Assignment: HW4 [40 points]

Assigned: 2006/10/25
Due: 2006/11/01

Solutions

P4.1 [$4 + 4 = 8$ points]

(a) Find the moment of inertia tensor I of a uniform cube of side s and mass M whose pivot is at a corner and whose sides are lined up along the axes of an orthonormal coordinate system.

(b) Find the principal axis system and the moments of inertia.

S4.1 **(a)** The elements of the moment-of-inertia tensor are

$$I_{ij} = \int_V (\delta_{ij} x^2 - x_i x_j)\rho dV \tag{1}$$

where V the volume and $\rho \equiv \frac{M}{V}$ is the uniform density of the rigid body. For the given cube,

$$\begin{aligned} I_{11} = I_{22} = I_{33} &= \rho \int_0^s dx_1 \int_0^s dx_2 \int_0^s dx_3 (x_1^2 + x_2^2) = \frac{2}{3}\rho s^5 = \frac{2}{3} M s^2 \\ I_{12} = I_{23} = I_{31} &= -\rho \int_0^s dx_1 \int_0^s dx_2 \int_0^s dx_3 (x_1 x_2) = -\frac{1}{4}\rho s^5 = -\frac{1}{4} M s^2 \end{aligned} \tag{2}$$

Therefore, the I matrix is

$$I = \frac{2}{3} M s^2 A \equiv \frac{2}{3} M s^2 \begin{bmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{bmatrix} \tag{3}$$

where $\alpha = -\dfrac{3}{8}$ and A is the 3×3 matrix.

(b) To find the principal axes, we digonalize A. The eigenvalue equation is

$$(1-\lambda)^3 - 3\alpha^2(1-\lambda) + 2\alpha^3 = (1-\lambda-\alpha)^2(1-\lambda+2\alpha) = 0. \tag{4}$$

whose solutions are

$$\lambda_1 = 2\alpha + 1 = \frac{1}{4}, \qquad \lambda_2 = \lambda_3 = 1 - \alpha = \frac{11}{8}. \tag{5}$$

The eigenvector belonging to λ_1 (not normalized) is $(1, 1, 1)$, and thus it lies along the diagonal of the cube: one of the principal axes of the cube pivoted at a corner is along its diagonal passing through that corner. Since the remaining two eigenvalues are degenerate, any two mutually perpendicular directions perpendicular to the first eigenvector can serve as the other two principal axes. The corresponding moments of inertia are

$$I_1 = \frac{1}{6} M s^2, \qquad I_2 = I_3 = \frac{11}{12} M s^2. \tag{6}$$

P4.2 [4 points]
The cube in Problem 1 rotates instantaneously about the edge that is lined up along the x_1 axis. Find the angle between the angular momentum $\mathbf{L}$ and the angular velocity $\vec{\omega}$.

S4.2 If $\vec{\omega}$ is along the x_1 axis, $\vec{\omega} = (\omega, 0, 0)$. In that coordinate system,

$$\mathbf{L} = \frac{2}{3}Ms^2 \begin{bmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{bmatrix} = \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix} = \frac{2}{3}Ms^2\omega \begin{bmatrix} 1 \\ \alpha \\ \alpha \end{bmatrix} = Ms^2\omega \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}. \tag{7}$$

Clearly, $\vec{\omega}$ is not an eigenvector of I and $\mathbf{L}$ is not parallel to $\vec{\omega}$. The angle θ between $\mathbf{L}$ and $\vec{\omega}$ is given by

$$\cos\theta = \frac{\mathbf{L}\cdot\vec{\omega}}{L\omega} = \frac{\frac{2}{3}}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2}} = 0.8835, \tag{8}$$

or, $\theta = 0.4876$.

P4.3 [4 points]
Consider the symmetric dumbbell rotating in a "double cone" about its CM as shown in Fig. 4.3: two equal point masses m connected by a massless inextensible link of length 2ℓ. Find the angular momentum of the system and the torque required to maintain the motion.

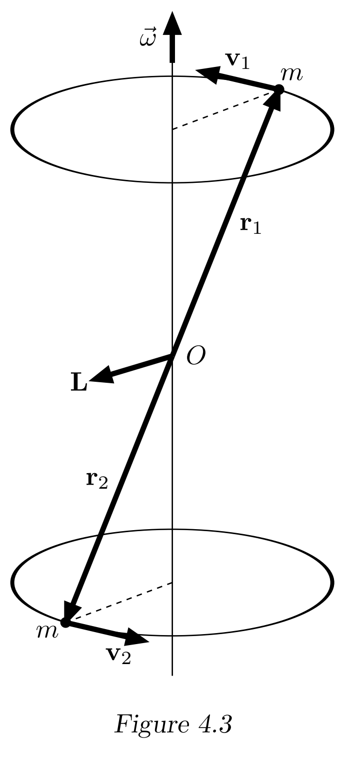


Figure 4.3

S4.3 The relation connecting $\mathbf{r}_\alpha$, $\mathbf{v}_\alpha$ ($\alpha = 1, 2$) and $\vec{\omega}$ is

$$\mathbf{v}_\alpha = \tilde{\omega} \times \mathbf{r}_\alpha \tag{9}$$

and the relation connecting $\mathbf{r}_\alpha$, $\mathbf{v}_\alpha$ ($\alpha = 1, 2$) and $\mathbf{L}$ is

$$\mathbf{L} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times \ \mathbf{v}_\alpha. \tag{10}$$

