Assignment: HW6 [40 points]

Assigned: 2006/11/10
Due: 2006/11/17

P6.1 [$4 + 3 + 3 = 10$ points]

Consider a particle of mass m moving in two dimensions in a potential well. Let us choose the origin of our coordinate system at the minimum of this well. The well would be termed *isotropic* if the potential did not depend on the polar angle.

(a) First, consider the anisotropic potential in a given Cartesian coordinate system:

$$V(x_1, x_2) = \frac{k}{2}(x_1^2 + x_2^2) + k' x_1 x_2; \qquad k > k' > 0. \tag{1}$$

Find the eigenfrequencies and normal modes, preferably by reasoning rather than brute-force matrix diagonalization. Give a physical interpretation of the normal modes.

(b) Use a qualitative physics-based argument to write down two independent constants of the motion. Verify your choice using the Poisson bracket equation

$$\dot{u} = \{u, H\}_{\text{PB}} + \frac{\partial u}{\partial t}, \tag{2}$$

where $u = u(q, p, t)$ and H is the Hamiltonian.

(c) The oscillator becomes isotropic if $k' = 0$. Again use a qualitative physics-based argument to write down an additional independent constant of motion if $k' = 0$, and verify your choice with the PB equation above.

P6.2 [$5 + 1 + 2 = 8$ points]

(a) Verify the Poisson bracket equation

$$\{L_i, L_j\} = \epsilon_{ijk} L_k \tag{3}$$

among the Cartesian components of angular momentum of a spherical pendulum of mass m in a gravitaional field of acceleration $\vec{g}$ pointing opposite to the pole. ϵ_{ijk} represents the Levi-Civita tensor[1].
Hint: Start with expressing the Lagrangian in spherical coordinates: $\mathcal{L} = \mathcal{L}(\theta, \phi, \dot{\theta}, \dot{\phi})$.

(b) Likewise, verify

$$\{p_\theta, p_\phi\} = 0 \tag{4}$$

for the spherical pendulum.

[1] In 3 dimensions, the (antisymmetric) Levi-Civita tensor is defined as $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$, all other $\epsilon_{ijk} = 0$. In n dimensions $\epsilon_{123\ldots n}$ and its even permutations (i.e., even number of swapping of adjacent indices) are 1, odd permutations -1, all others 0.

(c) The mathematical machinery of Poisson brackets evidently tells us that some perpendicular momentum components are valid canonical momenta (e.g., p_θ and p_ϕ), while others are not (e.g., the Cartesian components of angular momentum above). Explain the physics behind this.

P6.3 [$2 + 5 + 2 + 1 = 10$ points]
Consider a system with a time-dependent Hamiltonian

$$H(q, p, t) = H_0(q, p) - \epsilon q \sin(\omega t), \tag{5}$$

where ϵ and ω are known constants and $\dfrac{\partial H_0}{\partial t} = 0$.

(a) Derive Hamilton's canonical equations of motion for the system.

(b) Use a canonical transformation generating function $G(q, P, t)$ to find a new Hamiltonian H' and new canonical variables Q, P such that $H'(Q, P) = H_0(q, p)$.
Hint: The partial differential equations do not tell us how q and P are related in the generating function. We can take an educated guess though. $G = qP - \dfrac{\epsilon q}{\omega}\cos(\omega t)$ works.

(c) Verify that Hamilton's canonical equations of motion are invariant under the transformation.

(d) Suggest a possible physical interpretation of the time-dependent term in H.

P6.4 [4 points]
Show that canonical transformations leave the physical dimension of the product $p_i q_i$ unchanged, i.e., $[P_i Q_i] = [p_i q_i]$. Let Φ be the generating functin for a canonical transformation. Show that

$$[P_i Q_i] = [p_i q_i] = [\Phi] = [Ht], \tag{6}$$

where H is the Hamiltonian and t the time.

P6.5 [$4 + 4 = 8$ points]
The Hamiltonian $H = \dfrac{p^2}{2m} + \dfrac{m\omega^2 q^2}{2}$ describes a simple harmonic oscillator of mass m and frequency ω. Introducing the transformation

$$x_1 \equiv \omega\sqrt{m}q, \qquad x_2 \equiv \frac{p}{\sqrt{m}}, \qquad \tau \equiv \omega t, \tag{7}$$

we obtain $H = \frac{1}{2}(x_1^2 + x_2^2)$.

(a) What is the generating function $\hat{\Phi}_1(x_1, y_1)$ for the canonical transformation $\{x_1, x_2\} \to \{y_1, y_2\}$ that corresponds to the function $\Phi(q, Q) = \dfrac{m\omega q^2}{2}\cot Q$?

(b) Calculate the matrix $M_{ij} \equiv \dfrac{\partial x_i}{\partial y_j}$ and confirm that $\det \mathbf{M} = 1$ and $\mathbf{M}^T \epsilon \mathbf{M} = \epsilon$ (ϵ is the antisymmetric matrix used in the lectures to put the coordinates q_i and momenta p_i in a single array w_μ).

Hint: $y_1 = Q$, $y_2 = \omega P$, where Q and P are the new generalized coordinates and momenta, respectively.