Assignment: HW6 [40 points]

# Solutions

Assigned: 2006/11/10
Due: 2006/11/17

**P6.1** [$4 + 3 + 3 = 10$ points]
Consider a particle of mass $m$ moving in two dimensions in a potential well. Let us choose the origin of our coordinate system at the minimum of this well. The well would be termed *isotropic* if the potential did not depend on the polar angle.

(a) First, consider the anisotropic potential in a given Cartesian coordinate system:

$$V(x_1, x_2) = \frac{k}{2}(x_1^2 + x_2^2) + k' x_1 x_2; \qquad k > k' > 0. \tag{1}$$

Find the eigenfrequencies and normal modes, preferably by reasoning rather than brute-force matrix diagonalization. Give a physical interpretation of the normal modes.

(b) Use a qualitative physics-based argument to write down two independent constants of the motion. Verify your choice using the Poisson bracket equation

$$\dot{u} = \{u, H\}_{\text{PB}} + \frac{\partial u}{\partial t}, \tag{2}$$

where $u = u(q, p, t)$ and $H$ is the Hamiltonian.

(c) The oscillator becomes isotropic if $k' = 0$. Again use a qualitative physics-based argument to write down an additional independent constant of motion if $k' = 0$, and verify your choice with the PB equation above.

**S6.1** (a) The potential funcion describes an ellipsoid (i.e. the equipotential contours are ellipses in the $(x_1,\ x_2)$ plane). The $x_1 x_2$ term in $V$ indicates that oscillations along $x_1$ and $x_2$ are coupled: those are not normal coordinates. The symmetrical dependence of $V$ on $x_1$, $x_2$ suggests, however, that a $\frac{\pi}{4}$ rotation is a sensible guess for a set of normal Cartesian coordinates:

$$q_1 = \frac{1}{\sqrt{2}}(x_1 + x_2), \qquad q_2 = \frac{1}{\sqrt{2}}(x_1 - x_2), \tag{3}$$

whence,

$$x_1 = \frac{1}{\sqrt{2}}(q_1 + q_2), \qquad x_2 = \frac{1}{\sqrt{2}}(q_1 - q_2). \tag{4}$$

In terms of the new coordinates and velocities, the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) \\ &= \frac{1}{4}m((\dot{q}_1 + \dot{q}_2)^2 + (\dot{q}_1 - \dot{q}_2)^2) \\ &= \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2), \end{aligned} \tag{5}$$

and the potential energy is

$$\begin{aligned} V &= \frac{1}{2}k(x_1^2 + x_2^2) + k' x_1 x_2 \\ &= \frac{1}{4}k((q_1+q_2)^2 + (q_1-q_2)^2) + \frac{1}{2}k'(q_1^2 - q_2^2) \\ &= \frac{1}{2}k(q_1^2+q_2^2) + \frac{1}{2}k'(q_1^2-q_2^2) \\ &= \frac{1}{2}((k+k')q_1^2 + (k-k')q_2^2). \end{aligned} \tag{6}$$

So, the kinetic energy tensor is diagonal in $\dot{q}_1$, $\dot{q}_2$ and potential energy tensor is diagonal in $q_1$, $q_2$. Thus, the system executes independent simple harmonic motions in each normal coordinate $q_1$ and $q_2$ with the respective angular frequencies

$$\omega_1 = \sqrt{\frac{k+k'}{m}} \qquad \text{and} \qquad \omega_2 = \sqrt{\frac{k-k'}{m}}. \tag{7}$$

The first normal mode represents a SHM along the steepest slope inside the potential well (the minor axis of the ellipse), while the second represents that along the least steep slope, which is perpendicular to the first (the major axis of the ellipse).

