Assignment: HW7 [40 points]

Assigned: 2006/11/15
Due: 2006/11/22

Solutions

P7.1 [$6+8=14$ points]
A particle of mass m can move in one dimension under the influence of two springs connected to two walls that are a distance a apart, as shown in Fig. 7.1. The springs obey Hooke's law, have zero unstretched lengths and spring constants k_1 and k_2 respectively.

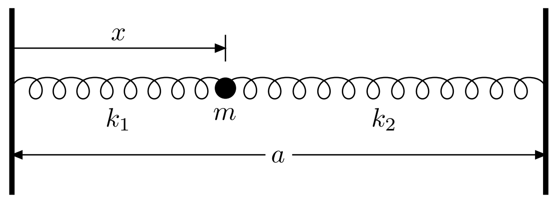

Figure 7.1

(a) Let x be the length of the first spring, and b its equilibrium value. Using the displacement from the equilibrium, $q = x - b$, as the generalized coordinate, find the Hamiltonian and the total energy of the system, and examine if these quantities are conserved.

(b) Now consider the coordinate transformation

$$Q = q - b\sin(\omega t), \tag{1}$$

where ω is some constant (not necessarily the natural frequency of the system). Find the Hamiltonian and the total energy of the system in terms of Q and its conjugate momentum P, and examine if these quantities are conserved.

S7.1 (a) The equilibrium condition requires the forces exerted on the mass by the two springs to cancel each other:

$$k_1 b = k_2(a-b), \quad \text{or,} \quad b = \frac{k_2}{k_1+k_2}a. \tag{2}$$

Thus, the potential energy of the particle is

$$\begin{aligned} V &= \frac{1}{2}(k_1x^2 + k_2(a-x)^2) \\ &= \frac{1}{2}(k_1(q+b)^2 + k_2(a-q-b)^2) \\ &= \frac{1}{2}(kq^2 + \frac{k_1k_2}{k}a^2), \end{aligned} \tag{3}$$

where $k = k_1 + k_2$, and the kinetic energy is

$$T = m\dot{q}^2 \tag{4}$$

We can drop the constant term $\dfrac{k_1 k_2}{k} a^2$ from the potential energy and write the Lagrangian as

$$L(q, \dot{q}) = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2, \tag{5}$$

and the total energy as

$$E = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} k q^2. \tag{6}$$

The momentum corresponding to q is

$$\begin{aligned} p &= \frac{\partial L}{\partial \dot{q}} = m\dot{q} \\ \text{or,} \qquad \dot{q} &= \frac{p}{m}, \end{aligned} \tag{7}$$

which gives the Hamiltonian

$$H(q, p) = p\dot{q} - L = \frac{p^2}{2m} + \frac{kq^2}{2}. \tag{8}$$

Neither the Lagrangian nor the Hamiltonian has any explicit time dependence. Therefore, both the total energy and the Hamiltonian are conserved.

(b) Under the transformation,

$$q = Q + b\sin(\omega t) \tag{9}$$

and

$$\dot{q} = \dot{Q} + b\omega\cos(\omega t). \tag{10}$$

Therefore, from Eq. 5,

$$L(Q, \dot{Q}) = \frac{m}{2}(\dot{Q} + b\omega\cos(\omega t))^2 - \frac{k}{2}(Q + b\sin(\omega t))^2, \tag{11}$$

and

$$E = \frac{m}{2}(\dot{Q} + b\omega\cos(\omega t))^2 + \frac{k}{2}(Q + b\sin(\omega t))^2, \tag{12}$$

Thus,

$$\begin{aligned} P &= \frac{\partial L}{\partial \dot{Q}} = m(\dot{Q} + b\omega\cos(\omega t)) \\ \text{or,} \qquad \dot{Q} &= \frac{P}{m} - b\omega\cos(\omega t), \end{aligned} \tag{13}$$

Therefore,

$$E = \frac{P^2}{2m} + \frac{k}{2}(Q + b\sin(\omega t))^2, \tag{14}$$

and

$$\begin{aligned} H(Q, P) &= P\dot{Q} - L \\ &= P\left(\frac{P}{m} - b\omega\cos(\omega t)\right) - \frac{P^2}{2m} + \frac{k}{2}(Q + b\sin(\omega t))^2 \\ &= \frac{P^2}{2m} + \frac{k}{2}(Q + b\sin(\omega t))^2 - b\omega P\cos(\omega t) \\ &= E - b\omega P\cos(\omega t). \end{aligned} \tag{15}$$

Now,

$$\frac{dE}{dt} = \frac{P\dot{P}}{m} + k(Q+b\sin(\omega t))(\dot{Q}+b\omega\cos(\omega t)) = \frac{P}{m}(\dot{P}+k(Q+b\sin(\omega t))), \tag{16}$$

but,

$$\dot{P} = -\frac{\partial H}{\partial Q} = -k(Q+b\sin(\omega t)). \tag{17}$$

So, the RHS of Eq. 16 vanishes, i.e.,

$$\frac{dE}{dt} = 0. \tag{18}$$

So, the energy is conserved, as expected. However, the new Hamiltonian depends explicitly on t, and is therefore not conserved.