Note that $\vec{\omega}$ is directed along the axis of rotation while $\mathbf{L}$ is perpendicular to the line connecting the two masses. Thus, the angular momentum vector $\mathbf{L}$ is not constant in time but rotates with an angular velocity $\vec{\omega}$ in such a way that it traces out a cone whose axis is the axis of rotation:

$$\mathbf{N} = \dot{\mathbf{L}} \neq 0, \tag{11}$$

i.e., to keep the dumbbell rotating as shown in Fig. 4.3, a constant external torque must be applied. In this case, $m_1 = m_2 = m$, $|\mathbf{r}_1| = |\mathbf{r}_2| = \ell$. Hence, using Eqs. 9 and 10, we get

$$|\mathbf{L}| = 2m\ell^2\omega \sin\theta \tag{12}$$

where θ is the angle between the axis of rotation and the line connecting the two masses. The direction of $\mathbf{L}$ is perpendicular to both.

P4.4 [8 points]
Find the characteristic frequencies of the coupled circuits in Fig. 4.4. Comment on the two modes of oscillation (*Hint: only one mode is damped*). Examine how the damped mode depends on the relation between R^2 and $\dfrac{L}{C}$.

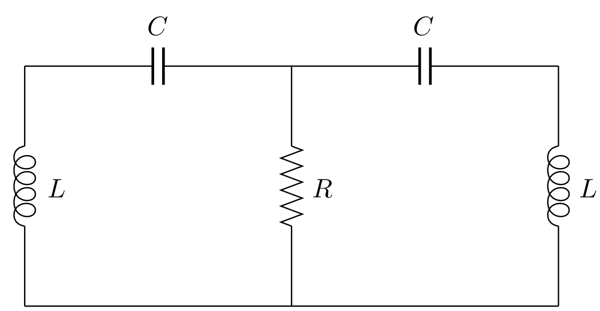
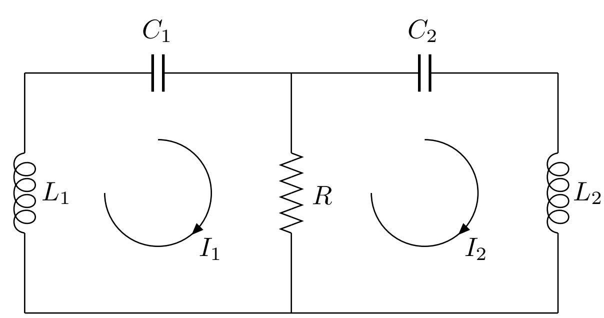


Figure 4.4

S4.4 Let us start from a more general circuit shown in the figure below.



Let the charge on C_1 and C_2 be Q_1 and Q_2 respectively. Then, using Kirchhoff's law,

$$\begin{aligned} L_1\dot{I}_1 + R(I_1 - I_2) + \frac{Q_1}{C_1} &= 0, \\ L_2\dot{I}_2 + R(I_2 - I_1) + \frac{Q_2}{C_2} &= 0. \end{aligned} \tag{13}$$

Taking time derivatives of Eqs. 13 and using $\dot{Q} = I$,

$$\begin{aligned} L_1\ddot{I}_1 + R(\dot{I}_1 - \dot{I}_2) + \frac{I_1}{C_1} &= 0, \\ L_2\ddot{I}_2 + R(\dot{I}_2 - \dot{I}_1) + \frac{I_2}{C_2} &= 0. \end{aligned} \tag{14}$$

Let us try solutions of the form

$$\begin{aligned} I_1(t) &= A_1 e^{-i\omega t}, \\ I_2(t) &= A_2 e^{-i\omega t}. \end{aligned} \tag{15}$$

Using these trial solutions in Eqs. 14 gives the secular equation (by setting the determinant of the coefficients of the A's to 0):

$$\left(\omega^2 L_1 - \frac{1}{C_1} - i\omega R\right)\left(\omega^2 L_2 - \frac{1}{C_2} - i\omega R\right) + \omega^2 R^2 = 0. \tag{16}$$

