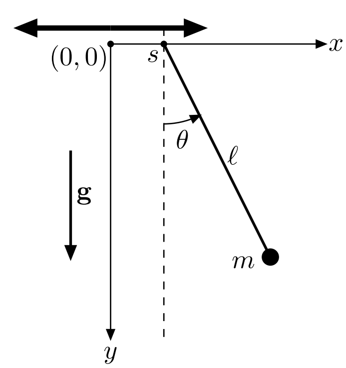
Assignment: HW3 [40 points]

Assigned: 2006/10/04
Due: 2006/10/11

P3.1 [7 points]
A smooth rod of length ℓ rotates in a plane with a constant angular velocity ω about an axis fixed at one end of the rod and perpendicular to the plane of motion. A bead of mass m, free to move along the rod, is initially positioned at the fixed end of the rod and given a slight push such that its initial speed directed towards the other end of the rod is $\omega\ell$. Using Lagrange's method, find the time it takes the bead to reach the other end of the rod.

P3.2 [7 points]
Using Lagrange's method, find the two-dimensional equation of motion of a pendulum of mass m suspended at the end of a massless rod of length ℓ in a gravitational field of uniform acceleration $\mathbf{g}$, whose point of support is executing a simple harmonic motion in the direction perpendicular to gravity, as shown in the figure below, i.e., the coordinates of the point of support are given as functions of time by

$$x_s(t) = x_0 \cos(\omega t); \qquad y_s(t) = 0.$$



Use θ, the angle between the pendulum and the direction of gravity, as the generalized coordinate, and express your answer in terms of θ (and its time derivatives). Assume θ to be small and use the corresponding approximations to simplify your answer. Compare your result to the equation of motion of a forced harmonic oscillator.

P3.3 [6+3 = 9 points]

(a) Obtain the Hamiltonian and the canonical equations for a particle in a central force field (in 3 dimensions).

(b) Take two of the initial conditions to be $p_\phi(0) = 0$ and $\phi(0) = 0$ (this is essentially the choice of a particular spherical coordinate system). Discuss the resulting simplification of the canonical equations.

P3.4 [$5 + 5 = 10$ points]

The 3-dimensional motion of a particle of mass m is described by the Lagrangian function

$$L = \frac{m}{2}\dot{x}_i^2 + \omega l_3, \tag{1}$$

where l_3 represents the third (z) component of the angular momentum, and ω is the corresponding constant angular velocity.

(a) Find the equations of motion, write them in terms of the complex variable $u \equiv x_1 + ix_2$, and of x_3, and solve them.

(b) Find the *kinetic* and *canonical* momenta, and construct the Hamiltonian. Show that the particle has only kinetic energy, and that the canonical momenta are conserved.

P3.5 [$3 + 4 = 7$ points]

Invariance under time translations and Noether's theorem. The theorem of E. Noether can be applied to the case of translations in *time* by means of the following procedure. Make t a coordinate-like variable by parametrizing both q and t as functions of a common independent variable τ:

$$q_i = q_i(\tau) \qquad (i = 1, \ldots, n); \qquad t = t(\tau), \tag{2}$$

and by defining a new Lagrangian function in terms of the old one:

$$\tilde{L}\left(q_i, t, \frac{dq_i}{d\tau}, \frac{dt}{d\tau}\right) \equiv L\left(q_i, \frac{1}{\frac{dt}{d\tau}}\frac{dq_i}{d\tau}, t\right)\frac{dt}{d\tau} \tag{3}$$

(a) Show that amilton's variational principle applied to $\tilde{L}$ yields the same equations of motion as it does for L.

(b) Assume L to be invariant under time translations:

$$h^s(q_i, t) = (q_i, t + s). \tag{4}$$

Apply Noether's theorem to $\tilde{L}$ and find the constant of motion corresponding to the invariance.