2.4 Parity transformation

An extremely simple group is one that has only two elements: $\{e, P\}$. Obviously, $P^{-1} = P$, so $P^2 = e$, with e represented by the unit $n \times n$ matrix in an n-dimensional representation. Thus, if $|\psi\rangle$ is an eigenstate of P, then

$$P|\psi\rangle = \pm|\psi\rangle \tag{2.65}$$

We could represent P as a phase-changing operator $P = e^{in\pi}$, where n is an integer. Such a phase would be additive for a composite system. However, the common practice is just to keep track of the sign, which then becomes a multiplicative quantum number.

The most familiar example of a multiplicative quantum number is *parity*, or *space inversion*, given by the Lorentz transformation in Eq. 1.12. Since both $\mathbf{x}$ and $\mathbf{p}$ change sign under a parity transformation, $\mathbf{J} = \mathbf{r} \times \mathbf{p}$ does not. Since $[P, p] \neq 0$, momentum eigenstates of particles in motion are not eigenstates of P, but one would expect stationary systems described as eigenstates of H and J^2, J_3 to be. Indeed, it had been known that parity is conserved in electromagnetic and strong interactions. However, to everyone's suprise, it was found in the 1950's that parity is violated (i.e. $[P, H] \neq 0$) in weak interactions. Indeed, it is a maximal violation: in the limit of a massless neutrino, which can only be produced in weak interactions, its spin 3-vector always points opposite to its momentum 3-vector, i.e. in a *left-handed helicity* state.[6]

It is useful to know how fields transform under parity. Since the single particle states are obtained by applying the field operator to the vacuum, this tells us the parity transformation properties of the state. For integral spin, if the transformation of the field is the same as that of a spatial tensor of the same rank (e.g. a scalar or a vector), then the field is said to have *natural* spin-parity. If the field transforms with an extra minus sign (e.g. a pseudoscalar or a pseudovector), then it is said to have unnatural spin-parity. The photon has natural spin-parity since the polarization vectors are ordinary 4-vectors. For spin-$\frac{1}{2}$ there is no analogy to space tensors, but we can (and will, in the next chapter) examine how the current behaves under parity transformation. We state, without proof for now, that we must assign opposite intrinsic parities to the fermions and antifermions. Thus, in the massless limit, all antineutrinos produced in weak interactions are right-handed.

One of the clearest manifestation of the maximal parity violation can be seen in the decay of charged pions, which is a weak process. Even though energy-wise a larger phase space is open for the decay $\pi^- \to e^-\bar{\nu}_e$ than $\pi^- \to \mu^-\bar{\nu}_\mu$, the former is exceedingly rare: $B(\pi^- \to e^-\bar{\nu}_e) = 1.23 \times 10^{-4}$ vs $B(\pi^- \to \mu^-\bar{\nu}_\mu) = 0.999877$ (No other 2-body decay is kinematically feasible)! This is explained by the fact that for spin (=0 for pions) to be conserved in the process, the ℓ^- and the $\bar{\nu}_\ell$ must have the same helicity. This is harder to achieve in the $e^-\bar{\nu}_e$ channel since the electron, being $\sim$200 times lighter than the muon, gets a much higher boost in the π^- rest frame. Consequently, the phase space is severly squeezed in the

[6] The helicity of a massive particle is not absolute - it can be flipped by a Lorentz boost.

$e^-\bar{\nu}_e$ channel because the boost needed for a frame to observe the e^- and the $\bar{\nu}_e$ in the opposite helicity states is much larger than for the μ^- and the $\bar{\nu}_\mu$.

2.5 Charge Conjugation

Charge conjugation is an operation that converts a particle to its antiparticle:

$$C|\psi\rangle = |\bar{\psi}\rangle, \tag{2.66}$$

resulting in the inversion of *all internal quantum numbers*, i.e. electric charge, isospin, color, lepton number, baryon number, ..., without alterning mass, momentum, and spin. This is another example of a group with only two elements, with

$$C^2|\psi\rangle = |\psi\rangle, \tag{2.67}$$

so the eigenvalues of C are ± 1. Unlike P, however, only particles that are their own antiparticles are eigenstates of C.