(b) The total energy $E = T + V$ of any simple harmonic oscillator is a constant of the motion. In this example, the energy associated with each of the two normal modes,

$$E_1 = \frac{m}{2}\dot{q}_1^2 + \frac{k+k'}{2}q_1^2 \qquad \text{and} \qquad E_2 = \frac{m}{2}\dot{q}_2^2 + \frac{k-k'}{2}q_2^2 \tag{8}$$

is conserved separately.
Since the potential is conservative, $H = E_1 + E_2$. Denoting the momentum conjugate to $q_i$ by $p_i = \dfrac{\partial L}{\partial \dot{q}_i} = \dfrac{\partial(T-V)}{\partial \dot{q}_i} = m\dot{q}_i$, we have for the first normal mode,

$$\begin{aligned} \dot{E}_1 &= \{E_1, H\} + \frac{\partial E_1}{\partial t} \\ &= \{E_1, E_1 + E_2\} \\ &= \{E_1, E_1\} + \{E_1, E_2\} \\ &= \{E_1, E_2\} \\ &= \frac{\partial E_1}{\partial q_i}\frac{\partial E_2}{\partial p_i} - \frac{\partial E_1}{\partial p_i}\frac{\partial E_2}{\partial q_i} \\ &= \frac{\partial E_1}{\partial q_1}\frac{\partial E_2}{\partial p_1} + \frac{\partial E_1}{\partial q_2}\frac{\partial E_2}{\partial p_2} - \frac{\partial E_1}{\partial p_1}\frac{\partial E_2}{\partial q_1} - \frac{\partial E_1}{\partial p_2}\frac{\partial E_2}{\partial q_2} \\ &= 0. \end{aligned} \tag{9}$$

Similarly, $\dot{E}_2 = 0$.

(c) If $k' = 0$, then the potential is isotropic in the $x_1$, $x_2$ plane. The rotational symmetry implies that the polar angle is cyclic. Consequently, the angular momentum $\ell_3 = m(q_1 p_2 - q_2 p_1)$ is a constant of the motion.[1] With the Hamiltonian simplified to

$$H = \frac{1}{2}\left(\frac{p_1^2 + p_2^2}{m} + k(q_1^2 + q_2^2)\right), \tag{10}$$

we explicitly verify the conservation of the angular momentum:

$$\begin{aligned} \dot{\ell}_3 &= \{\ell_3, H\} + \frac{\partial \ell_3}{\partial t} \\ &= \{\ell_3, H\} \\ &= \frac{\partial \ell_3}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \ell_3}{\partial p_i}\frac{\partial H}{\partial q_i} \\ &= p_2 p_1 - p_1 p_2 + mkq_2 q_1 - mkq_1 q_2 \\ &= 0. \end{aligned} \tag{11}$$

**P6.2** [$5 + 1 + 2 = 8$ points]

(a) Verify the Poisson bracket equation

$$\{L_i, L_j\} = \epsilon_{ijk} L_k \tag{12}$$

among the Cartesian components of angular momentum of a spherical pendulum of mass $m$ in a gravitaional field of acceleration $\vec{g}$ pointing opposite to the pole. $\epsilon_{ijk}$ represents the Levi-Civita tensor[2].
*Hint: Start with expressing the Lagrangian in spherical coordinates:* $\mathcal{L} = \mathcal{L}(\theta, \phi, \dot{\theta}, \dot{\phi})$.

(b) Likewise, verify

$$\{p_\theta, p_\phi\} = 0 \tag{13}$$

for the spherical pendulum.

(c) The mathematical machinery of Poisson brackets evidently tells us that some perpendicular momentum components are valid canonical momenta (e.g., $p_\theta$ and $p_\phi$), while others are not (e.g., the Cartesian components of angular momentum above). Explain the physics behind this.

**S6.2** Let the distance between the bob and the fixed support point be $\ell$.

(a) Then, the Lagrangian is

$$\mathcal{L}(\theta, \phi, \dot{\theta}, \dot{\phi}) = T - V = \frac{m}{2}\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + mg\ell\cos\theta. \tag{14}$$

This gives the conjugate momenta

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \qquad \text{and} \qquad p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m\ell^2\sin^2\theta\dot{\phi}, \tag{15}$$

[1] The subscript "3" after $\ell$ merely states that the angular momentum is perpendicular to the plane of motion.

