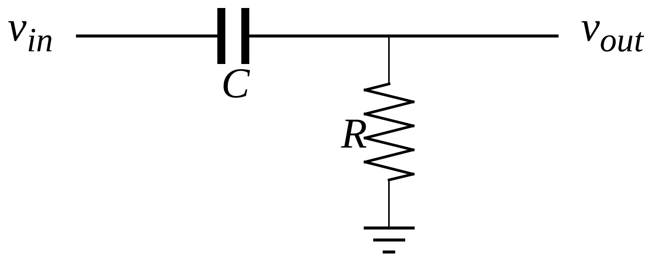
Filter Circuits

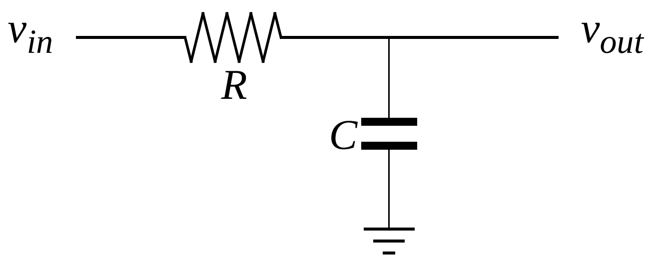
- Passive filters with a single resistor and capacitor are called one-pole filters.
- The high-pass filter selects frequencies above a breakpoint frequency $\omega_B = 1/RC$.



$$v_{out} = \frac{R}{R + 1/j\omega C} v_{in} = \frac{j\omega RC}{j\omega RC + 1} v_{in}$$

$$A = \left| \frac{v_{out}}{v_{in}} \right| = \frac{\omega^2 R^2 C^2}{\sqrt{1 + \omega^2 R^2 C^2}}$$

- For small ω, A goes as ω or 6 dB/octave; $\phi = \tan^{-1}(1/\omega RC)$; for small ω, ϕ approaches $+90°$.
- The low-pass filter selects frequencies below a breakpoint frequency $\omega_B = 1/RC$.



$$v_{out} = \frac{1/j\omega C}{R + 1/j\omega C} v_{in} = \frac{1}{j\omega RC + 1} v_{in}$$

$$A = \left| \frac{v_{out}}{v_{in}} \right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

- For large ω, A goes as $1/\omega$ or 6 dB/octave; $\phi = \tan^{-1}(-\omega RC)$; for large ω, ϕ approaches $-90°$.

Transfer Function

- The complex gain for a filter is the *transfer function*.
- For a high-pass filter it is,

$$\frac{v_{out}}{v_{in}} = \frac{R}{R + 1/j\omega C} = \frac{j\omega/\omega_B}{1 + j\omega/\omega_B} \equiv H(j\omega)$$

with the breakpoint frequency $\omega_B = 1/RC$.

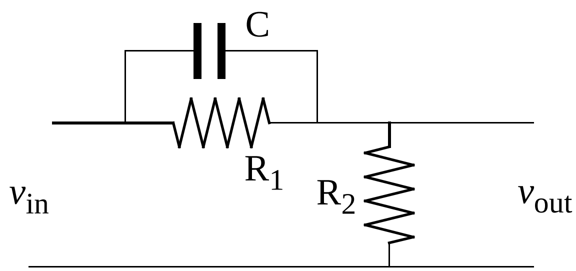
- The transfer function describes behavior as a function of frequency.
- Again for the high-pass filter, the real gain $G(\omega) = |H(j\omega)|$

$$G(\omega) = \frac{\omega/\omega_B}{\sqrt{1 + (\omega/\omega_B)^2}}$$

falls off below ω_B at 20 dB/decade or 6 dB/octave.

Speed-up Capacitor

- Consider that a capacitor looks like an open connection to low *f* and like a short circuit at high *f* .



$$v_{out} = \frac{R_2}{R_2 + Z_1} v_{in}$$

- The circuit is a resistor divider with R_1 replaced with Z_1 which includes a capacitor.

$$Z_1 = \frac{R/j\omega C}{R + 1/j\omega C} = \frac{R}{j\omega RC + 1}$$

- The expectation is that at high *f*, the divider has $Z_1 = 0$.

