Chapter 2

Symmetries, Groups, and Conservation Laws

The dynamical properties and interactions of a system of particles and fields are derived from the *principle of least action*, where the *action* is a 4-dimensional Lorentz-invariant integral of the corresponding Lagrangian density. The general theorem called *Noether's theorem* dictates that to every symmetry of the Lagrangian there is a conserved current. It is a key ingredient in the construction of theories in particle physics. Symmetries appear in many ways in the studies of particle interactions: gauged (local) and global symmetries, exact and approximate symmetries, explicitly realized and spontaneously broken symmetries. The branch of mathematics devoted to the study of symmetries is called *Group theory*. It will be useful to familiarize ourselves with some basic concepts of group theory.

2.1 Groups and Representations

Definitions A group is a set G on which a law of composition " $\cdot$ " is defined with the following properties:

1. *Closure*: if x_1 and x_2 are in G, so is $x_1 \cdot x_2$;

2. *Identity*: there is an identity element e in G such that $x \cdot e = e \cdot x = x$ for any x in G;

3. *Inverse*: for every x in G, there is an inverse element x^{-1} in G such that $x \cdot x^-1 = x^-1 \cdot x = e$;

4. *Associativity*: for every x_1, x_2, and x_3 in G, $(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$.

A group is said to be commutative or *Abelian* if $x_1 \cdot x_2 = x_2 \cdot x_1$ for all x_1, x_2 in G. Otherwise, it is *non-Abelian*.

A group may have a finite or infinite number of elements. For example, the set of all real numbers form a continuous Abelian group with an infinite number of elements under the composition law of arithmetic addition. The set of all possible permutations of 3 labelled objects is an example of a discrete non-Abelian group with a finite number of elements:

$$\begin{array}{rcll} (\ ) & : & (a,b,c) \to (a,b,c), & \\ (12) & : & (a,b,c) \to (b,a,c), & \\ (23) & : & (a,b,c) \to (a,c,b), & \\ (31) & : & (a,b,c) \to (c,b,a), & \\ (123) & : & (a,b,c) \to (c,a,b) & \text{(cyclic permutation)}, \\ (321) & : & (a,b,c) \to (b,c,a). & \end{array} \tag{2.1}$$

The permutation group is an example of a *transformation group* on a physical system. In quantum mechanics, a transformation of the system is associated with a unitary operator in the Hilbert space.[1] Thus, a transformation group of a quantum mechanical system is associated with a mapping of the group into a set of unitary operators. So, for each x in G there is a $D(x)$ which is a unitary (linear) operator. Furthermore, the mapping must preserve the composition law

$$D(x_1)D(x_2) = D(x_1 \cdot x_2) \tag{2.2}$$

for all x_1, x_2 in G. A mapping which satisfies Eq. 2.2 is called a *representation* of the group G.[2] For example, the mapping

$$D(x) = e^{-ipx}, \tag{2.3}$$

is a representation of the additive group of real numbers because

$$e^{-ipx_1} e^{-ipx_2} = e^{-ip(x_1+x_2)}. \tag{2.4}$$

The following mapping is a representation of the permutation group on 3 labelled objects:

$$\begin{aligned} D(\ ) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & D(12) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ D(23) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & D(31) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ D(123) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & D(321) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.5}$$

[1] We will ignore the possibility of antiunitary operators, which are irrelevant in our context.

[2] Unitarity is not required in the definition of representation.

For example, the composition $(12)\cdot(23) = (123)$ is mapped into the matrix multiplication

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.6}$$

Thus, in any representation of a group, the composition law is realized by multiplication of (finite- or infinite-dimensional) matrices that the group elements map into. Such a mapping is not necessarily one-to-one. When it is, we call it the *fundamental representation.*

Group theory makes it possible to determine many properties of any representation from the abstract properties of the group. It is convenient to view representations both as abstract linear operators and as matrices. The connection is as follows: let $|i\rangle$ be an orthonormal basis in the space on which $D(g)$ acts as a linear operator. then

$$D(g)_{ij} = \langle i|D(g)|j\rangle. \tag{2.7}$$

So,

$$D(g)|i\rangle = \sum_j |j\rangle\langle j|D(g)|i\rangle = \sum_j |j\rangle D(g)_{ji}. \tag{2.8}$$

Two representations are *equivalent* if they are related by a *similarity transformation*

$$D_2(x) = SD_1(x)S^{-1}, \tag{2.9}$$

with a fixed operator S for all x in G.