It is clear from Eq. 16 that the oscillations will be damped because ω will have an imaginary part (The resistor in the circuit dissipates energy). The problem at hand is a special case in which $L_1 = L_2 = L$ and $C_1 = C_2 = C$, which simplifies the secular equation to

$$\left(\omega^2 L - \frac{1}{C} - i\omega R\right)^2 + \omega^2 R^2 = 0. \tag{17}$$

with solutions

$$\begin{aligned} \omega_1 &= \pm\frac{1}{\sqrt{LC}}, \\ \omega_2 &= \frac{i}{L}\left(R \pm \sqrt{R^2 - \frac{L}{C}}\right). \end{aligned} \tag{18}$$

Hence, the general solution for $I_1(t)$ is

$$\begin{aligned} I_1(t) =& A_{1,1}^{+} e^{i\sqrt{\frac{1}{LC}}t} + A_{1,1}^{-} e^{-i\sqrt{\frac{1}{LC}}t} \\ &+ e^{-\frac{R}{L}t}\left(A_{1,2}^{+} e^{\sqrt{\frac{R^2}{L^2} - \frac{1}{LC}}t} + A_{1,2}^{-} e^{-\sqrt{\frac{R^2}{L^2} - \frac{1}{LC}}t}\right), \end{aligned} \tag{19}$$

and similarly for $I_2(t)$. Mode 1, represented by ω_1, is purely oscillatory with no damping. This is the case when I_1 and I_2 are equal in magnitude and sense, cancelling each other in R. Mode 2, represented by ω_2, is where I_1 and I_2 enforce each other in R. Examination of the exponents inside the parentheses in Eq. 19 shows that if $R^2 < \frac{L}{C}$, then there will be damped oscillations of I_1 and I_2 (underdamping), whereas if $R^2 > \frac{L}{C}$, then the currents will decrease monotonically without oscillation (overdamping).

P4.5 [10 points]
A mass M moves horizontally along a smooth rail. A pendulum of mass m hangs from M by a massless rod of length ℓ in a uniform vertical gravitational field $\mathbf{g}$ as shown in Fig. 4.5. Ignore all terms of order θ^3 and higher in expansions of trigonometric functions, as well as terms of order $\theta^2\dot{\theta}$ and higher in the Lagrangian. Find the eigenfrequencies and describe the normal modes.

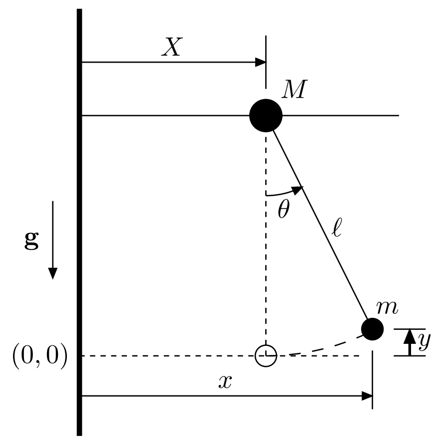


Figure 4.5

S4.5 The coordinates of the pendulum are given by

$$\begin{aligned} x &= X + \ell \sin\theta, \\ y &= \ell - \ell\cos\theta. \end{aligned} \tag{20}$$

and it's velocity (by taking time derivatives of Eq. 20) by

$$\begin{aligned} \dot{x} &= \dot{X} + \ell\dot{\theta}\cos\theta, \\ \dot{y} &= \ell\dot{\theta}\sin\theta. \end{aligned} \tag{21}$$

Hence, the kinetic energy is

$$\begin{aligned} T &= \frac{M}{2}\dot{X}^2 + \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \\ &= \frac{M}{2}\dot{X}^2 + \frac{m}{2}(\dot{X}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{X}\dot{\theta}\cos\theta), \end{aligned} \tag{22}$$

and the potential energy

$$U = mgy = mg\ell(1 - \cos\theta). \tag{23}$$

For $\theta \ll 1$, $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$. Using this approximation, and further negelcting terms of order $\theta^2\dot{\theta}$ and higher, Eqs. 22 and 23 reduce to

$$T = \frac{(M+m)}{2}\dot{X}^2 + \frac{m}{2}(\ell^2\dot{\theta}^2 + 2\ell\dot{X}\dot{\theta}) \tag{24}$$

and

$$U = mgy = \frac{m}{2}g\ell\theta^2 \tag{25}$$

respectively. Hence, choosing X and θ as the generalized coordinates,

$$\mathbf{m} = \begin{pmatrix} M+m & m\ell \\ m\ell & m\ell^2 \end{pmatrix}, \tag{26}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & mg\ell \end{pmatrix}, \tag{27}$$

and the eigenfrequencies are given by the equation

$$|\mathbf{A} - \mathbf{m}\omega^2| = 0$$

$$\text{or,} \qquad \begin{vmatrix} -(M+m)\omega^2 & -m\ell\omega^2 \\ -m\ell\omega^2 & mg\ell - m\ell^2\omega^2 \end{vmatrix} = 0, \tag{28}$$

$$\text{or,} \qquad \omega^2[\omega^2 M\ell^2 - mg\ell(m+M)] = 0,$$

i.e.,

$$\begin{aligned} \omega_1 &= 0, \\ \omega_2 &= \sqrt{\frac{g}{\ell}\frac{(M+m)}{M}}. \end{aligned} \tag{29}$$

