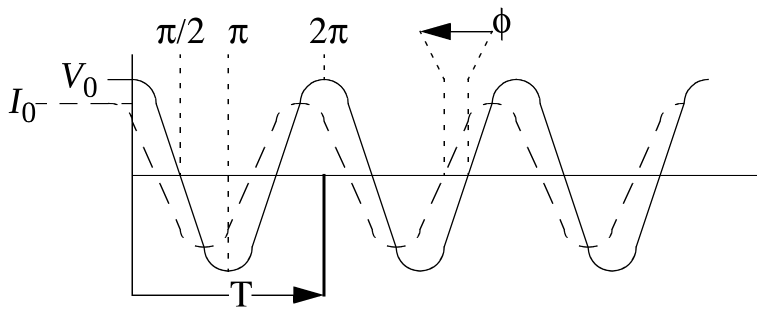
AC Circuits

- Pure AC voltage varies in time.

$$v(t) = V_0 \cos \omega t = V_0 e^{j\omega t}$$

- The AC current also varies in time with the same frequency but may have a different phase.

$$i(t) = I_0 \cos(\omega t + \phi) = I_0 e^{j(\omega t + \phi)}$$



- The RMS voltage and current, $V_{\rm rms}$, $I_{\rm rms}$, are related to the amplitude.

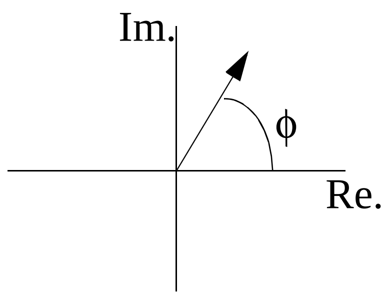
$$V_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\langle V_0^2 \cos^2 \omega t \rangle} = \frac{V_0}{\sqrt{2}}$$

$$I_{rms} = \sqrt{\langle i^2 \rangle} = \sqrt{\langle I_0^2 \cos^2(\omega t + \phi) \rangle} = \frac{I_0}{\sqrt{2}}$$

- The phase doesn't matter for rms measurement.

Phasors

- A complex number can be represented as a magnitude and angle in the complex plane.



- The imaginary unit in electronics is j to avoid confusion with current. $j = \sqrt{-1}$
- Euler's formula for an exponential in $j\phi$ links to sinusoidal behavior.

$$e^{j\phi} = \cos\phi + j\sin\phi$$

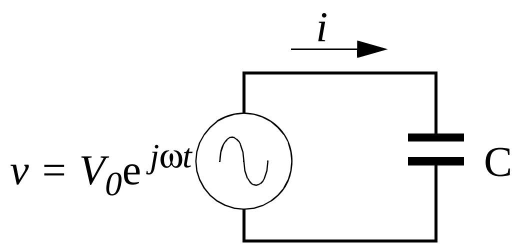
- Trigonometric formulas can be replaced by exponential ones.

$$\cos\phi = Re(e^{j\phi})$$

- AC signals are represented by a vector in the complex plane and are called *phasors*.

Capacitors

- The voltage-current relationship for a capacitor in an AC circuit comes from the stored charge.



$$v = \frac{Q}{C} \qquad \frac{dv}{dt} = \frac{i}{C}$$

- For a pure AC signal, there is a simple voltage-current relationship.

$$v = \frac{1}{j\omega C} i$$

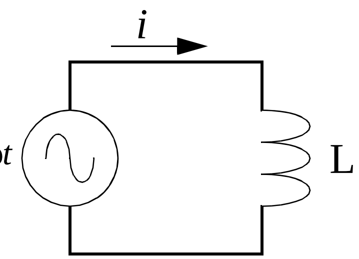
- This looks like a complex version of Ohm's law.
- The capacitive reactance is a complex impedance.

$$Z_C = \frac{1}{j\omega C}$$

Inductors

- An inductor creates a voltage in response to a changing current.

$$v = V_0 e^{j\omega t}$$



$$v = L\frac{di}{dt}$$

- This gives rise to another complex current-voltage relationship.

$$v = j\omega L i$$

- The inductive reactance is a complex impedance.

$$Z_L = j\omega L$$

- Capacitive reactances are negative on the imaginary axis and inductive reactances are positive on the imaginary axis.

