Assignment: HW8 [40 points]

Assigned: 2006/11/20
Due: 2006/11/27

Solutions

P8.1 [$5 + 3 = 8$ points]

(a) Show that the combination of two successive Lorentz transformations with parallel velocities β_1 and β_2 is equivalent to a single Lorentz transformation with velocity $\beta = \dfrac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}$.

(b) Show that the set of all Lorentz transformations with this composition rule forms a group.

S8.1 (a) Since the directions perpendicular to the common velocity direction do not enter the picture, we can ignore them, and treat the problem at hand as a two-dimensional one:

$$\begin{aligned}
\mathbf{\Lambda} &= \mathbf{\Lambda_2}\mathbf{\Lambda_1} \\
&= \gamma_2 \begin{pmatrix} 1 & -\beta_2 \\ -\beta_2 & 1 \end{pmatrix} \gamma_1 \begin{pmatrix} 1 & -\beta_1 \\ -\beta_1 & 1 \end{pmatrix} \\
&= \gamma_1\gamma_2 \begin{pmatrix} 1 + \beta_1\beta_2 & -(\beta_1 + \beta_2) \\ -(\beta_1 + \beta_2) & 1 + \beta_1\beta_2 \end{pmatrix} \\
&= \gamma_1\gamma_2(1 + \beta_1\beta_2) \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix},
\end{aligned} \tag{1}$$

where $\beta \equiv \dfrac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}$.

Now,

$$\begin{aligned}
\gamma &= (1 - \beta^2)^{-\frac{1}{2}} \\
&= \left(1 - \left(\frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}\right)^2\right)^{-\frac{1}{2}} \\
&= \left(\frac{1 + 2\beta_1\beta_2 + \beta_1^2\beta_2^2 - \beta_1^2 - 2\beta_1\beta_2 - \beta_2^2}{(1 + \beta_1\beta_2)^2}\right)^{-\frac{1}{2}} \\
&= (1 + \beta_1\beta_2)((1 - \beta_1^2)(1 - \beta_2^2))^{-\frac{1}{2}} \\
&= \gamma_1\gamma_2(1 + \beta_1\beta_2).
\end{aligned} \tag{2}$$

So, using Eq. 2 in Eq. 1,

$$\mathbf{\Lambda} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}, \tag{3}$$

which represents a single Lorentz boost of β.

(b) Clearly,

$$-1 < \beta < 1 \quad \text{for all} \quad -1 < \beta_1 < 1, \quad -1 < \beta_2 < 1. \tag{4}$$

The identity element of the group corresponds to $\beta = 0$:

$$\mathbf{\Lambda}(\beta = 0) = \mathbf{1}. \tag{5}$$

While the inverse relationship is satisfied by

$$(\mathbf{\Lambda}(\beta))^{-1} = \mathbf{\Lambda}(-\beta). \tag{6}$$

The composition rule thus gives us the continuous Lorentz boost group.

P8.2 [$8 + 2 + 2 = 12$ points]

(a) A rocket propels itself along a straight line in empty space by burning its fuel in such a way that the velocity of the exhaust gases ejected from the nozzle is v_0 in the instantaneous (inertial) rest frame of the rocket. If the rocket starts from rest with a total mass M_i (icluding fuel), find the relationship between its velocity and mass during its journey.

(b) Typical chemical fuels yield exhaust speeds of the order of 10^3 m/s. Let us imagine we had a fuel that gives $v_0 = 3 \times 10^5$ m/s. What initial mass of fuel would the rocket need in order to attain a final velocity of $0.1c$ for a final mass of 1 ton?

(c) Matter-antimatter fuel yields $v_0 = c \equiv 1$ (the exhaust consists of photons). What initial mass of fuel would be required in this case (in order to attain a final velocity of $0.1c$ for a final mass of 1 ton)?

S8.2 (a) Consider the rocket when it is at velocity $\beta = \dfrac{v}{c} \equiv v$ and mass M ejecting an infinitesimal mass dm of exhaust gas with velocity $\beta_0 = v_0/c \equiv v_0$, increasing its own velocity by dv, and mass by dM in the process (dM is negative). As an instructive exercise, let us solve the problem using relativistic 4-velocities u, u_0, u' instead of v, v_0, v'. This actually makes the algebra much cleaner. We can always transform back to v's or β's at the end. We shall use the parametric representations based on the rapidity η: [1]

$$u = \sinh\eta; \qquad \gamma = \cosh\eta; \qquad \beta = \tanh\eta. \tag{8}$$

[1] Given any one of the 3 relations as the definition, you can easily deduce the other two from any two of the following:

$$\gamma = (1-\beta^2)^{-\frac{1}{2}}; \qquad u = \gamma\beta; \qquad \gamma = (1+u^2)^{\frac{1}{2}}. \tag{7}$$