[2] In 3 dimensions, the (antisymmetric) Levi-Civita tensor is defined as $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$, all other $\epsilon_{ijk} = 0$. In $n$ dimensions $\epsilon_{123\ldots n}$ and its even permutations (i.e., even number of swapping of adjacent indices) are 1, odd permutations $-1$, all others 0.

and the Hamiltonian

$$H(\theta, \phi, p_\theta, p_\phi) = p_\theta \dot{\theta} + p_\phi \dot{\phi} - \mathcal{L} = \frac{1}{2m\ell^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - mg\ell\cos\theta. \tag{16}$$

Now, the Cartesian coordinates can be expressed in terms of the sphericals:

$$\begin{aligned} x &= \ell \sin\theta\cos\phi, \\ y &= \ell\sin\theta\sin\phi, \\ z &= \ell\cos\theta. \end{aligned} \tag{17}$$

And the Cartesian momenta are

$$\begin{aligned} p_x &= m\dot{x} = m\ell(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi) = \frac{p_\theta}{\ell}\cos\theta\cos\phi - \frac{p_\phi\sin\phi}{\ell\sin\theta}, \\ p_y &= m\dot{y} = m\ell(\dot{\theta}\cos\theta\sin\phi - \dot{\phi}\sin\theta\cos\phi) = \frac{p_\theta}{\ell}\cos\theta\sin\phi + \frac{p_\phi\cos\phi}{\ell\sin\theta}, \\ p_z &= m\dot{z} = m\ell\dot{\theta}\sin\theta = -\frac{p_\theta}{\ell}\sin\theta. \end{aligned} \tag{18}$$

Now we can compute the Cartesian angular momentum components:

$$\begin{aligned} L_x &= yp_z - zp_y \\ &= \ell\sin\theta\sin\phi\left(-\frac{p_\theta}{\ell}\sin\theta\right) - \ell\cos\theta\left(\frac{p_\theta}{\ell}\cos\theta\sin\phi + \frac{p_\phi\cos\phi}{\ell\sin\theta}\right) \\ &= -p_\theta\sin\phi - p_\phi\cot\theta\cos\phi. \end{aligned} \tag{19}$$

Similarly,

$$L_y = zp_x - xp_z = p_\theta\cos\phi - p_\phi\cot\theta\sin\phi, \tag{20}$$

and

$$L_z = xp_y - yp_x = p_\phi. \tag{21}$$

So,

$$\begin{aligned} \{L_y, L_z\} &= \frac{\partial L_y}{\partial\theta}\frac{\partial L_z}{\partial p_\theta} + \frac{\partial L_y}{\partial\phi}\frac{\partial L_z}{\partial p_\phi} - \frac{\partial L_y}{\partial p_\theta}\frac{\partial L_z}{\partial\theta} - \frac{\partial L_y}{\partial p_\phi}\frac{\partial L_z}{\partial\phi} \\ &= 0 + \frac{\partial L_y}{\partial\phi} + 0 + 0 \\ &= -p_\theta\sin\phi - p_\phi\cot\theta\cos\phi \\ &= L_x. \end{aligned} \tag{22}$$

Similarly it can be shown that $\{L_z, L_x\} = L_y$ and $\{L_x, L_y\} = L_z$. These results are compactified in

$$\{L_i, L_j\} = \epsilon_{ijk}L_k. \tag{23}$$

(b)

$$\{p_\theta, p_\phi\} = \frac{\partial p_\theta}{\partial\theta}\frac{\partial p_\phi}{\partial p_\theta} + \frac{\partial p_\theta}{\partial\phi}\frac{\partial p_\phi}{\partial p_\phi} - \frac{\partial p_\theta}{\partial p_\theta}\frac{\partial p_\phi}{\partial\theta} - \frac{\partial p_\theta}{\partial p_\phi}\frac{\partial p_\phi}{\partial\phi} = 0. \tag{24}$$

(c) The orientation of a rigid body in space is fully specified by two independent coordinates, e.g. $\theta$ and $\phi$. It is therefore impossible to find 3 independent components of angular momentum. Consequently, the Poisson brackets $\{L_i, L_j\}$ do not vanish.