Classical electrodynamics is invariant under charge conjugation. The potentials and fields all change their signs so as to leave the forces unaffected. Since the field changes its sign, its quantum, the photon, has a charge conjugation eigenvalue of -1. In general, a fermion-antifermion system with orbital angular momentum l and total spin s constitutes an eigenstate of C with eigenvalue $(-1)^{l+s}$. This is the basis of classification of mesons, which are quark-antiquark bound states, in terms of J^{PC}.

Like P, C is a multiplicative quantum number that is conserved in the strong and electromagnetic interactions, but not in weak interactions. This can be readily seen as a consequence of parity violation in weak processes: applying C to any process involving a (massless left-handed) neutrino will result in one with a left-handed antineutrino, which does not exist. It was once argued by some that charge conjugation should be considered an integral part of a more general definition of "parity" amounting to CP by our definitions, which would be conserved in weak interactions. But a superbly reasoned prediction followed by a landmark experiment on mixing of the neutral kaon with its own antiparticle ($\Delta S = 2$) established CP violation in weak processes, albeit extremely mild compared to C or P violations separately. Subsequent studies of semileptonic decays of K_L^0, the longer-lived (near-symmetric) admixture of the two pseudoscalar CP eigenstates, showed even more dramatic evidence of CP violation through a slight imbalance in the decay fractions to $\pi^+e^-\bar{\nu}_e$ and $\pi^-e^+\nu_e$. This is a process that makes an absolute distinction between matter and antimatter, and provides an unambiguous, convention-free definition of positive charge: it is the charge carried by the lepton preferentially produced in the decay of K_L^0.[7]

[7] However, the level of CP violation within the standard model seems to fall far short of explaining the observed degree of preponderance of matter over antimatter in today's universe.

2.6 Lagrangian Density, Field Equations, and Conserved Currents

A particle theory is set up by defining the dynamical variables $\phi_j(x^\mu)$ that are functions of space-time. These *fields* are described by the Lagrangian density, which is a function of the fields and their first derivatives only

$$\mathcal{L} = \mathcal{L}(\phi_j, \partial_\mu \phi_j), \tag{2.68}$$

where $j = 1, 2, \ldots, n$ label the fields and/or different components of a field. The variational principle contends that the action integral

$$S = \int \mathcal{L}(\phi(x)) d^4x \tag{2.69}$$

is stationary with respect to any changes in ϕ that vanish on the boundary. Then, in a manner analogous to the one in classical mechanics where the Lagrangian is a function of space-time directly, it can be shown that the variational principle leads to the equations of motion

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)} - \frac{\partial \mathcal{L}}{\partial \phi_j} = 0. \tag{2.70}$$

The elementary particles of a theory appear as the solutions of the field equations resulting from the associated Lagrangian.[8] For example, in quantum electrodynamics, the photon is the quantum of the electromagnetic field, represented by the vector potential A^μ. The electron is represented by the fermion field ψ. The Lagrangian contains the fundamental interactions of the theory. For electrodynamics, that is the $J_\mu A^\mu$ term in the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu. \tag{2.71}$$

(Exercise: show that Maxwell's equations follow from this Lagrangian).

It is the potential energy parts of the Lagrangian that specify the theory. The kinetic energy parts are general and only depend on the spins of the particles. The potential energy terms specify the forces. These terms are collectively called the *interaction Lagrangian.*

Consider an infinitesimal change $\delta\phi_j$ in fields ϕ_j that is a symmetry of $\mathcal{L}$ in the the sense that

$$\mathcal{L}(\phi_j + \delta\phi_j) = \mathcal{L}(\phi_j). \tag{2.72}$$

[8]Composite objects may appear as bound states of the elementary particles.