P7.2 $[4+2=6$ points$]$
Let (q,p) be the phase-space coordinates of a system with one degree of freedom.

(a) Under what conditions is the transformation

$$Q = \alpha\frac{p}{q}; \qquad P = \beta q^2 \tag{19}$$

canonical (α and β are constants)?

(b) Find a suitable generating function of type 1, $F_1(q,Q)$, for the above transformation.

S7.2 (a) First, let us find the inverse transformations $q(Q,P)$ and $p(Q,P)$:

$$q = \sqrt{\frac{P}{\beta}}; \qquad p = \frac{qQ}{\alpha} = \frac{Q}{\alpha}\sqrt{\frac{P}{\beta}}. \tag{20}$$

Now use the conditions that yield non-zero partial derivatives:

$$\begin{aligned} \left(\frac{\partial P}{\partial q}\right)_p &= -\left(\frac{\partial p}{\partial Q}\right)_P \\ \Rightarrow \quad 2\beta q &= -\frac{1}{\alpha}\sqrt{\frac{P}{\beta}} = -\frac{q}{\alpha}. \end{aligned} \tag{21}$$

Therefore, the transformation is canonical if

$$2\alpha\beta = -1. \tag{22}$$

(b) If $F_1(q,Q)$ is a generating function, then

$$p = \left(\frac{\partial F_1}{\partial q}\right)_Q \tag{23}$$

Thus using Eq. 20 and the condition 22,

$$F_1(q,Q) = \frac{q^2 Q}{2\alpha} = -\beta q^2 Q. \tag{24}$$

Note that the generating function is not unique.

P7.3 [6 points]

Consider the (continuous and regular) one-parameter group of canonical transformations ψ_θ defined as the solution to the differential equation

$$\frac{\partial \psi^\mu(\omega_0;\theta)}{\partial\theta} = \epsilon^{\mu\nu}\left[\frac{\partial\phi(\omega)}{\partial\omega^\nu}\right]_{\omega^\mu=\psi^\mu(\omega_0;\theta)} = \epsilon^{\mu\nu}\frac{\partial\phi(\omega_0;\theta)}{\partial\psi^\nu}, \tag{25}$$

where ϕ^μ are some functions of ω, independent of θ. Indeed, ϕ^μ is the inverse of the ratio of the infinitesimal change in the parameter θ and the corresponding change in the phase space coordinate ω^μ:

$$\omega'^\mu = \omega^\mu + \delta\theta\phi^\mu(\omega). \tag{26}$$

Show that a function $A(\omega) = A(\psi(\omega_0;\theta)) = A_\theta(\omega_0)$ will obey the differential equation

$$\frac{\partial A_\theta(\omega_0)}{\partial\theta} = \{A_\theta(\omega_0), \phi(\omega_0)\}_{\omega_0}, \tag{27}$$

which then leads to the power series solution

$$A(\omega) = A_\theta(\omega_0) = A(\omega_0) + \theta\{A_\theta(\omega_0),\phi(\omega_0)\} + \frac{\theta^2}{2!}\{\{A_\theta(\omega_0),\phi(\omega_0)\},\phi(\omega_0)\} + \cdots. \tag{28}$$

S7.3

$$\begin{aligned}\frac{\partial A_\theta(\omega_0)}{\partial\theta} &= \frac{\partial A(\psi(\omega_0;\theta))}{\partial\theta}\\ &= \frac{\partial A(\psi)}{\partial\psi^\lambda}\frac{\partial\psi^\lambda}{\partial\theta}\\ &= \epsilon^{\lambda\mu}\frac{\partial A(\psi)}{\partial\psi^\lambda}\frac{\partial\phi(\psi)}{\partial\psi^\mu}\\ &= \{A(\psi),\phi(\psi)\}_\psi\\ &= \{A_\theta(\omega_0),\phi(\omega_0)\}_{\omega_0},\end{aligned} \tag{29}$$

where we have used Eq.25 in the 3rd step. In the last step we switched from ψ to ω_0 in evaluating the Poisson Brackets and made use of the fact that ϕ is independent of θ:

$$\phi(\psi(\omega_0;\theta)) = \phi(\omega_0). \tag{30}$$

P7.4 [$10+4=14$ points]

A particle of mass m moves in one dimension under a potential $V(x) = kx^{-2}$, where x is the Cartesian coordinate and k a constant.

