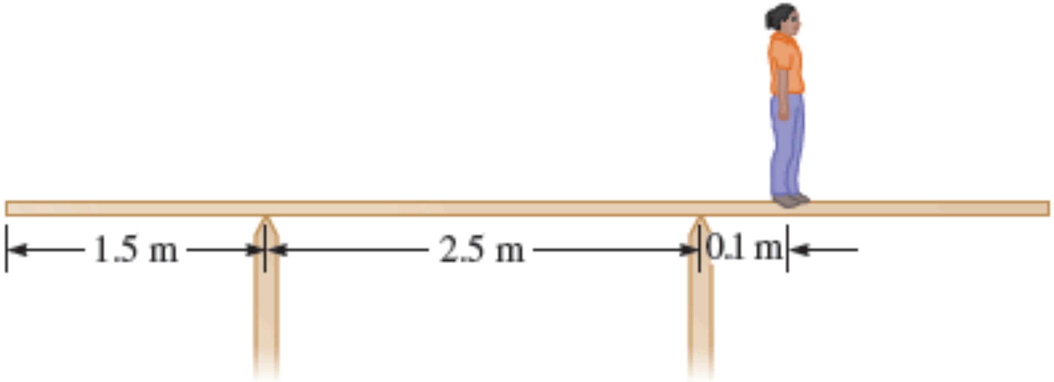
Find the cross product **A** x **B** in each case:

a) **A** = 15.0 ihat and **B** = 15.0 ihat
b) **A** = 15.0 ihat and **B** = 15.0 jhat
c) **A** = 15.0 ihat and **B** = -15.0 jhat

Can the dot product and the cross product between two vectors both be zero? Explain

The propeller of an aircraft accelerates from rest with an angular acceleration $\alpha = 4t+6$, where $\alpha$ is in rad/s^2 and t is in seconds. What is the angle in radians though which the propeller rotates from $t = 1.00$ s to $t = 6.00$ s?

A uniform plank 6.0 m long rests on two supports, 2.5 m apart, as shown in the figure below. The gravitational force on the plank is 100 N. The left end of the plank is 1.5 m to the left of the left support, so that the plank is not centered on the supports. A person is standing on the plank 0.1 meter to the right of the right support. The gravitational force on this person is 800.0 N. How far to the right can the person walk before the plank begins to tip?



The blades on modern wind turbines are typically 20 to 40 meters in length. Two different turbines have their blades attached with one end at the center of rotation. Consider two points, each at the outer end of a blade on each of the turbines, one with a blade length of 20.0m, and the other with a length of 40.0m. Each blade is rotating with a constant angular speed of 3.88 rad/s, and the rotation is considered for 10.0 s.

a) What is the speed of each point in meters per second?
b) What is the angular distance traveled by each point?
c) What is the translational distance traveled by each point?
d) What is the magnitude of the centripetal acceleration that would be experienced by an object located at each point?

DJs use a turntable in applying their trade, often using their hand to speed up or slow down a disc record so as to produce a desired change in sound. Suppose a DJ wants to slow down a record initially rotating clockwise (as viewed from above) with an angular speed of 33.0 rpm to an angular speed of 22.0 rpm. The record has a rotational inertia of 0.012 kg m^2, and a radius of 0.15 m.

a) What angular acceleration is necessary if he wishes to accomplish this feat in exactly 0.65 s with a constant acceleration?
b) How many revolutions does the record go through during this change in speed?
c) If the DJ applies a vertical force with his finger to the edge of the record, with what force must he push so as to slow the record in the above time? Assume the coefficient of kinetic friction between his finger and the record is 0.50, and ignore the mass of the finger.

An ice skater spins on one foot. Is he rotating? If so, describe the axis of rotation and explain your answer. Afterwards, the skater skates around a closed, circular track. Is he rotating? If so, describe the axis of rotation and explain your answer.