A representation is *reducible* if it is equivalent to a representation D' with block-diagonal form:

$$D'(x) = SD(x)S^{-1} = \begin{pmatrix} D_1'(x) & 0 \\ 0 & D_2'(x) \end{pmatrix}, \tag{2.10}$$

whence the vector space on which D' acts breaks up into two orthogonal subspaces, each of which is mapped into itself by all the operators in $D'(x)$. The representation D' is said to be the *direct sum* of D_1' and D_2',

$$D' = D_1' \oplus D_2'. \tag{2.11}$$

A representation is *irreducible* if it is not reducible, that is if it cannot be put into a block-diagonal form by any similarity transformation. Any finite dimensional representation of a finite group is completely reducible into a direct sum of irreducible representations.

Group elements are rarely dealt with as abstract mathematical objects. Instead, a representation is used to obtain the composition table which is, in a sense, the group. For the groups of our interest (in the realm of quantum theories of particles and fields), all irreducible representations are equivalent to representations by unitary operators.

A *Lie group* is a group of unitary operators that are labeled by a set of continuous real parameters with a composition law that depends smoothly on the parameters. If the volume of the parameter space of a Lie group is finite, then it is called a *compact Lie group*. Any element of a compact Lie group can be obtained from the identity element by continuous changes in the parameters and can be expressed as $e^{i\alpha_a X_a}$, where α_a $(a = 1 \ldots n)$ are real parameters and X_a are linearly independent hermitian operators (a sum over the repeated index a is implied). The X_a are a basis of a vector space spanned by the linear combinations $\alpha_a X_a$, called the *generators* of the group. Any function of the generators that commutes with all generators of a Lie group is called a *Casimir operator* of that group.

Note that the space of the group generators is different from the space on which the generators act, which is some as yet unspecified Hilbert space. For the compact Lie groups, the space on which the generators act are finite dimensional, so the X_a can be expressed as finite hermitian matrices.

Generators have two nice features. First, since the generators form a vector space, unlike the group elements, they can be multiplied by numbers and added to obtain other generators. Second, they satisfy simple commutation relations which determine (almost) the full structure of the group. Consider the composition

$$e^{i\lambda X_b} e^{i\lambda X_a} e^{-i\lambda X_b} e^{-i\lambda X_a} = 1 + \lambda^2 [X_a, X_b] + \cdots$$

Because of the properties of group composition, the result corresponds to another group element and can be written as $e^{i\beta_c X_c}$. As $\lambda \to 0$, we must have $\lambda^2 [X_a, X_b] \to i\beta_c X_c$. Writing $\beta_c = \lambda^2 f_{abc}$, we get

$$[X_a, X_b] = i f_{abc} X_c. \tag{2.12}$$

The constants f_{abc} are called the *structure constants* of the group. The structure constants reflect the group composition law. This can be seen as follows. It is always possible to define

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} \equiv e^{i\delta_c X_c}, \tag{2.13}$$

where δ_c is determined by α, β and f:

$$\delta_c = \alpha_c + \beta_c - \frac{1}{2} f_{abc} \alpha_a \beta_b + \cdots \tag{2.14}$$

The generators also satisfy the *Jacobi identity*:

$$[X_a, [X_b, X_c]] + \text{ cyclic permutations } = 0. \tag{2.15}$$

This is obvious for the representation, since then the X_a are just linear operators, but in fact it is true for the abstract group generators. In terms of the structure constants, the Jacobi identity becomes

$$f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0. \tag{2.16}$$

If we define a set of matrices T_a

$$(T_a)_{bc} \equiv -if_{abc}, \tag{2.17}$$

Then, after simular definitions for T_b and T_c, Eq. 2.16 can be rewritten as

$$[T_a, T_b] = if_{abc}T_c. \tag{2.18}$$

In other words, the structure constants themselves generate a representation of the algebra. The representation generated by the structure constants is called the *adjoint representation.* The *dimension* of a representation is the dimension of the vector space on which it acts. The dimension of the adjoint representation is just the number of generators, which is the number of real parameters necessary to describe a group element.

The generators and the commutation relations define the *Lie algebra* associated with the Lie group. Every representation of the group defines a representation of the algebra. The generators in the representation, when exponentiated, give the operators of the group representation. The definitions of equivalence, reducibility and irreducibility can be transferred unchanged from the group to the algebra.

Spacetime symmetries like rotations in an Euclidean space are particularly obvious examples of transformation groups. Other important transformation groups include the Lorentz group of special relativity and the Poincaré group (Lorentz boost plus translations and rotations). However, these are not compact groups. The nature of their representations is different from that of the groups which involve changes in particle identities, with no connection to the structure of space and time. These groups are associated with internal symmetries, and are the primary objects of our interest.

