Harmonic Oscillators

- $F=-kx$ or $V=cmx^2$. Arises often as first approximation for the minimum of a potential well
- Solve directly through “calculus” (analytical)
- Solve using group-theory like methods from relationship between $x$ and $p$ (algebraic)

Classically $F=ma$ and

$$\frac{d^2x}{dt^2} + \frac{cm}{m} x = 0 \iff x = \sin 2\pi vt \quad \nu = \frac{1}{2\pi} \sqrt{\frac{c}{m}}$$

Define

$$\frac{-\hbar^2 d^2\psi}{2mdx^2} + \frac{c}{2} x^2 \psi = E \psi \quad \alpha = \frac{2\pi m \nu}{\hbar} \quad \beta = \frac{2mE}{\hbar^2} \quad u = \sqrt{\alpha} x$$

$$\Rightarrow \frac{d^2\psi}{du^2} + \left( \frac{\beta}{\alpha} - u^2 \right) \psi = 0$$
Harmonic Oscillators - Guess

- Can use our solution to finite well to guess at a solution
- Know lowest energy is 1-node, second is 2-node, etc. Know will be Parity eigenstates (odd, even functions)
- Could try to match at boundary but turns out in this case can solve the diff.eq. for all $x$ - as no abrupt changes in $V(x)$
Harmonic Oscillators

- Solve by first looking at large $|u|$

\[
\frac{d^2\psi}{du^2} - u^2 \psi = 0 \implies \psi = Ae^{-u^2/2} + Be^{u^2/2}
\]

*but* $B = 0$ as $\psi(\infty) = 0$

- for smaller $|u|$ assume

\[
\psi(u) = Ae^{-u^2/2} H(u)
\]

\[
\frac{d^2H}{du^2} - 2u \frac{dH}{du} + (\frac{\beta}{\alpha} - 1) H = 0
\]

- Hermite differential equation. Solved in 18th/19th century. Its constraints lead to energy eigenvalues

- Solve with Series Solution technique
Hermite Equation

\[ \frac{d^2H}{du^2} - 2u \frac{dH}{du} + \left( \frac{\beta}{\alpha} - 1 \right)H = 0 \]

\[ H(u) = a_0 + a_1u + a_2u^2 \ldots = \sum_{l=0}^{\infty} a_l u^l \]

\[ \frac{dH}{du} = \sum_{l=1}^{\infty} l a_l u^{l-1} \quad \frac{d^2H}{du^2} = \sum_{l=2}^{\infty} (l - 1) l a_l u^{l-2} \]

• put \( H, \frac{dH}{du}, \frac{d^2H}{du^2} \) into Hermite Eq.
• Rearrange so that you have

\[
A + Bu + Cu^2 + Du^3 \ldots = 0
\]

• for this to hold requires that \( A = B = C = \ldots = 0 \) gives

\[
A \text{ term: } 2a_2 + \left( \frac{\beta}{\alpha} - 1 \right) a_0 = 0
\]

\[
B \text{ term: } 6a_3 - 2a_1 + \left( \frac{\beta}{\alpha} - 1 \right) a_1 = 0
\]
Hermite Equation

- Get recursion relationship
- if any $a_l=0$ then the series can end (higher terms are 0). Gives eigenvalues
  \[ a_{l+2} = \frac{-(\beta/\alpha - 1 - 2l)}{(l+1)(l+2)} \ a_l \]
- easiest if define odd or even types

- $a_1 = 0 \ a_0, a_2, ..., \text{non-zero (even)}$
- $a_0 = 0 \ a_1, a_3, ..., \text{non-zero (odd)}$

ends when $\frac{\beta}{\alpha} - 1 - 2l = 0$ gives

\[ \frac{2mE}{\hbar^2} = \frac{2E}{\hbar \nu} = 2l + 1 \ (\nu = \text{class. freq.}) \]

\[ E_l = \hbar \nu \left( \frac{1}{2} + l \right) \quad l = 0, 1, 2, ... \]
Hermite Equation

- Get recursion relation

\[ a_{l+2} = \frac{-(\frac{\beta}{\alpha} - 1 - 2l)}{(l+1)(l+2)} a_l \]