Subsituting the eigenfrequencies in the equation

$$\sum_j (A_{j,k} - \omega_r^2 m_{j,k}) a_{j,r} = 0 \tag{30}$$

gives

$$\begin{aligned} a_{2,1} &= 0 && (k=2, \quad r=1), \\ a_{1,2} &= -\frac{m\ell}{m+M} a_{2,2} && (k=2, \quad r=2). \end{aligned} \tag{31}$$

Thus the expressions of the generalized coordinates in terms of the normal coordinates,

$$\begin{aligned} X &= a_{1,1}\eta_1 + a_{1,2}\eta_2, \\ \theta &= a_{2,1}\eta_1 + a_{2,2}\eta_2 \end{aligned} \tag{32}$$

become

$$\begin{aligned} X &= a_{1,1}\eta_1 - \frac{m\ell}{m+M} a_{2,2}\eta_2, \\ \theta &= a_{2,2}\eta_2 \end{aligned} \tag{33}$$

which can be inverted to yield the normal coordinates in terms of the generalized coordinates:

$$\begin{aligned} \eta_2 &= \frac{1}{a_{2,2}}\theta, \\ \eta_1 &= \frac{1}{a_{1,1}}\left(X + \frac{m\ell}{m+M}\theta \right). \end{aligned} \tag{34}$$

P4.6 [6 points]

Three oscillators of equal mass m moving in one dimension are coupled such that the potential energy of the system is given by

$$U = \frac{1}{2}\left[\kappa_1(x_1^2 + x_3^2) + \kappa_2 x_2^2 + \kappa_3(x_1x_2 + x_2x_3)\right] \tag{35}$$

where

$$\kappa_3 = \sqrt{2\kappa_1\kappa_2}. \tag{36}$$

Find the eigenfrequencies by solving the secular equation. What is the physical interpretation fo the zero-frequency mode?

S4.6

$$\mathbf{A} = \begin{pmatrix} \kappa_1 & \frac{1}{2}\kappa_3 & 0 \\ \frac{1}{2}\kappa_3 & \kappa_2 & \frac{1}{2}\kappa_3 \\ 0 & \frac{1}{2}\kappa_3 & \kappa_1 \end{pmatrix} \tag{37}$$

$$\mathbf{m} = m\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{38}$$

Thus, the secular equation is

$$\begin{aligned} & |\mathbf{A} - \mathbf{m}\omega^2| = 0 \\ \text{or,} \quad & \begin{vmatrix} \kappa_1 - m\omega^2 & \frac{1}{2}\kappa_3 & 0 \\ \frac{1}{2}\kappa_3 & \kappa_2 - m\omega^2 & \frac{1}{2}\kappa_3 \\ 0 & \frac{1}{2}\kappa_3 & \kappa_1 - m\omega^2 \end{vmatrix} = 0 \\ \text{or,} \quad & (\kappa_1 - m\omega^2)^2(\kappa_2 - m\omega^2) - \frac{1}{2}\kappa_3^2(\kappa_1 - m\omega^2)^2 = 0 \end{aligned} \tag{39}$$

Since $\kappa_3 = \sqrt{2\kappa_1\kappa_2}$, Eq. 39 reduces to

$$\begin{aligned} & (\kappa_1 - m\omega^2)[(\kappa_1 - m\omega^2)(\kappa_2 - m\omega^2) - \kappa_1\kappa_2] = 0 \\ \text{or,} \quad & m\omega^2(\kappa_1 - m\omega^2)[(\kappa_1 + \kappa_2) - m\omega^2] = 0. \end{aligned} \tag{40}$$

Therefore, the eigenfrequencies are

$$\begin{aligned} \omega_1 &= 0, \\ \omega_2 &= \sqrt{\frac{\kappa_1}{m}}, \\ \omega_3 &= \sqrt{\frac{\kappa_1 + \kappa_2}{m}}. \end{aligned} \tag{41}$$

For the zero-frequency mode, we have

$$\ddot{\eta}_1 + \omega_1^2\eta_1 = 0 \quad \Rightarrow \quad \ddot{\eta}_1 = 0 \quad \Rightarrow \quad \eta_1(t) = at + b. \tag{42}$$

That is, the zero-frequency mode corresponds to a pure translation of the system, with no oscillation.