**P6.3** $[2+5+2+1=10$ points$]$
Consider a system with a time-dependent Hamiltonian

$$H(q,p,t) = H_0(q,p) - \epsilon q \sin(\omega t), \tag{25}$$

where $\epsilon$ and $\omega$ are known constants and $\dfrac{\partial H_0}{\partial t} = 0$.

(a) Derive Hamilton's canonical equations of motion for the system.

(b) Use a canonical transformation generating function $G(q,P,t)$ to find a new Hamiltonian $H'$ and new canonical variables $Q, P$ such that $H'(Q,P) = H_0(q,p)$.
*Hint: The partial differential equations do not tell us how $q$ and $P$ are related in the generating function. We can take an educated guess though. $G = qP - \frac{\epsilon q}{\omega}\cos(\omega t)$ works.*

(c) Verify that Hamilton's canonical equations of motion are invariant under the transformation.

(d) Suggest a possible physical interpretation of the time-dependent term in $H$.

**S6.3** (a) Hamilton's canonical equations of motion are

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = \frac{\partial H_0}{\partial p} \\ \text{and} \quad \dot{p} &= -\frac{\partial H}{\partial q} = -\frac{\partial H_0}{\partial q} + \epsilon \sin(\omega t) \end{aligned} \tag{26}$$

(b) The generating function has to satisfy the following conditions:

$$p = \frac{\partial G}{\partial q}, \quad Q = \frac{\partial G}{\partial P}, \quad \text{and} \quad H' = H + \frac{\partial G}{\partial t}. \tag{27}$$

The last one means

$$\frac{\partial G}{\partial t} = H' - H = \epsilon q \sin(\omega t). \tag{28}$$

While $G$ cannot be fully determined from this, the first two relations in Eq. 27 suggest that the identity transformation $q_i P_i$ be added to the time integral of $\epsilon q \sin(\omega t)$. So,

$$G = qP - \frac{\epsilon q}{\omega}\cos(\omega t) \tag{29}$$

is a sensible trial function. This leads to

$$\begin{aligned} P &= p + \frac{\epsilon}{\omega}\cos(\omega t), \\ Q &= q, \\ \text{and} \quad H'(Q,P,t) &= H(q,p,t) + \epsilon q \sin(\omega t) \\ &= H_0 - \epsilon q \sin(\omega t) + \epsilon q \sin(\omega t) \\ &= H_0, \end{aligned} \tag{30}$$

where the transformed Hamiltonian can be written as

$$H'(Q,P) = H_0(q,p) = H_0(Q, P - \frac{\epsilon}{\omega}\cos(\omega t)). \tag{31}$$

So, our trial function passes the requirements.

(c)

$$\frac{\partial H'}{\partial P} = \frac{\partial H_0}{\partial p}\frac{\partial p}{\partial P} + \frac{\partial H_0}{\partial q}\frac{\partial q}{\partial P} = \frac{\partial H_0}{\partial p} + 0 = \dot{q} = \dot{Q}, \tag{32}$$

and

$$-\frac{\partial H'}{\partial Q} = -\frac{\partial H_0}{\partial q}\frac{\partial q}{\partial Q} - \frac{\partial H_0}{\partial p}\frac{\partial p}{\partial Q} = -\frac{\partial H_0}{\partial q} - 0 = \dot{p} - \epsilon\sin(\omega t) = \dot{P}. \tag{33}$$

Therefore, the transformation is canonical and Hamilton's equations of motion remain invariant.

(d) The difference between the old and the new Hamiltonians, $\epsilon q \sin(\omega t)$, must have the dimensions of energy, and $\epsilon\sin(\omega t)$ has the form of a generalized force. Thus, in the transformed system, the particle experiences a sinusoidal force that has an angular frequency $\omega$. The variation of the force in phase space is undetermined.

**P6.4** [4 points]
Show that canonical transformations leave the physical dimension of the product $p_i q_i$ unchanged, i.e., $[P_i Q_i] = [p_i q_i]$. Let $\Phi$ be the generating functin for a canonical transformation. Show that

$$[P_i Q_i] = [p_i q_i] = [\Phi] = [Ht], \tag{34}$$

where $H$ is the Hamiltonian and $t$ the time.