For this rectilinear motion, we can ignore two of the 3 spatial dimensions. So, the increase in the rocket's velocity in its own instantaneous rest frame is related to that in its initial rest frame as

$$du = \gamma du'. \tag{9}$$

The conservation of momentum in the rocket's instantaneous rest frame gives (to the first order),

$$dM + \gamma_0 dm = 0 \text{ for } \quad \mu = 0, \tag{10}$$

and,

$$M du' - u_0 dm = 0 \text{ for } \quad \mu = 1. \tag{11}$$

Combining the two, and using the 4-velocity $u^\mu \equiv \gamma v^\mu$ we get[2]

$$\begin{aligned} & M du' = u_0 dm = \gamma_0 \beta_0 dm = -\beta_0 dM, \\ \text{or,} \qquad & -\frac{dM}{M} = \frac{du'}{\beta_0} = \frac{1}{\beta_0}\frac{du}{\gamma} = \frac{1}{\beta_0}\frac{d(\sinh\eta)}{\cosh\eta} = \frac{1}{\beta_0} d\eta. \end{aligned} \tag{12}$$

Since $u = 0 \Rightarrow \eta = 0$, indefinite integration of both sides gives

$$\begin{aligned} & -\int_{M_i}^{M} \frac{dM}{M} = \frac{1}{\beta_0}\int_0^\eta d\eta \\ \text{or,} \qquad & [\ln M]_{M_i}^{M} = -\frac{1}{\beta_0}[\eta]_0^\eta = \frac{\eta}{\beta_0} \\ \text{or,} \qquad & \frac{M}{M_i} = e^{-\frac{\eta}{\beta_0}} = (e^\eta)^{-\frac{1}{\beta_0}}. \end{aligned} \tag{13}$$

Now we can switch back, for familiarity's sake, to traditional (non-relativistic) definition of velocities, β etc.

$$e^\eta = \sinh\eta + \cosh\eta = u + \gamma = \gamma(\beta + 1) = \frac{1+\beta}{\sqrt{1-\beta^2}} = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}. \tag{14}$$

Substituting this expression for e^η in Eq. 13 we get the relation between the rocket's (non-relativistic) velocity in its initial rest frame and its mass:

$$M = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2\beta_0}} M_i. \tag{15}$$

(b) For $\beta_0 = 3 \times 10^5$ m/s$=10^{-3}$, $\beta = 0.1$ and $M = 1$ ton, we get from Eq. 15,

$$M_i = \left(\frac{1.1}{0.9}\right)^{500} \times 1 \text{ ton} = 3.76 \times 10^{43} \text{ ton}. \tag{16}$$

This is several orders of magnitude greater than the mass of the entire Milky Way galaxy (including its dark matter content), which is about $6 \times 10^{11} M_\odot$, where $M_\odot \approx 2 \times 10^{30}$ kg is the mass of our sun.

[2]We could stick to $\frac{u_0}{\gamma_0}$ if we wanted to be purists, but since β_0 is constant and it keeps the algebra simpler, we'll indulge in a little harmless promiscuity and keep β_0.

(c)

P8.3 [5 points]
A particle moving with velocity $\vec{\beta}$ decays "in flight" into two particles. Determine the relation between the angle of emergence of either daughter particle and its energies in the laboratory frame and the rest frame of the parent particle.

S8.3 We have to deal with two inertial frames: the laboratory frame, and the rest frame of the original particle. We shall use the subscripts L and C, respectively, for various quantities pertaining to any one of the daughter particles in these two frames. Let us choose the Cartesian space axes such that $\vec{\beta} = \beta\hat{\mathbf{x}_L}$, all particle motions are confined to the $x_L y_L$ plane, and the two sets of axes coincide at the event of the decay.

Let the angle of emergence of any one of the decay particles with respect to $\vec{\beta}(\equiv \beta_L)$ be θ_L. Then,

$$E_C = \gamma(E_L - \beta p_L \cos\theta_L) \quad \text{or,} \quad \cos\theta_L = \frac{\gamma E_L - E_C}{\gamma\beta\sqrt{E_L^2 - m^2}}, \tag{17}$$

where m is the mass of the particle (since $E^2 - \vec{p}\cdot\vec{p} = m^2$).

For the determination of E_L from $\cos\theta_L$ we get the quadratic equation

$$E_L^2(1 - \beta^2\cos^2\theta_L) - 2E_L E_C\sqrt{1-\beta^2} + E_C^2(1-\beta^2) + \beta^2 m^2\cos^2\theta_L = 0. \tag{18}$$

P8.4 [7 points]
Determine the maximum energy which can be carried off by any one of the decay particles, when a particle of mass m_0 at rest decays into three particles with masses m_1, m_2, and m_3.

S8.4 The decay particle in question has its maximum energy when the system of the other two has the least possible mass, which happens when the latter are moving with the same velocity. Say, particle 1 moves at a speed $\beta_{1\max}$ when particles 2 and 3 move together with speeds β'. The kinematics of particle 1 is the same as it would be if it were recoiling against a single sister particle of mass m_2+m_3. Since there is no motion in 2 of the 3 space dimensions, we can ignore those. Conservation of momentum dictates

$$p_i^\mu = p_f^\mu, \tag{19}$$

where the subscripts i and f denote the initial and the final states, respectively.