We can then write using Eq. 2.70,

$$\begin{aligned} 0 &= \delta\mathcal{L}(\phi) = \mathcal{L}(\phi_j + \delta\phi_j) - \mathcal{L}(\phi_j) = \frac{\partial\mathcal{L}}{\partial\phi_j}\delta\phi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta(\partial_\mu\phi_j) \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\right)\delta\phi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\partial_\mu\delta\phi_j \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j\right). \end{aligned} \tag{2.73}$$

Thus the conserved current is

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j. \tag{2.74}$$

We will be particularly interested in transformations of the ϕ fields that are homogeneous, linear, and unitary. Such a symmetry takes the form

$$\delta\phi = i\epsilon_a T^a\phi, \tag{2.75}$$

where the T^a are a set of $n \times n$ Hermitian matrices accting on the space of the ϕ's and the ϵ_a are infinitesimal parameters. Equation 2.75 is the infinitesimal form of a unitary transformation

$$\phi \quad \rightarrow \quad \phi' = \exp(i\epsilon_a T^a)\phi \simeq (1 + i\epsilon_a T^a)\phi. \tag{2.76}$$

Let us consider for the moment only *global symmetries*, in which the parameters ϵ_a are independent of x, but otherwise arbitrary. Then the following is true:

$$\delta(\partial^\mu\phi) = i\epsilon_a T^a\partial^\mu\phi, \tag{2.77}$$

which looks just like Eq. 2.75. We say $\partial^\mu\phi$ transforms (under the symmetry operation) like ϕ.

In this case, the conserved currents take a particularly simple form. Taking out the infinitesimal parameters, we can write the conserved currents as

$$J^a_\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}T^a\phi. \tag{2.78}$$

So far we have not put the quantum into the quantum field theory. When we do, the fields ϕ_j and their conjugate momenta

$$\Pi_j = \frac{\partial\mathcal{L}}{\partial(\partial^0\phi_j)} \tag{2.79}$$

satisfy the *equal time (anti-)commutation* relations for (fermion) boson fields:

$$\begin{aligned} [\phi_j(x), \phi_k(y)]^{\mathrm{ET}}_\pm &= [\Pi_j(x), \Pi_k(y)]^{\mathrm{ET}}_\pm = 0 \\ [\phi_j(x), \Pi_k(y)]^{\mathrm{ET}}_\pm &= i\delta_{jk}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \end{aligned} \tag{2.80}$$

where "ET"stands for equal time, $x^0 = y^0$; and $[A, B]_\pm = AB \pm BA$.

We will deal with situations where the matrices are organized into an algebra that closes under commutation relations of Eq. 2.18. The time components of the currents are related to charges

$$Q^a = \int d^3x J_0^a(x), \tag{2.81}$$

which satisfy Eq. 2.18. It follows after some algebra that

$$[Q^a, Q^b] = if^{abc}Q^c. \tag{2.82}$$

The existence of these charges and their associates currents is an important consequence of the symmetry. Note that nothing in the derivation requires the interpretation to be in terms of electric charge. Indeed, particles have a number of charges, of which at least some can be related to conserved currents. Let us examine a few examples.

Example 1

The simplest example is that of a real scalar (i.e., a spinless) field ϕ of mass m, described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2), \tag{2.83}$$

which leads to the Klein-Gordon wave equation

$$\partial^\mu\partial_\mu\phi = m^2\phi. \tag{2.84}$$

Identifying the free-particle 4-momentum with the 4-gradient in space-time $p^\mu = \partial^\mu = \dfrac{\partial}{\partial_\mu x}$, we see that this is merely the relativistic rendition of the Schrödinger equation. The solution (the field amplitude), except for a normalization constant, is given by

$$\phi(x^\mu) = e^{-ip^\mu x_\mu}. \tag{2.85}$$

Interactions of the field with other particles requires the introduction of a *source* term into the field equations. The simplest example modifies the field equations as follows:

$$(\partial^\mu\partial_\mu - m^2)\phi = \rho. \tag{2.86}$$

Since ϕ is a Lorentz scalar, ρ must be so as well. If the source is localized, static and unperturbed by the interactions (a very heavy particle at rest would be a good approximation), then choosing the origin at the source,