Gain

- The unit of gain A is the decibel (dB).
- Decibels are a logarithmic measure,

  Voltage and current, $A_{dB} = 20 \log_{10} A$

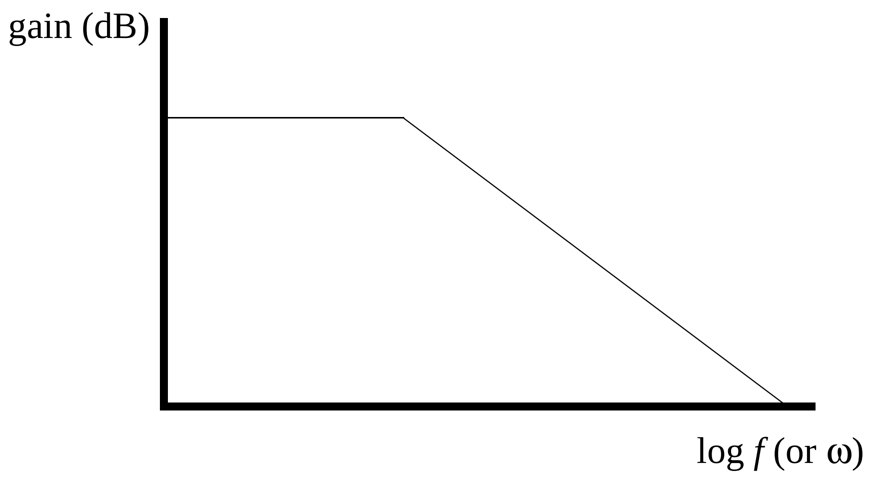
  Power = voltage*current, $A_{dB} = 10 \log_{10} A$
- There are some useful approximations for amplitude gain.

  A factor of 10 is a 20 dB measure

  A factor of 2 is about a 6 dB measure

  Neagtive dB is a reduction in magnitude

- A Bode plot displays gain versus the log of the frequency.



Complex Gain

- The complex gain includes the magnitude of gain and the phase shift.

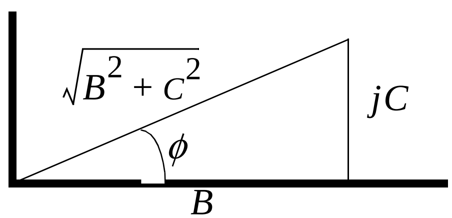
- The absolute magnitude (a real) is the magnitude of the gain.

$$|A| = |B + jC| = \sqrt{(B + jC)(B - jC)} = \sqrt{B^2 + C^2}$$

- The angle in the complex plane is the phase shift.

$$\tan\phi = C/B$$

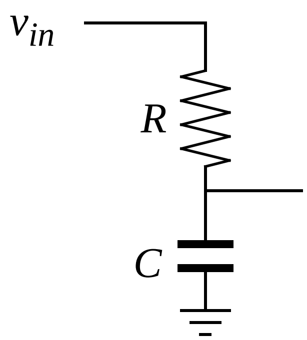
- Graphically:



$$\phi = \operatorname{atan}\frac{C}{B}$$

Low-pass Filter

- The low-pass filter is a common element in AC circuits.
- It is a voltage divider with complex impedance.



$$v_{out} = \frac{1/j\omega C}{R + 1/j\omega C} v_{in} = \frac{1}{1 + j\omega RC} v_{in}$$

$$A = \left|\frac{v_{out}}{v_{in}}\right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$A = (1+\omega^2 R^2 C^2)^{-1/2}$; for large ω, A goes as $1/\omega$ or 6 dB/octave.

