Mid-term Exam: Solutions

Date: 2006/10/23 Time: 1 hr 15 min

30 points

Q 1 [3+1+1=5 points]
Consider the following transformations of the coordinate system:

$$\begin{aligned} \{t, \mathbf{r}\} &\xrightarrow{E} \{t, \mathbf{r}\}, \\ \{t, \mathbf{r}\} &\xrightarrow{P} \{t, -\mathbf{r}\}, \\ \{t, \mathbf{r}\} &\xrightarrow{T} \{-t, \mathbf{r}\}, \end{aligned}$$

as well as the transformation PT that is generated by performing first T and then P.

(a) Write the combination of every pair of the above transformations in the form of a matrix (truth table), and thus show that $\{E, P, T, PT\}$ form a group.

(b) Identify the inverse of each element.

(c) Is the group abelian, or non-abelian?

A 1 **(a)**

	E	P	T	PT
E	E	P	T	PT
P	P	E	PT	T
T	T	PT	E	P
PT	PT	T	P	E

(b) Every element is its own inverse.

(c) Abelian.

Q 2 [4+4+5+2 = 15 points]
Consider the coplanar double-pendulum suspended from a fixed support and placed in a uniform gravitational field $\mathbf{g}$ as shown in Fig. 1.

(a) Write down the Lagrangian.

(b) Determine the equations of motion under the following small-angle approximations:

$$\begin{aligned} &\sin\theta_1 \approx \theta_1, &&\sin\theta_2 \approx \theta_2 \\ &\cos\theta_1 \approx 1 - \frac{\theta_1^2}{2}, &&\cos\theta_2 \approx 1 - \frac{\theta_2^2}{2}, &&\cos(\theta_1 - \theta_2) \approx 1. \end{aligned}$$

(c) From the characteristic equation, find the eigenfrequencies.

(d) What happens in the limit $\dfrac{m_1}{m_2} \to \infty$?

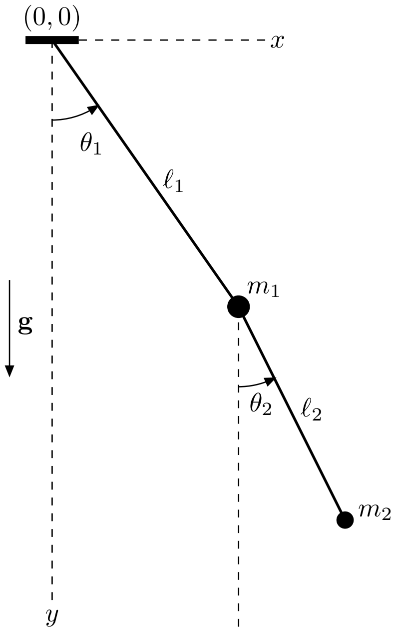

Figure 1

A 2 **(a)** We choose θ_1 and θ_2, as shown in the figure, as the generalized coordinates. The cartesian coordinates of m_1 are

$$x_1 = \ell_1 \sin\theta_1, \qquad y_1 = \ell_1 \cos\theta_1, \tag{1}$$

and those of m_2 are

$$x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2, \qquad y_2 = \ell_1 \cos\theta_1 + \ell_2 \cos\theta_2. \tag{2}$$

The total kinetic energy is

$$\begin{aligned} T = T_1 + T_2 =& \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) \\ =& \frac{m_1}{2}\ell_1^2\dot{\theta}_1^2 + \frac{m_2}{2}\left(\ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2)\right) \end{aligned} \tag{3}$$

and the total potential energy is

$$U = U_1 + U_2 = -g\,(m_1 y_1 + m_2 y_2) = -\,(m_1 + m_2)\,g\ell_1\cos\theta_1 - m_2 g\ell_2\cos\theta_2. \tag{4}$$

Hence, the Lagrangian is

$$\begin{aligned} L = T - U =& \frac{1}{2}\,(m_1 + m_2)\,\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\ell_2^2\dot{\theta}_2^2 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) \\ &+ (m_1 + m_2)g\ell_1\cos\theta_1 + m_2 g\ell_2\cos\theta_2. \end{aligned} \tag{5}$$

(b) Under the given small-angle approximations, after dropping the additive constants from L, Eq. 5 reduces to

$$\begin{aligned} L =& T - U \\ =& \frac{1}{2}\left(m_1+m_2\right)\ell_1^2\dot\theta_1^2 + \frac{1}{2}m_2\ell_2^2\dot\theta_2^2 + m_2\ell_1\ell_2\dot\theta_1\dot\theta_2 \\ & -\frac{1}{2}(m_1+m_2)g\ell_1\theta_1^2 - \frac{1}{2}m_2 g\ell_2\theta_2^2. \end{aligned} \tag{6}$$

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = 0 \tag{7}$$

with $q = \theta_1, \theta_2$, i.e.,

$$\begin{aligned} &(m_1+m_2)\ell_1\ddot\theta_1 + m_2\ell_2\ddot\theta_2 + (m_1+m_2)g\theta_1 = 0, \\ &\ell_1\ddot\theta_1 + \ell_2\ddot\theta_2 + g\theta_2 = 0. \end{aligned} \tag{8}$$