Examples L=0, L=1, L=2

\[ a_0 = 1; a_1 = 0 \quad \frac{\beta}{\alpha} = 1; \quad l = 0 \quad \Rightarrow \quad a_2 = 0 \]
\[ a_0 = 0; a_1 = 1 \quad \frac{\beta}{\alpha} = 3; \quad l = 1 \quad \Rightarrow \quad a_3 = 0 \]
\[ a_0 = 1; a_1 = 0 \quad \frac{\beta}{\alpha} = 5; \quad l = 2 \quad \Rightarrow \quad a_4 = 0; \]
\[ a_2 = \frac{-(5-1-0)}{(0+1)(0+2)} a_0 = -2a_0 \]
Hermite Equation - wavefunctions

- Use recursion relationship to form eigenfunctions. Compare to first 2 for infinite well
  - First
    \[ a_0 = 1 \quad a_{n>0} = 0 \quad \Rightarrow \quad \psi_0(u) = A_0 e^{-u^2/2} \]
    \[ e^{-u^2/2} = 1 - \frac{u^2}{2} + \left(\frac{u^2}{2}\right)^2 \frac{1}{2} \ldots \quad \cos u = 1 - \frac{u^2}{2} + \frac{u^4}{24} \ldots \]
  - Second
    \[ a_1 = 1 \quad a_0, a_{n>1} = 0 \quad \Rightarrow \quad \psi_1(u) = A_1 u e^{-u^2/2} \]
    \[ u e^{-u^2/2} = u - \frac{u^3}{2} + \frac{u^5}{8} \ldots \quad \sin u = u - \frac{u^3}{6} + \frac{u^5}{120} \ldots \]
  - Third
    \[ a_0 = 1 \quad a_2 = -2 \quad a_1, a_{n>2} = 0 \quad \Rightarrow \]
    \[ \psi_2(u) = A_2 (1 - 2u^2) e^{-u^2/2} \]

Note: \[ \psi(x) = \pm \psi(-x) \]
H.O. Example

• A particle starts in the state:

\[ \Psi(x,0) = A(1 - 2u)^2 e^{-u^2/2} \]
\[ u = \sqrt{\frac{m\omega}{\hbar}} x \]

• Split among eigenstates

\[ \Psi(x,0) = A(1 - 4u + 4u^2) e^{-u^2/2} \]
\[ \Psi_0(x,t) = A e^{-u^2/2} e^{-iE_0t/\hbar} \]
\[ \Psi_1(x,t) = B e^{-u^2/2} e^{-i3E_0t/\hbar} \]
\[ \Psi_2(x,t) = C(1 - 2u^2) e^{-u^2/2} e^{-i5E_0t/\hbar} \]

• From t=0

\[ \Psi(x,0) = (-2(1 - 2u^2) - 4u + 3)e^{-u^2/2} \]
\[ \Rightarrow \Psi(x,t) = D(3\Psi_0 - 4\Psi_1 - 2\Psi_2) \]

• what is the expectation value of the energy?

\[ \langle E \rangle = \frac{3^2 E_0 + 4^2 E_1 + 2^2 E_2}{9 + 16 + 4} = 2.7 E_0 \]
H.O. Example 2

- A particle starts in the state:
  \[ \Psi(x, 0) = A(1 - 4u + 4u^2)e^{-u^2/2} \]
  \[ \Psi(x, t) = D(3\psi_0 e^{-iE_0t/\hbar} - 4\psi_1 e^{-i3E_0t/\hbar} - 2\psi_2 e^{-i5E_0t/\hbar}) \]

- At some later time T the wave function has evolved to a new function. Determine T
  \[ \Psi(x, T) = A(1 + 4u + 4u^2)e^{-u^2/2} \]
  \[ \Psi(x, T) = D(3\psi_0 e^{-iE_0T/\hbar} - 4\psi_1 e^{-i3E_0T/\hbar} - 2\psi_2 e^{-i5E_0T/\hbar}) \]
  \[ = De^{-iE_0T/\hbar} (3\psi_0 - 4\psi_1 e^{-i2E_0T/\hbar} - 2\psi_2 e^{-i4E_0T/\hbar}) \]

- Solve by inspection. Need:
  \[ e^{-i2E_0T/\hbar} = -1 \text{ and } e^{-i4E_0T/\hbar} = 1 \]
  \[ \Rightarrow 2E_0T/\hbar = \pi \]