Filter Circuits

- Passive filters with a single resistor and capacitor are called one-pole filters.

- The high-pass filter selects frequencies above a breakpoint frequency $\omega_B = \frac{1}{RC}$.

  \[ v_{out} = \frac{R}{R + 1/j\omega C}v_{in} = \frac{j\omega RC}{j\omega RC + 1}v_{in} \]

  \[ A = \left| \frac{v_{out}}{v_{in}} \right| = \frac{\omega^2 R^2 C^2}{\sqrt{1 + \omega^2 R^2 C^2}} \]

  - For small $\omega$, $A$ goes as $\omega$ or 6 dB/octave; $\phi = \tan^{-1}(1/\omega RC)$; for small $\omega$, $\phi$ approaches $+90^\circ$.

- The low-pass filter selects frequencies below a breakpoint frequency $\omega_B = \frac{1}{RC}$.

  \[ v_{out} = \frac{1/j\omega C}{R + 1/j\omega C}v_{in} = \frac{1}{j\omega RC + 1}v_{in} \]

  \[ A = \left| \frac{v_{out}}{v_{in}} \right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \]

  - For large $\omega$, $A$ goes as $1/\omega$ or 6 dB/octave; $\phi = \tan^{-1}(\omega RC)$; for large $\omega$, $\phi$ approaches $-90^\circ$. 
Transfer Function

- The complex gain for a filter is the *transfer function*.
- For a high-pass filter it is,

\[
\frac{v_{out}}{v_{in}} = \frac{R}{R + 1/j\omega C} = \frac{j\omega/\omega_B}{1 + j\omega/\omega_B} \equiv H(j\omega)
\]

with the breakpoint frequency \( \omega_B = 1/RC \).

- The transfer function describes behavior as a function of frequency.
- Again for the high-pass filter, the real gain \( G(\omega) = |H(j\omega)| \)

\[
G(\omega) = \frac{\omega/\omega_B}{\sqrt{1 + (\omega/\omega_B)^2}}
\]

falls off below \( \omega_B \) at 20 dB/decade or 6 dB/octave.
Speed-up Capacitor

- Consider that a capacitor looks like an open connection to low \( f \) and like a short circuit at high \( f \).

\[
\begin{align*}
v_{in} & \rightarrow \frac{C}{R_1} \rightarrow \frac{1}{R_2} \rightarrow v_{out}
\end{align*}
\]

\[
v_{out} = \frac{R_2}{R_2 + Z_1} v_{in}
\]

- The circuit is a resistor divider with \( R_1 \) replaced with \( Z_1 \) which includes a capacitor.

\[
Z_1 = \frac{R/j\omega C}{R + 1/j\omega C} = \frac{R}{j\omega RC + 1}
\]

- The expectation is that at high \( f \), the divider has \( Z_1 = 0 \).

\[
v_{out} = \frac{R_2}{R_2 + R_1/(j\omega R_1 C + 1)} v_{in} = \frac{j\omega R_1 R_2 C + R_2}{j\omega R_1 R_2 C + R_2 + R_1} v_{in}
\]

\[
A = \frac{(\omega R_1 R_2 C)^2 + (R_2)^2}{(\omega R_1 R_2 C)^2 + (R_2 + R_1)^2}
\]

- For \( \omega << 1/R_1 C \), \( A = R_2/(R_1 + R_2) \); \( \omega >> 1/R_1 C \), \( A = 1 \).

- High frequencies are enhanced, so a pulse edge becomes sharper.
Two-Pole Filters

- Two RC high-pass filters can be placed in series.

\[
\begin{align*}
H(j\omega) &= R \left( \frac{R}{2R + 1/j\omega C} \right) \left( \frac{1}{1/j\omega C + \frac{R(R + 1/j\omega C)}{2R + 1/j\omega C}} \right) \\
H(j\omega) &= \frac{-\omega^2 R^2 C^2}{1 + 3j\omega RC - \omega^2 R^2 C^2} = \frac{-(\omega/\omega_B)^2}{1 + 3j(\omega/\omega_B) - (\omega/\omega_B)^2}
\end{align*}
\]

- The gain varies as \(\omega^2\).

\[
G(\omega) = \frac{(\omega/\omega_B)^2}{\sqrt{1 + 7(\omega/\omega_B)^2 + (\omega/\omega_B)^4}}
\]

- This is a second-order filter.
- More poles further increase the rapidity of fall off and add phase shifts.
RLC Filter

- A second-order low-pass filter can be made with a resistor and capacitor.

\[
H(j\omega) = \frac{1}{j\omega LC} = \frac{1}{j\omega L + R + 1/j\omega C} = \frac{1}{1 + j\omega/Q\omega_0 - (\omega/\omega_0)^2}
\]

where \(\omega_0^2 = 1/LC\) and \(Q = \omega_0 L/R\).

- The circuit is equivalent to a damped driven harmonic oscillator.