$\phi = \tan^{-1}(-\omega RC)$; for large ω, ϕ approaches $-90°$.

Fourier Series

- A periodic signal with a period T can be expressed as $f(t) = f(t + nT)$
- A fourier series represents this as a sum of sine and cosine functions.
- The periodic function can be expressed with $\omega = 2\pi/T$.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

- Instead of sines and cosines, the function can be represented in complex notation.

$$f(t) = \sum_{n=-\infty}^{\infty} [c_n e^{jn\omega t}]$$

Fourier Coefficients

- The Fourier coefficients are found by integrating the signal over any one period.

- The first term $a_0/2$ is the DC component of $f(t)$.

$$\int_0^T f(t)\,dt = \frac{a_0 T}{2}$$

- The coeffiecients a_n, b_n represent the strength of that frequency component.
- The cosine terms are symmetric and the sine terms are antisymmetric.

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt$$

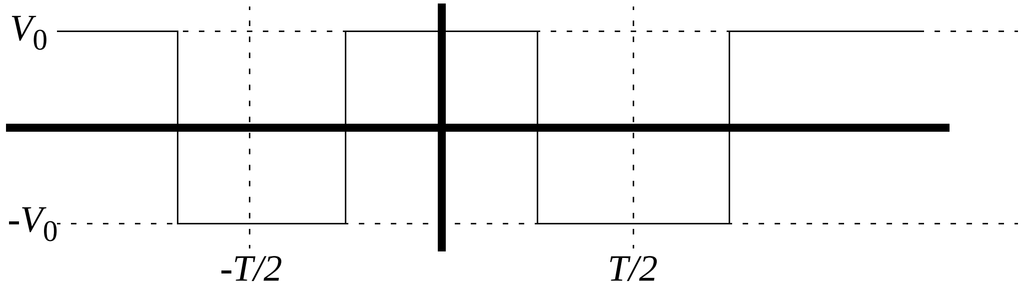
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt$$

- The coefficients c_n for the complex form are similar.

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega t}\,dt$$

Square Wave

- A symmetric square wave can be simply decomposed into a Fourier series.



- The function has been set up as a symmetric function around t = 0.
- The period is split into two symmetric halves.

$$v(t) = V_0 \qquad -\frac{T}{4} < t < \frac{T}{4}$$

$$v(t) = -V_0 \qquad |t| > \frac{T}{4}$$

- Integration can be done on each half separately.

Square Wave Coefficients

- Since the form is symmetric the sine terms will all be 0.

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} v(t)\sin(n\omega t)dt = 0$$

- The symmetric terms will be 2 times the value of the positive side.

$$a_n = \frac{4}{T}\int_{0}^{T/4} V_0\cos(n\omega t)dt + \frac{4}{T}\int_{T/4}^{T/2} -V_0\cos(n\omega t)dt$$

$$a_n = \frac{4V_0}{Tn\omega}\sin(n\omega T/4) + \frac{4V_0}{Tn\omega}\sin(n\omega T/4)$$