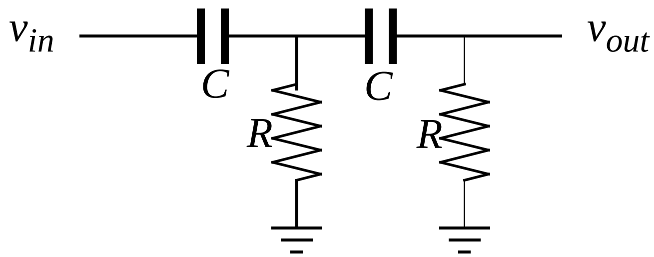
$$v_{out} = \frac{R_2}{R_2 + R_1/(j\omega R_1 C + 1)} v_{in} = \frac{j\omega R_1 R_2 C + R_2}{j\omega R_1 R_2 C + R_2 + R_1} v_{in}$$

$$A = \sqrt{\frac{(\omega R_1 R_2 C)^2 + (R_2)^2}{(\omega R_1 R_2 C)^2 + (R_2 + R_1)^2}}$$

- For $\omega \ll 1/R_1C$, $A = R_2/(R_1 + R_2)$; $\omega \gg 1/R_1C$, $A = 1$.
- High frequencies are enhanced, so a pulse edge becomes sharper

Two-Pole Filters

- Two RC high-pass filters can be placed in series.



$$H(j\omega) \;=\; R\left(\frac{R}{2R + 1/j\omega C}\right)\left(\frac{1}{1/j\omega C + \dfrac{R(R + 1/j\omega C)}{2R + 1/j\omega C}}\right)$$

$$H(j\omega) \;=\; \frac{-\omega^2 R^2 C^2}{1 + 3j\omega RC - \omega^2 R^2 C^2} \;=\; \frac{-(\omega/\omega_B)^2}{1 + 3j(\omega/\omega_B) - (\omega/\omega_B)^2}$$

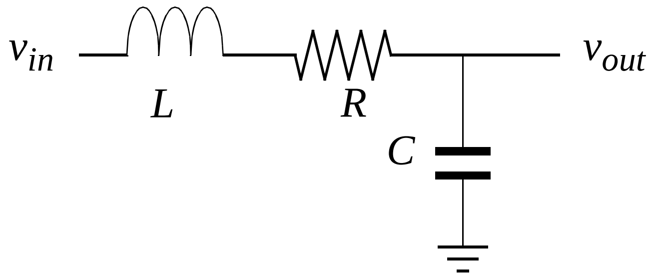
- The gain varies as ω^2.

$$G(\omega) \;=\; \frac{(\omega/\omega_B)^2}{\sqrt{1 + 7(\omega/\omega_B)^2 + (\omega/\omega_B)^4}}$$

- This is a *second-order* filter.
- More poles further increase the rapidity of fall off and add phase shifts.

RLC Filter

- A second-order low-pass filter can be made with a resistor and capacitor.



$$H(j\omega) = \frac{1/j\omega C}{j\omega L + R + 1/j\omega C} = \frac{1}{1 + j\omega/Q\omega_0 - (\omega/\omega_0)^2}$$

where $\omega_0^2 = 1/LC$ and $Q = \omega_0 L/R$.

- The circuit is equivalent to a damped driven harmonic oscillator.
- There is a damping factor $d_0 = 1/Q = R/\omega_0 L$.

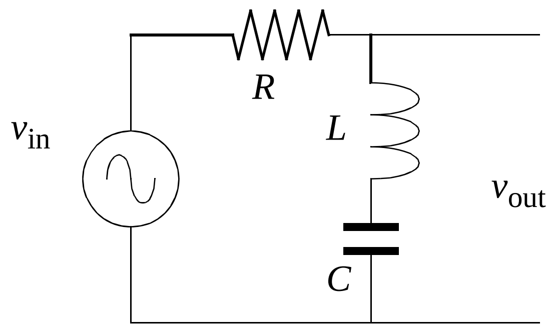
$$H(j\omega) = \frac{1}{1 + jd_0(\omega/\omega_0) - (\omega/\omega_0)^2}$$

$$G(\omega) = \frac{1}{\sqrt{d_0^2(\omega/\omega_0)^2 + [1 - (\omega/\omega_0)^2]^2}}$$

- As a second-order filter, the gain varies as ω^2 above ω_0.