**S6.4** Set $w_\alpha = (q_1, \ldots, q_n, p_1 \ldots, p_n)$, $w_\beta = (Q_1, \ldots, Q_n, P_1 \ldots, P_n)$, and define

$$M_{\alpha\beta} \equiv \frac{\partial w_\beta}{\partial w_\alpha}, \tag{35}$$

and

$$\epsilon = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix}. \tag{36}$$

Then the equation that relates $\frac{\partial q_i}{\partial Q_k}$ to $\frac{\partial P_k}{\partial p_i}$, $\frac{\partial q_j}{\partial P_l}$ to $\frac{\partial Q_l}{\partial p_j}$ etc. is given by

$$\mathbf{M}^T \epsilon \mathbf{M} = \epsilon. \tag{37}$$

From this it follows that $[P_i Q_i] = [p_i q_i] = [\Phi]$. If $\Phi(w_\alpha, w_\beta)$ is the generating function of the canonical transformation, as $\tilde{H} = H + \frac{\partial \Phi}{\partial t}$, the function $\Phi$ has thte dimension of the product $Ht$. The last part of the assertion then follows from the canonical equations.

**P6.5** [$4 + 4 = 8$ points]

The Hamiltonian $H = \dfrac{p^2}{2m} + \dfrac{m\omega^2 q^2}{2}$ describes a simple harmonic oscillator of mass $m$ and frequency $\omega$. Introducing the transformation

$$x_1 \equiv \omega\sqrt{m}q, \qquad x_2 \equiv \frac{p}{\sqrt{m}}, \qquad \tau \equiv \omega t, \tag{38}$$

we obtain $H = \frac{1}{2}(x_1^2 + x_2^2)$.

(a) What is the generating function $\hat{\Phi}_1(x_1, y_1)$ for the canonical transformation $\{x_1, x_2\} \to \{y_1, y_2\}$ that corresponds to the function $\Phi(q, Q) = \dfrac{m\omega q^2}{2}\cot Q$?

(b) Calculate the matrix $M_{ij} \equiv \dfrac{\partial x_i}{\partial y_j}$ and confirm that $\det \mathbf{M} = 1$ and $\mathbf{M}^T \epsilon \mathbf{M} = \epsilon$ ($\epsilon$ is the antisymmetric matrix used in the lectures to put the coordinates $q_i$ and momenta $p_i$ in a single array $w_\mu$).

*Hint: $y_1 = Q$, $y_2 = \omega P$, where $Q$ and $P$ are the new generalized coordinates and momenta, respectively.*

**S6.5** (a) We can wrirte

$$\tilde{H} = H + \frac{\partial \Phi}{\partial \tau}, \tag{39}$$

such that $[\Phi] = [H] = [x_1 x_2] = [\omega p q]$. The new generalized coordinate $y_1 = Q$ is dimensionless. As $y_1 y_2$ has the same dimension as $x_1 x_2$, $y_2$ must have the dimension of $H$. So, $y_2 = \omega P$. Therefore,

$$\hat{\Phi}_1(x_1, y_1) = \frac{1}{2} x_1^2 \cot y_1. \tag{40}$$

(b)

$$x_2 = \frac{\partial \hat{\Phi}}{\partial x_1} = x_1 \cot y_1, \qquad y_2 = -\frac{\partial \hat{\Phi}}{\partial y_1} = \frac{x_1^2}{2\sin^2 y_1}, \tag{41}$$

or,

$$x_1 = \sqrt{2y_2}\sin y_1, \qquad x_2 = \sqrt{2y_2}\cos y_1. \tag{42}$$

Thus,

$$M_{\alpha\beta} = \frac{\partial x_\alpha}{\partial y_\beta} = \begin{pmatrix} (2y_2)^{\frac{1}{2}}\cos y_1 & (2y_2)^{-\frac{1}{2}}\sin y_1 \\ -(2y_2)^{\frac{1}{2}}\sin y_1 & (2y_2)^{-\frac{1}{2}}\cos y_1 \end{pmatrix} \tag{43}$$

It is easy to verify that $\det \mathbf{M} = 1$ and $\mathbf{M}^T \epsilon \mathbf{M} = \epsilon$.