$$a_n = \frac{8V_0}{2\pi n}\sin(2\pi n/4) = \frac{4V_0}{\pi n}\sin(n\pi/2)$$

Low Order Expansion

- Consider the first few terms of the square wave series

$$a_n = \frac{4V_0}{\pi n}\sin\left(\frac{n\pi}{2}\right)$$

$a_1 = 4V_0/\pi$

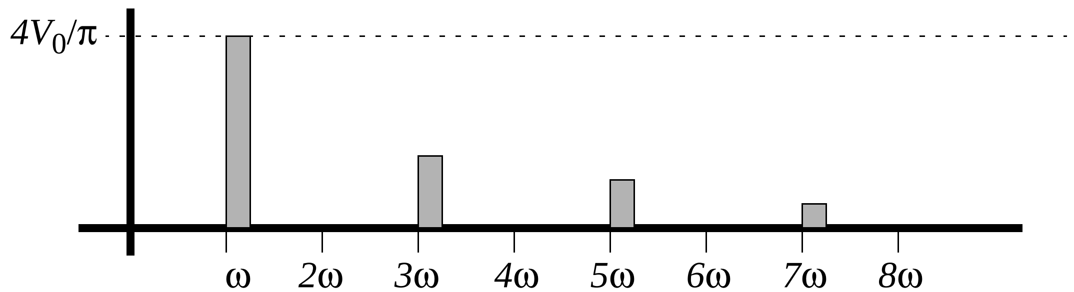
$a_2 = 0$

$a_3 = 4V_0/3\pi$

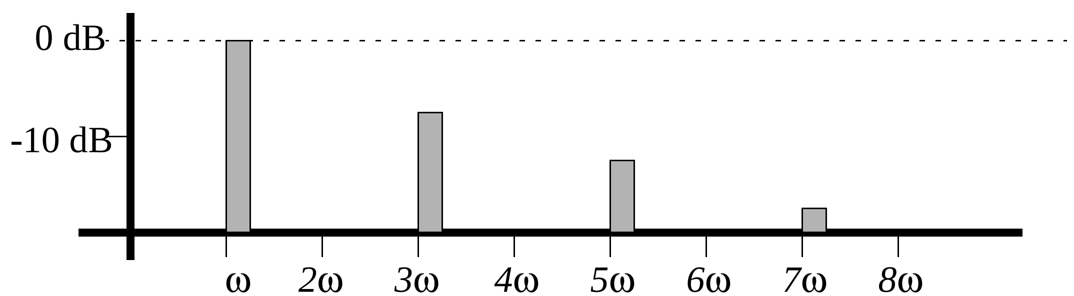
$a_4 = 0$

$a_5 = 4V_0/5\pi$

$a_6 = 0$

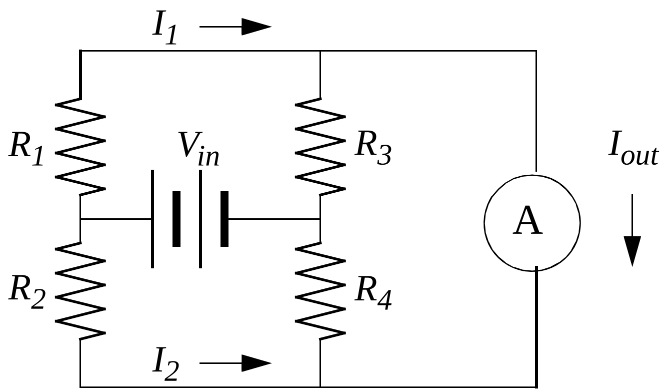


- The signal can be displayed in terms of the frequencies and strengths.



Bridge Circuits

- The Wheatstone bridge is a DC circuit.



- If the current meter shows $I_{out} = 0$, there are two balance equations.

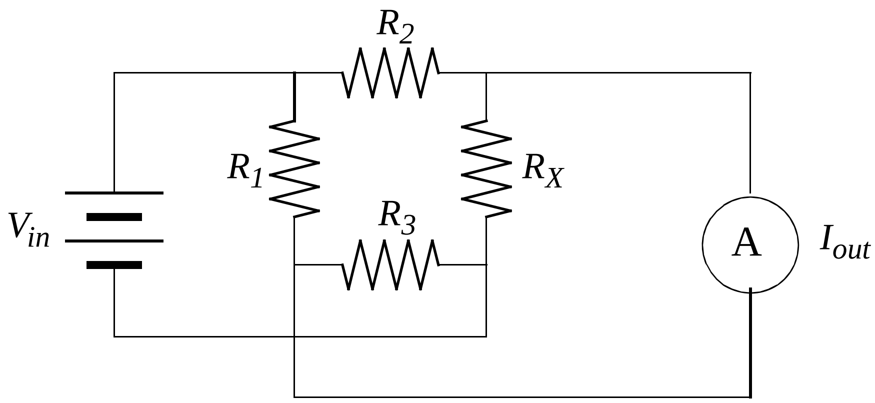
$$I_1 R_1 = I_2 R_2 \qquad I_1 R_3 = I_2 R_4$$

- These can be combined and R_4 can be found in terms of the other resistors.

$$R_4 = R_2 R_3 / R_1$$

Bridge Circuits

- The Wheatstone bridge is typically used to measure an unknown resistance R_X.