Series RLC Circuit

- An RLC circuit can form a notch filter that only negates a narrow band of frequency.



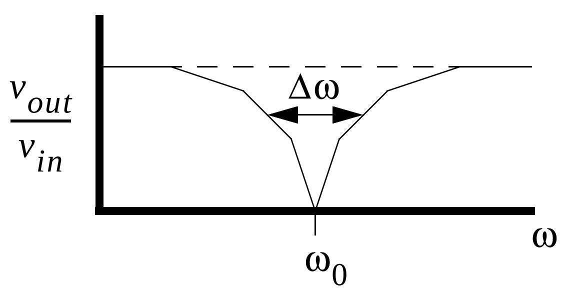
$$v_{out} = \frac{Z_{LC}}{R + Z_{LC}} v_{in}$$

$$Z_{LC} = 1/j\omega C + j\omega L = \frac{1 - \omega^2 LC}{j\omega C}$$

- The series impedance can be calculated and inserted to find the gain. $A = \frac{v_{out}}{v_{in}} = \frac{1 - \omega^2 LC}{j\omega RC + 1 - \omega^2 LC}$

- The width of the filtered region is the Q value. $Q = \frac{\omega_0}{\Delta\omega} = \frac{L\omega_0}{R} = R\sqrt{\frac{L}{C}}$

- A graph of the behavior shows the notch.

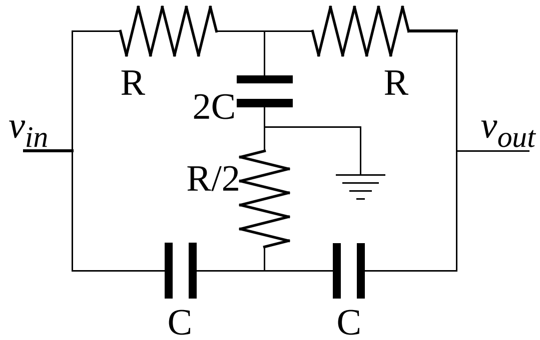


$$\Delta\omega = R/L$$

$$\omega_0 = 1/\sqrt{LC}$$

Twin-T Filter

- A notch filter can be built with Combines two 2-pole passive filters. One is low pass, and one is high pass.



$$v_{out-LP} = \left(\frac{1/j\omega C}{R + 1/j\omega C}\right)^2 v_{in}$$

$$v_{out-HP} = \left(\frac{R}{R + 1/j\omega C}\right)^2 v_{in}$$

The combined effect of the two filters is:

$$v_{out} = \left[\left(\frac{1}{1 + j\omega RC}\right)^2 + \left(\frac{j\omega RC}{1 + j\omega RC}\right)^2\right] v_{in}$$

$$v_{out} = \left[\frac{1 - (\omega RC)^2}{(1 + j\omega RC)^2}\right] v_{in}$$

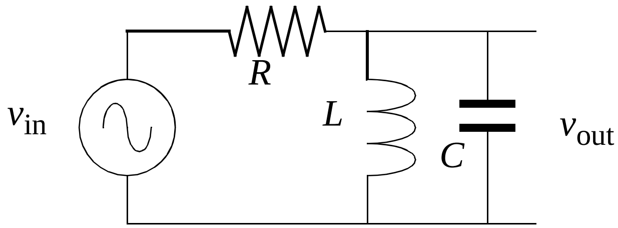
At $\omega = 1/RC$, the gain is 0.

Low frequencies are shifted by –90°

High frequencies are shifted by +90°

Parallel RLC Circuit

- If the inductor and capacitor are in parallel there is a positive resonance.