$$\rho = g\delta^3(\mathbf{x}). \tag{2.87}$$

Since ρ is not time dependent, Eq. 2.86 becomes

$$(-\nabla^2 - m^2)\phi = g\delta^3(\mathbf{x}). \tag{2.88}$$

Writing

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{\phi}(\mathbf{k}) \tag{2.89}$$

and the inverse Fourier transform

$$\tilde{\phi}(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \phi(\mathbf{x}) \tag{2.90}$$

yields (since $\nabla^2 \to -\mathbf{k}^2$)

$$(\mathbf{k}^2 + m^2)\tilde{\phi}(\mathbf{k}) = \frac{g}{2\pi^{\frac{3}{2}}}. \tag{2.91}$$

Substituting this into Eq. 2.89, we get

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + m^2}. \tag{2.92}$$

Evaluation of the integral yields the time-independent solution

$$\phi(r) = \frac{g}{4\pi} \frac{e^{-mr}}{r}, \tag{2.93}$$

which is called the *Yukawa potential*. We see that the strength of the potential at a given point is determined by the *coupling constant* g, and m. A large value of m is gives a short range of interaction. This is the reason, in fact, of the weakness of the weak interactions. It is an example of a general result that high-mass physics is hard to see at low energies, since it corresponds only to phenomena at very short distances.

If we had removed the constraint that the source be time-independent, the denominator in the integrand on the RHS of Eq. 2.92 would simply be modified to $m^2 - k^2$, where $k^2 = k^\mu k_\mu$. Indeed, such a denominator appears as a propagator whenever a particle is exchanged in an interaction. This conforms to the general interpretation, in a quantum field theory, that all interactions are due to the exchange of field quanta. The concepts of force and of interaction are used interchangeably. Usually the matrix elements are written in the momentum space. Then from Eq. 2.92, the momentum space quantity representing the exchanged particle of mass m is

$$\frac{1}{k^2 - m^2}. \tag{2.94}$$

This is called a *propagator*, which will show up whenever we write the matrix element. The complete propagator also has a phase factor and a numerator that depends on the spin of the exchanged particle, but for most calculation these can be considered as technical details that do not affect the qualitative results.

Example 2

Some interesting physics emerges if we consider a system of two real scalar fields, ϕ_1 and ϕ_2, that have the same mass m. Then we can expect from Eq. 2.83 that

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1 \partial^\mu \phi_1 - m^2 \phi_1^2) + \frac{1}{2}(\partial_\mu \phi_2 \partial^\mu \phi_2 - m^2 \phi_2^2). \tag{2.95}$$

We can combine ϕ_1 and ϕ_2 into a single complex scalar field ϕ by writing

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2). \tag{2.96}$$

Then

$$\phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \tag{2.97}$$

So, the Lagrangian can be rewritten as

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \tag{2.98}$$

Note that ϕ has the same mass m as ϕ_1 and ϕ_2, and ϕ and ϕ^* are normalized to the same total amplitude as ϕ_1 and ϕ_2 are.

The current density is

$$J^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*), \tag{2.99}$$

satisfying the continuity equation

$$\partial_\mu J^\mu = 0. \tag{2.100}$$

The field (wave) equations are

$$(\partial^\mu \partial_\mu - m^2)\phi = (\partial^\mu \partial_\mu - m^2)\phi^* = 0. \tag{2.101}$$

Example 3

If there were an Abelian vector (spin-1) field B^μ, like the electromagnetic field, but massive, the Lagrangian given in Eq. 2.71 would acquire an additional term

$$\frac{1}{2} m^2 B^\mu B_\mu, \tag{2.102}$$

to accommodate a mass term will in the wave equation. If we see a term $B^\mu B_\mu$ appear in a Lagrangian, we can identify its coefficient as $\frac{m^2}{2}$. Such a term explicitly violates gauge symmetry. Thus, the gauge symmetry forbids a non-zero mass for the photon.