- If the user balances the current meter until $I_{out} = 0$.
- The unknown resistance comes from the balance equation.

$$R_X = R_2 R_3 / R_1$$

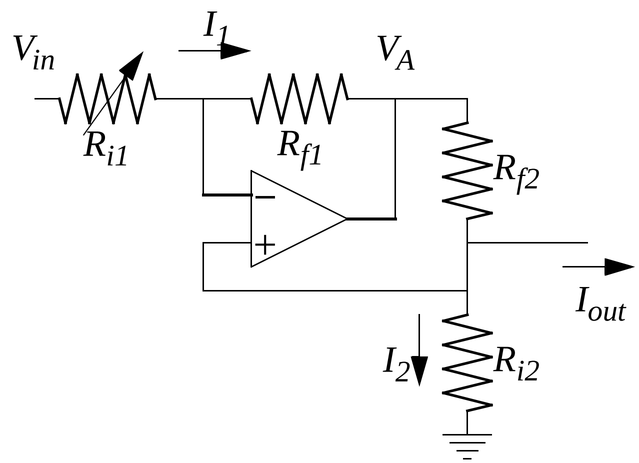
Op-Amp Bridge

- An op-amps can be used to generate a balance condition.
- Negative feedback sets the voltages on the input of the op-amp.

$$V_{inv} = V_{non} = V_{in} - I_1 R_{i1} = I_2 R_{i2}$$

- The resistr network sets the voltage on the output of the op-amp.

$$V_A = V_{in} - I_1 R_{i1} - I_1 R_{f1} = (I_2 + I_{out}) R_{f2} + I_2 R_{i2}$$



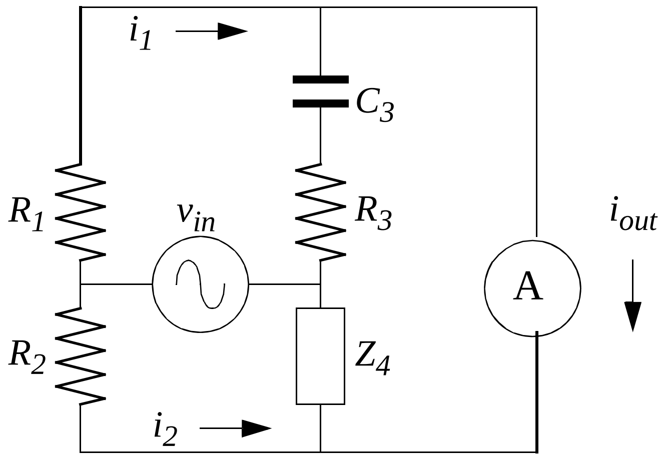
- The variable resistor can be adjusted until the resistors meet the following ratio: $\frac{R_{f2}}{R_{f1}} = \frac{R_{i2}}{R_{i1}}$
- The current only depends on R_{i2} and V_{in}.

$$I_{out} = -\frac{V_{in}}{R_{i2}}$$

- Measuring the current gives the value of the resistor R_{i2}.

AC Bridges

- A DC bridge can be modified to measure complex impedances



- Match the real and imaginary parts at $Z_4 = a + jb$.

$$R_1(a + jb) = R_2(R_3 - j/\omega C_3)$$

- The imaginary part measures the capacitance or inductance.

$$b = -R_2/\omega R_1 C_3$$

- The real part measures any intrinsic resistance; and can be used to calculate the Q.

$$a = R_2 R_3 / R_1$$