- There is a damping factor \(d_0 = 1/Q = R/\omega_0 L\).

\[
H(j\omega) = \frac{1}{1 + jd_0(\omega/\omega_0) - (\omega/\omega_0)^2}
\]

\[
G(\omega) = \frac{1}{\sqrt{d_0^2(\omega/\omega_0)^2 + [1 - (\omega/\omega_0)^2]^2}}
\]

- As a second-order filter, the gain varies as \(\omega^2\) above \(\omega_0\).
Series RLC Circuit

- An RLC circuit can form a notch filter that only negates a narrow band of frequency.

\[
Z_{LC} = \frac{1}{j\omega C} + j\omega L = \frac{1 - \omega^2 LC}{j\omega C}
\]

- The series impedance can be calculated and inserted to find the gain.
  \[ A = \frac{v_{out}}{v_{in}} = \frac{1 - \omega^2 LC}{j\omega RC + 1 - \omega^2 LC} \]

- The width of the filtered region is the Q value.
  \[ Q = \frac{\omega_0}{\Delta\omega} = \frac{L\omega_0}{R} = R \sqrt{\frac{L}{C}} \]

- A graph of the behavior shows the notch.

\[
\Delta\omega = \frac{R}{L} \\
\omega_0 = \frac{1}{\sqrt{LC}}
\]
**Twin-T Filter**

- A notch filter can be built with Combines two 2-pole passive filters. One is low pass, and one is high pass.

\[
v_{out-LP} = \left( \frac{1/j\omega C}{R + 1/j\omega C} \right)^2 v_{in}
\]

\[
v_{out-HP} = \left( \frac{R}{R + 1/j\omega C} \right)^2 v_{in}
\]

The combined effect of the two filters is:

\[
v_{out} = \left[ \left( \frac{1}{1 + j\omega RC} \right)^2 \left( \frac{j\omega RC}{1 + j\omega RC} \right)^2 \right] v_{in}
\]

\[
v_{out} = \left[ \frac{1 - (\omega RC)^2}{(1 + j\omega RC)^2} \right] v_{in}
\]

At \( \omega = 1/RC \), the gain is 0.

Low frequencies are shifted by \(-90^\circ\)

High frequencies are shifted by \(+90^\circ\)
**Parallel RLC Circuit**

- If the inductor and capacitor are in parallel there is a positive resonance.

\[
Z_{LC} = \frac{j\omega L/j\omega C}{1/j\omega C + j\omega L} = \frac{j\omega L}{1 - \omega^2 LC}
\]

\[
v_{out} = \frac{Z_{LC}}{R + Z_{LC}} v_{in}
\]

- The impedance can be calculated and inserted to find the gain.

\[
\frac{v_{out}}{v_{in}} = \frac{j\omega L}{R(1 - \omega^2 LC) + j\omega L} = \frac{\omega^2 L^2}{R^2 (1 - \omega^2 LC)^2 + \omega^2 L^2}
\]

- The filter selects only a narrow range of frequencies.
Filter Jargon - Time Domain

• There are a number of terms used to describe the behavior of signals as a function of time.
  • **Risetime**: time to get to 90% of the signal value.
  • **Overshoot**: percent signal passes signal value.
  • **Settling time**: time to stay within \( \varepsilon \) of signal value.

• The effect of filter damping in the time domain is like a damped harmonic oscillator.

Overdamped \( (d_0 > \sqrt{2}) \) rises slowly.

Underdamped \( (d_0 < \sqrt{2}) \) rises quickly, but there is a ringing overshoot.
Filter Jargon - Frequency Domain

- Filter behavior is also studied as a function of frequency.
- **Passband**: Unattenuated region 0 to -3 dB.
- **Cutoff frequency**: edge of passband.
- **Ripple band**: passband that is not flat in frequency.
- **Skirt**: transition region from -3 to -40 dB.
- **Stopband**: frequencies with attenuation greater than -40 dB.
  Steeper skirts require more poles - higher order filter

- Damping has an effect in frequency as well as time.

High frequency ringing shows up as extra gain at resonant frequency.
Overdamped circuits have extra non-uniform gain in the passband.
Butterworth Filter

- Certain transfer functions give special properties to the behavior and have special names.
- A Butterworth filter is designed to give maximum flatness in the passband, so there is a critically damped response \((d_0^2 = 2)\) in the frequency domain.
- This creates ringing in time domain in exchange for uniform frequency response.
- The gain of a Butterworth filter is an approximation in terms of a cutoff frequency \(\omega_C\):
  \[
  G(j\omega)^2 = |H(j\omega)|^2 = \frac{A^2}{1 + (\omega^2 / \omega_C^2)^n}
  \]

- A butterworth filter can be made as a passive 3-pole circuit.

\[
|H(j\omega)|^2 = \left|\frac{1/j\omega C}{j\omega L_1 + 1/j\omega C} \left(\frac{R}{j\omega L_2 + R}\right)\right|^2 = \frac{1}{1 + (\omega^2 / \omega_C^2)^3}
\]

- For \((L_2/R)^2 = 2L_1C\), \(\omega_C = R^2/L_1^2L_2^2C^2\).
**Chebyshev Filter**

- A Chebyshev filter is designed to maximize the sharpness at the edge of the passband.

- The transfer function takes the following form.

\[
|H(j\omega)|^2 = \frac{A^2}{1 + \epsilon^2 C_n^2(\omega/\omega_0)}
\]

\(C_n\) is an \(n\)-th order Chebyshev polynomial: \(C_n(x) = \cos[n \cos x]\)

- This gives an underdamped response \((d_0 = 0.767)\)
- There is substantial ringing in the time domain.
Bessel Filter

- The Bessel filter gives an equal rise time independent of pulse height.

- The transfer function is as follows.

\[
|H(j\omega)|^2 = \frac{A^2}{1 + \varepsilon^2 B_n^2(\omega/\omega_0)}
\]

where \( B_n \) is an \( n \)-th order Bessel function.

- This gives an overdamped response \((d_0 = 1.736)\)
- There is the softer rise in the frequency domain.