The structure constants depend on the choice of bases in the vector space of the generators. For the treatment of internal symmetries in this course, we will deal with unitary unimodular groups called $SU(n)$.[3] They belong to a class called *compact semisimple* Lie groups, for which one can choose a basis such that

$$\mathrm{Tr}(T_aT_b) = \lambda\delta_{ab} \tag{2.21}$$

for some positive real number λ. In this basis, the structure constants are completely antisymmetric, because one can write

$$f_{abc} = -i\lambda^{-1}\mathrm{Tr}([T_a, T_b], T_c), \tag{2.22}$$

[3] The unitary group $U(n)$ is the subgorup consisting of those elements A of the general linear group $GL(n, C)$, represented by $n \times n$ complex matrices, such that $AA^\dagger = \mathbf{1}$. The special unitary group $SU(n)$ is that subgroup of $U(n)$ for which $\det A = 1$. The latter condition requires the generators to be traceless since

$$\text{for} \quad \psi \quad \to \quad \psi' = U\psi \qquad \text{with} \qquad U = \exp\left(\frac{i}{2}\sum_a \alpha_a X_a\right), \tag{2.19}$$

$$\det U = \exp(\mathrm{Tr}(\log U)) = \exp(\frac{i}{2}\mathrm{Tr}(\alpha_a X_a)). \tag{2.20}$$

Since α_a are arbitrary numbers, $\det U = 1 \Rightarrow \mathrm{Tr}(X_a) = 0$.

whence the antisymmetry of the RHS is ensured by the cyclic property of the trace. Also in this basis, the generators in the adjoint representation are hermitian matrices. In fact, it can be shown that for compact Lie groups (as for finite groups) any representation is equivalent to a representation by hermitian operators and all irreducible representations are finite hermitian matrices. The $SU(n)$ group has $n^2 - 1$ generators (one less than the $U(n)$), of which $n-1$ can be simultaneously diagonalized.

2.2 The Group *SU(2)*

The simplest non-Abelian Lie algebra consists of three generators J_a; $a = 1, 2, 3$, with $f_{abc} = \varepsilon_{abc}$, resulting in the commutation relations

$$[J_1, J_2] = iJ_3, \qquad [J_2, J_3] = iJ_1, \qquad [J_3, J_1] = iJ_2. \tag{2.23}$$

This is the angular momentum algebra obeyed by the generators of the rotation group in 3 dimensions. They determine the properties of $SU(2)$, the unimodular unitary group that is the most frequently appearing symmetry in particle physics, as it describes not only spin, but also isospin symmetry, e.g. that between the proton and the neutron, and of the three charged states of the pion.

The $SU(2)$ matrices are complex 2×2 matrices

$$U = \exp\left(i \sum_{k=1}^{3} \phi^k J_k \right) = \begin{pmatrix} u^1{}_1 & u^1{}_2 \\ u^2{}_1 & u^2{}_2 \end{pmatrix} \tag{2.24}$$

with the constraints

$$U^\dagger = U^{-1}, \qquad \det U = 1. \tag{2.25}$$

In the fundamental representation, the $SU(2)$ algebra is realized by

$$J_i = \frac{1}{2}\sigma_i, \tag{2.26}$$

where the σ_i, are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.27}$$

The operator U operates on a complex two-component spinor $\psi = (\psi^2, \psi^2)$, which transforms under $SU(2)$ as

$$\psi' = U\psi \qquad \text{or,} \qquad (\psi')^i = \sum_{j=1}^{2} u^i{}_j \psi^j. \tag{2.28}$$

The metric tensor is the two-dimensional Levi-Civita tensor $\varepsilon_{ij} = \varepsilon^{ij}$. Using this metric, covariant spinors can be obtained from contravariat spinors and vice-versa:

$$\psi_i = \varepsilon_{ij}\psi^j, \qquad \psi^i = \varepsilon^{ij}\psi_j. \tag{2.29}$$

The invariance of the inner product of two spinors (ψ^1, ψ^2) and (ϕ^1, ϕ^2)

$$\phi^{1*}\psi^1 + \phi^{2*}\psi^2 \equiv \phi^{i*}\psi^i, \tag{2.30}$$

implies that the contravariant complex conjugate ψ^* transforms the same way as the covariant ψ:

$$\psi^{i*} \sim \varepsilon_{ij}\psi^j = \psi_i. \tag{2.31}$$

This property is called the reality of $SU(2)$. It means that the complex conjugate ψ^{i*} does not introduce any new representation.