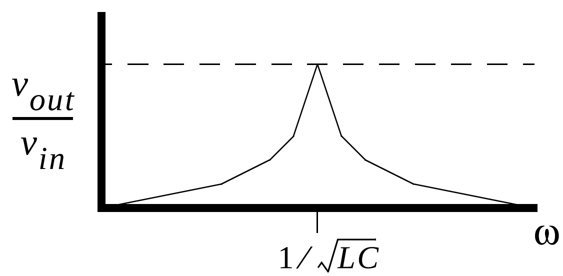
$$Z_{LC} = \frac{j\omega L/j\omega C}{1/j\omega C + j\omega L} = \frac{j\omega L}{1 - \omega^2 LC}$$

$$v_{out} = \frac{Z_{LC}}{R + Z_{LC}} v_{in}$$

- The impedance can be calculated and inserted to find the gain.

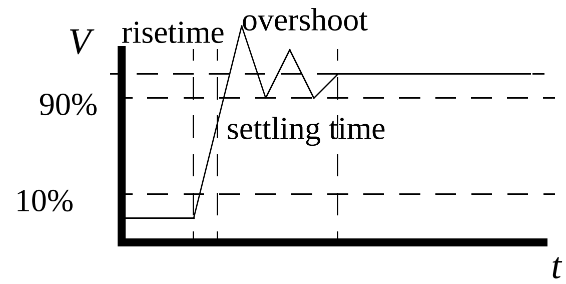
$$\frac{v_{out}}{v_{in}} = \frac{j\omega L}{R(1-\omega^2 LC) + j\omega L} = \frac{\omega^2 L^2}{R^2(1-\omega^2 LC)^2 + \omega^2 L^2}$$

- The filter selects only a narrow range of frequencies.

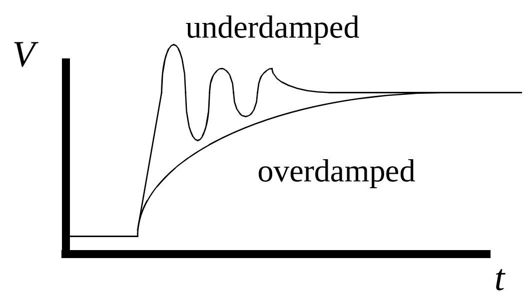


Filter Jargon - Time Domain

- There are a number of terms used to describe the behavior of signals as a function of time.
- *Risetime*: time to get to 90% of the signal value.
- *Overshoot*: percent signal passes signal value.
- *Settling time*: time to stay within ε of signal value.



- The effect of filter damping in the time domain is like a damped harmonic oscillator.

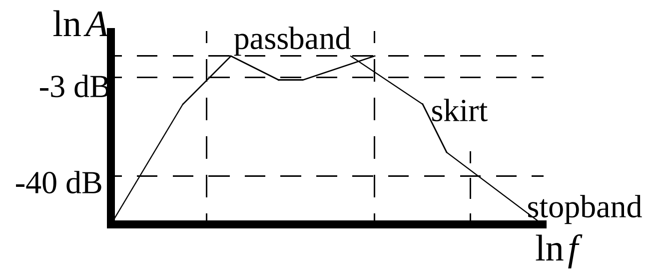


Overdamped ($d_0 > \sqrt{2}$) rises slowly.

Underdamped ($d_0 < \sqrt{2}$) rises quickly, but there is a ringing overshoot.

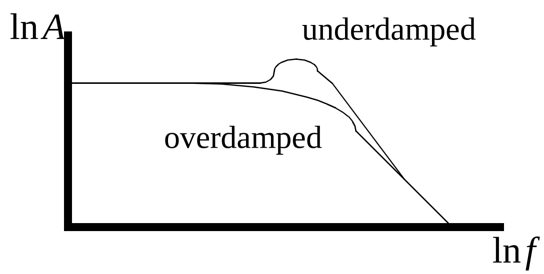
Filter Jargon - Frequency Domain

- Filter bahaior is also studied as a function of frequency.
- *Passband*: Unattenuated region 0 to -3 dB.
- *Cutoff frequency*: edge of passband.
- *Ripple band*: passband that is not flat in frequency.
- *Skirt*: transistion region from -3 to -40 dB.
- *Stopband*: frequencies with attenuation greater than -40 dB.



Steeper skirts require more poles - higher order filter

- Damping has an effect in frequency as well as time.