(c) Substituting trial solutions

$$\theta_i = A_i e^{-i\omega t} \tag{9}$$

we get

$$\begin{aligned} &A_1(m_1+m_2)(g-\ell_1\omega^2) - A_2\omega^2 m_2\ell_2 = 0, \\ &A_1\ell_1\omega^2 + A_2(g-\ell_2\omega^2) = 0 \end{aligned} \tag{10}$$

which leads to the characteristic equation

$$\begin{vmatrix} (m_1+m_2)g - \omega^2(m_1+m_2)\ell_1 & -\omega^2 m_2\ell_2 \\ -\omega^2\ell_1 & g-\omega^2\ell_2 \end{vmatrix} = 0$$

or,

$$\ell_1\ell_2 m_1\omega^4 - (m_1+m_2)(\ell_1+\ell_2)g\omega^2 + (m_1+m_2)g^2 = 0. \tag{11}$$

The roots of the characteristic equation are

$$\begin{aligned} \omega_{1,2}^2 =& \frac{g}{2m_1\ell_1\ell_2}[(m_1+m_2)(\ell_1+\ell_2) \\ & \pm\sqrt{m_1+m_2}\sqrt{(m_1+m_2)(\ell_1+\ell_2)^2 - 4m_1\ell_1\ell_2}]. \end{aligned} \tag{12}$$

(d) As $\dfrac{m_1}{m_2} \to \infty$, $\omega_1 \to \sqrt{\dfrac{g}{\ell_1}}$, $\omega_2 \to \sqrt{\dfrac{g}{\ell_2}}$, corresponding to independent oscillations of the two pendula.

Q 3 [10 points]
A homogeneous disk of mass M and radius R rolls without slipping on a horizontal surface and is attracted to a point a distance ℓ below the surface, as shown in Fig. 2.

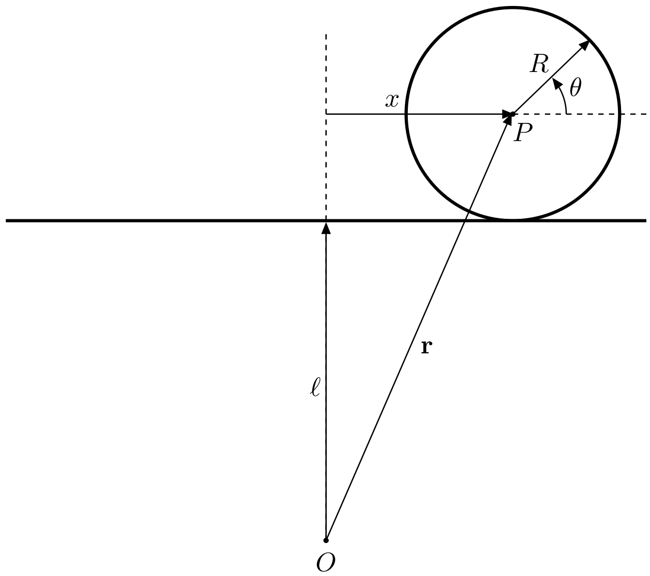

Figure 2

If the force of attraction is proportional to the distance between the disk's center of mass and and the center of the force, show that the disk will undergo a simple harmonic motion and find its frequency.

A 3 The force acting on the center of mass of the disk is

$$\mathbf{F} = -k\mathbf{r}. \tag{13}$$

Only the x component of the force,

$$F_x = -kr\sin\theta = -kx, \tag{14}$$

is allowed to do any work. This corresponds to an effective potential

$$U(x) = -\int_0^x F_x(u)du = \frac{1}{2}kx^2. \tag{15}$$

The kinetic energy of the disk is

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 \tag{16}$$

where I is the moment of inertia of the disk about the axis of rotation. The surface mass density of the disk is

$$\rho = \frac{M}{\pi R^2}. \tag{17}$$

Hence,

$$I = \int_M r^2 dm = \int_0^R r^2(\rho 2\pi r dr) = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R = \frac{MR^2}{2}. \tag{18}$$

Since,

$$dx = Rd\theta, \tag{19}$$

Using Eqs. 18 in Eq. 16 we get

$$T(x) = \frac{3}{4} M \dot{x}^2. \tag{20}$$

From Eqs. 20 and 15, we have the Lagrangian

$$L = T - U = \frac{3}{4} M \dot{x}^2 - \frac{1}{2} k x^2. \tag{21}$$

So, the equation of motion is

$$\begin{aligned} \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= 0, \\ \text{or,} \quad \frac{3}{2} M \ddot{x} + kx &= 0, \end{aligned} \tag{22}$$

which represents a simple harmonic oscillation along x

$$\ddot{x} + \omega^2 x = 0 \tag{23}$$

with angular frequency

$$\omega = \sqrt{\frac{2k}{3M}}. \tag{24}$$