For $\mu = 0$ we have

$$m_0 = \gamma_{1\max} m_1 + \gamma'(m_2 + m_3), \tag{20}$$

and, for $\mu = 1$,

$$0 = \gamma_{1\max}\beta_{1\max} m_1 - \gamma'\beta'(m_2 + m_3). \tag{21}$$

Now,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \Rightarrow \qquad \gamma^2\beta^2 = \gamma^2 - 1. \tag{22}$$

Thus Eq. 21 yields

$$\begin{aligned} & \gamma'\beta'(m_2+m_3) = \gamma_{1\max}\beta_{1\max}m_1 \\ \text{or,} \quad & (\gamma'^2-1)(m_2+m_3)^2 = (\gamma_{1\max}^2-1)m_1^2 \\ \text{or,} \quad & \gamma'^2(m_2+m_3)^2 = (\gamma_{1\max}^2-1)m_1^2 + (m_2+m_3)^2. \end{aligned} \tag{23}$$

We rearrange the terms in Eq. 20, square both sides, and use Eq. 23 to get the maximum energy of particle 1:

$$\begin{aligned} & m_0 - \gamma_{1\max}m_1 = \gamma'(m_2+m_3) \\ \Rightarrow \quad & m_0^2 - 2\gamma_{1\max}m_1m_0 + \gamma_{1\max}^2m_1^2 = (m_2+m_3)^2 + \gamma_{1\max}^2m_1^2 - m_1^2 \\ \Rightarrow \quad & E_{1\max} = \gamma_{1\max}m_1 = \frac{m_0^2+m_1^2-(m_2+m_3)^2}{2m_0}. \end{aligned} \tag{24}$$

P8.5 $[6+2=8$ points$]$

The Lagrangian that yields the correct relativistic equations of motion of a free particle can be expressed in the non-relativistitc form as

$$\mathcal{L}_0 = -\frac{1}{\gamma}m = -mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}} \simeq -mc^2 + \frac{1}{2}m\mathbf{v}^2. \tag{25}$$

where $c=1$.

(Note: This form of the Lagrangian refers to a fixed inertial frame. Thus, it is not a Lorentz invariant, which is why it is not used in practical applications. We choose to work with it in this problem only as an illustrative example. In this problem, $\dot{q} \equiv \frac{dq}{dt}$.)

Now consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}m\left(\psi\dot{\mathbf{q}}^2 - c_0^2\frac{(\psi-1)^2}{\psi}\right) \equiv \mathcal{L}(\dot{\mathbf{q}},\psi), \tag{26}$$

which contains the additional, dimensionless degree of freedom ψ. The parameter c_0 has the physical dimension of a velocity.

(a) Show that the extremum of the action integral yields a theory obeying special relativity for which c_0 is the maximal velocity (in other words, one obtains the Lagrangian in Eq. 25 with the velocity of light c replaced by c_0).

(b) What happens when $c_0 \to \infty$?

P8.5 As $\mathcal{L}$ does not depend on q, the equation of motion for this variable reads

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}} = m\frac{d}{dt}(\psi\dot{\mathbf{q}}) = 0. \tag{27}$$

In turn, $\mathcal{L}$ is independent of $\dot{\psi}$. The condition for the action integral to be extremal leads to the following equation for ψ:

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{1}{2} m \left(\dot{\mathbf{q}}^2 - c_0^2 \frac{\psi^2 - 1}{\psi^2} \right) = 0. \tag{28}$$

The solutions to this equation are

$$\psi_1 = \frac{c_0}{\sqrt{c_0^2 - \dot{\mathbf{q}}^2}}, \qquad \psi_2 = -\frac{c_0}{\sqrt{c_0^2 - \dot{\mathbf{q}}^2}}. \tag{29}$$

Insertion of ψ_1 into the Lagrangian function yields

$$\mathcal{L}(\dot{\mathbf{q}}, \psi = \psi_1) = \frac{1}{2} m \left(2c_0 \sqrt{c_0^2 - \dot{\mathbf{q}}^2} + 2c_0^2 \right) = -mc_0^2 \sqrt{1 - \frac{\dot{\mathbf{q}}^2}{c_0^2}} + mc_0^2. \tag{30}$$

This would be identical to the relativistic Lagrangian for a free particle if one puts $c_0 = c$.

If we let $c_0 \to \infty$, then $\psi_1 \to 1$, and the Lagrangian function reduces to that for a non-relativistic free particle: $\mathcal{L}_{\mathrm{NR}} = \frac{m}{2} \dot{\mathbf{q}}^2$. It is clear that Eq. 27 is the correct equation of motion in either case.

The second solution, ψ_2, must be excluded.

The term that is added to $\mathcal{L}_{\mathrm{NR}}$, $\frac{1}{2} m(\psi - 1) \left(\dot{\mathbf{q}}^2 - c_0^2 \frac{\psi - 1}{\psi} \right)$, takes care of the requirement that the velocity $\dot{\mathbf{q}}$ should not exceed c_0.