(a) Find $x(t)$ using the Poisson bracket form of the equation of motion for the quantity $y = x^2$ given the initial conditions $x(0) = x_0$, $p(0) = 0$.

(b) Show that the quantity $D = xp - 2Ht$ is a constant of the motion.

S7.4 (a) Using Eq. 28, we can write

$$f(t) = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathcal{H}^n f)_0 t^n = f(0) + (\mathcal{H}f)_0 t + \frac{1}{2!}(\mathcal{H}^2 f)_0 t^2 + \frac{1}{3!}(\mathcal{H}^3 f)_0 t^3 + \ldots, \tag{31}$$

where the operator $\mathcal{H}$ is defined as the Poisson bracket with the Hamiltonian:

$$\mathcal{H}f \equiv \{f, H\} \tag{32}$$

for any function $f(q, p, t)$. The subscript "0" indicates that the PB's are evaluated at $t = 0$.

Now,

$$H = \frac{p^2}{2m} + \frac{k}{x^2}. \tag{33}$$

So, for $y = x^2$ we have for the second term in the sum (since $\{f(x), g(x)\} = 0$,)

$$\mathcal{H}y = \{x^2, H\} = \frac{1}{2m}\{x^2, p^2\} = \frac{1}{2m}\left(\frac{\partial}{\partial x}x^2\right)\left(\frac{\partial}{\partial p}p^2\right) = \frac{2xp}{m}. \tag{34}$$

The next term is

$$\begin{aligned}
\mathcal{H}^2 y = \mathcal{H}(\mathcal{H}x^2) &= \mathcal{H}\left(\frac{2xp}{m}\right) \\
&= \frac{2}{m}\{xp, H\} \\
&= \frac{2}{m}(x\{p, H\} + p\{x, H\}) \\
&= \frac{2}{m}\left(x\left\{p, \frac{k}{x^2}\right\} + p\left\{x, \frac{p^2}{2m}\right\}\right) \\
&= \frac{2}{m}\left(-x\frac{\partial}{\partial x}\left(\frac{k}{x^2}\right) + \frac{p}{2m}\left(\frac{\partial}{\partial p}(p^2)\right)\right) \\
&= \frac{2}{m}\left(\frac{2k}{x^2} + \frac{p^2}{m}\right) \\
&= \frac{4}{m}H.
\end{aligned} \tag{35}$$

For $n = 3$,

$$\mathcal{H}^3 y = \mathcal{H}(\mathcal{H}^2 y) = \frac{4}{m}\mathcal{H}H = \frac{4}{m}\{H, H\} = 0. \tag{36}$$

Hence, all higher terms are zero, and the series terminates at $n = 2$. Thus the solution is a quadratic function in t:

$$\begin{aligned}
x^2(t) &= x_0^2 + \frac{2}{m}x_0 p_0 t + \frac{2}{m}H_0 t^2 \\
&= x_0^2 + \frac{2}{m}x_0 p_0 t + \frac{2}{m}\left(\frac{p_0^2}{2m} + \frac{k}{x_0^2}\right)t^2.
\end{aligned} \tag{37}$$

This is an example of how the Poisson brackets can be used to propagate a function in time (or any other parameter, for that matter) from given initial conditions. For $p_0 = 0$,

$$\begin{aligned}
x^2(t) &= x_0^2 + \frac{2k}{m x_0^2}t^2, \\
\text{or,} \qquad x(t) &= \left(x_0^2 + \frac{2k}{m x_0^2}t^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{38}$$

(b) The rate of change of D with t is given by

$$\frac{dD}{dt} = \{D, H\} + \frac{\partial D}{\partial t}. \tag{39}$$

Now, using Eq. 35,

$$\{D, H\} = \{xp, H\} - 2t\{H, H\} = -2H - 0 = -2H = -\frac{\partial D}{\partial t}, \tag{40}$$

or,

$$\frac{dD}{dt} = \{D, H\} + \frac{\partial D}{\partial t} = 0, \tag{41}$$

Hence, D is a constant of the motion.