High frequency ringing shows up as extra gain at resonant frequency.

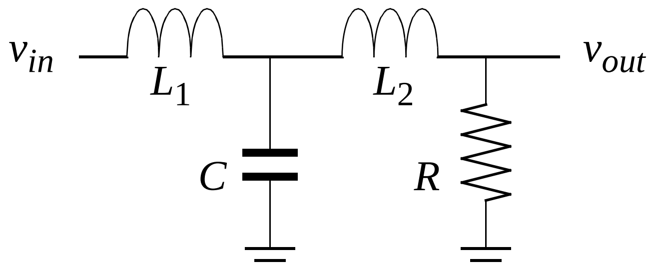
Overdamped circuits have extra non-uniform gain in the passband.

Butterworth Filter

- Certain transfer functions give special properties to the behavior and have special names.
- A Butterworth filter is designed to give maximum flattness in the passband, so there is a critically damped response ($d_0^2 = 2$) in the frequency domain.
- This creates ringing in time domain in exchange for uniform frequency response.
- The gain of a Butterworth filter is an approximation in terms of a cutoff frequency ω_C:

$$G(j\omega)^2 = |H(j\omega)|^2 = \frac{A^2}{1 + (\omega^2/\omega_C^{\,2})^n}$$

- A butterworth filter can be made as a passive 3-pole circuit.

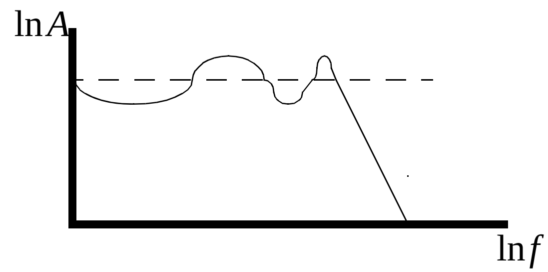


$$|H(j\omega)|^2 = \left|\left(\frac{1/j\omega C}{j\omega L_1 + 1/j\omega C}\right)\left(\frac{R}{j\omega L_2 + R}\right)\right|^2 = \frac{1}{1 + (\omega^2/\omega_C^{\,2})^3}$$

- For $(L_2/R)^2 = 2L_1C$, $\omega_C = R^2/L_1^2L_2^2C^2$.

Chebyshev Filter

- A Chebyshev fileter is designed to maximize the sharpness at the edge of the passband.

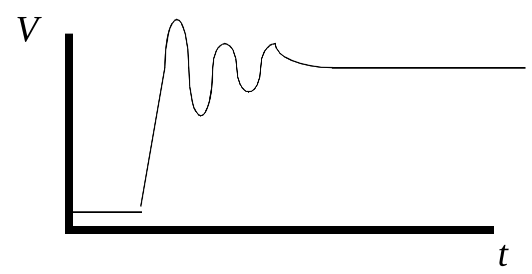


- The transfer function takes the following form.

$$|H(j\omega)|^2 = \frac{A^2}{1 + \varepsilon^2 C_n^{\ 2}(\omega/\omega_0)}$$

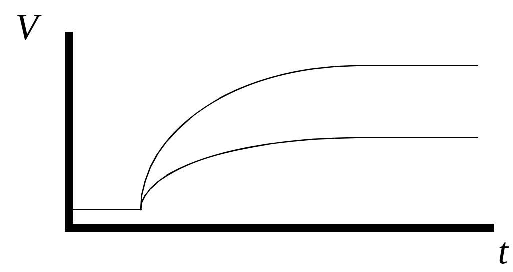
C_n is an n-th order Chebyshev polynomial: $C_n(x) = \cos[n \operatorname{acos} x]$

- This gives an underdamped response ($d_0 = 0.767$)
- There is substantial ringing in the time domain.



Bessel Filter

- The Bessel filter gives an equal rise time independent of pulse height.



- The transfer function is as follows.

$$|H(j\omega)|^2 = \frac{A^2}{1 + \varepsilon^2 B_n^2(\omega/\omega_0)}$$

where B_n is an n-th order Bessel function.

- This gives an overdamped response ($d_0 = 1.736$)
- There is the softer rise in the frequency domain.