The basis for the fundamental representation of $SU(2)$ is conventionally chosen to be the eigenvalues of J_3, that is, the column vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

describing a spin-$\frac{1}{2}$ particle of spin projection up and spin projection down along the 3-axis, respectively. The other two spin components combine to form raising and lowering operators

$$J^{\pm} \equiv \frac{1}{\sqrt{2}}(J_1 \pm iJ_2) \tag{2.32}$$

so called because when they act on an eigenstate of J_3, they raise or lower the eigenvalue by one unit (up to the highest or down to the lowest possible value). This is easily seen from the commutation relations

$$\begin{aligned} [J_3, J^{\pm}] &= \pm J^{\pm} \\ [J^+, J^-] &= J_3 \end{aligned} \tag{2.33}$$

So, if

$$J_3|m\rangle = m|m\rangle, \tag{2.34}$$

then

$$J_3 J^{\pm}|m\rangle = J^{\pm}J_3|m\rangle \pm J^{\pm}|m\rangle = (m \pm 1)J^{\pm}|m\rangle. \tag{2.35}$$

Suppose that a set of $|m\rangle$ forms an M-dimensional representation. The eigenvalues m are called *weights*. Let j be the highest weight. Then, by definition,

$$J^+|j\rangle = 0. \tag{2.36}$$

applying the lowering operator to $|m\rangle$, we find

$$J^-|m\rangle = N_m|m-1\rangle, \tag{2.37}$$

where N_m is a normalization constant which is determined as follows. From Eq. 2.37, we find

$$\langle m-1|J^-|m\rangle = N_m \qquad \Leftrightarrow \qquad \langle m|J^+|m-1\rangle = N_m^*. \tag{2.38}$$

By suitably choosing the phase of N_m, we have

$$J^-|m\rangle = N_m|m-1\rangle; \qquad J^+|m-1\rangle = N_m|m\rangle. \tag{2.39}$$

Taking the square of Eq. 2.37, we get

$$\begin{aligned} N_m^2 &= \langle m|J^+J^-|m\rangle \\ &= \langle m|J^-J^+|m\rangle + m \\ &= N_{m+1}^2 + m. \end{aligned} \tag{2.40}$$

Solving this recursion formula for N_m under the initial condition $N_j^2 = j$ we get

$$N_m = \sqrt{\frac{1}{2}(j+m)(j-m+1)}. \tag{2.41}$$

There are $2j$ coefficients that are non-zero and real for $-(j-1) \le m \le j$. From Eq. 2.37, N_m appears when a state $|m-1\rangle$ is created from $|m\rangle$ by applying J^-. Starting from $|j\rangle$, they are $|j-1\rangle, |j-2\rangle, \ldots |-j\rangle$. Adding to these the initial state $|j\rangle$, the total number of states is $M = 2j+1$. This completes the M-dimensional representation of $SU(2)$, with j corresponding to the total spin and m to the 3rd component of the spin. In the above we have not used the properties of the only Casimir operator

$$J^2 = {J_1}^2 + {J_2}^2 + {J_3}^2 \tag{2.42}$$

for the rotation group. There is an alternative way to derive the same result by using the commutation relations

$$[J^2, J_i] = 0. \tag{2.43}$$

The method shown here can be extended to $SU(3)$.

2.3 The Group *SU(3)*

Another symmetry group that has many manifestations in particle physics is $SU(3)$, the group of 3×3 unitary unimodular matrices. Its generators are 3×3 hermitian traceless matrices.[4] The standard basis in physics literature consists

[4] The tracelessness is a consequence of the condition that the determinant be 1.

of the 8 ($= 3^2 - 1$) Gell-Mann λ matrices:

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, &\qquad \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, &\qquad \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, &\qquad \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, &\qquad \lambda_8 &= \tfrac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{2.44}$$

The generators are

$$T_a = \frac{1}{2}\lambda_a, \tag{2.45}$$

normalized by Eq. 2.21 and satisfying the commutation relations

$$[T_a, T_b] = i f_{abc} T_c. \tag{2.46}$$

Clearly, T_1, T_2, and T_3 generate a $SU(2)$ subgroup of $SU(3)$. It is called the *isospin subgroup*, because in the physical application of uds (quark) flavor $SU(3)$, it represents isospin.

The structure constants of $SU(3)$ in the λ_i basis of Eq. 2.44 are fully antisymmetric under any pairwise interchange of indices, and the non-vanishing values are permutations of

$$\begin{aligned}
&f_{123} = 1, \\
&f_{458} = f_{678} = \tfrac{\sqrt{3}}{2}, \\
&f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \tfrac{1}{2}.
\end{aligned} \tag{2.47}$$

Just as in $SU(2)$, the fundamental representation of $SU(3)$ is based on the transformation

$$\psi' = U\psi \quad \text{or,} \quad \psi'^i = \sum_{j=1}^{3} u^i{}_j \psi^j, \tag{2.48}$$

but with $u^i{}_j$ as the components of the 3×3 special unitary matrix

$$U = e^{i\alpha_a T^a} \tag{2.49}$$

However, unlike the $SU(2)$ case, the $SU(3)$ representation (ψ^i) is not real, i.e. the complex conjugate transforms as

$$\psi'^{*i} = {u^i{}_j}^* \psi'^{*j} \tag{2.50}$$

which is independent of Eq. 2.49. This is because the metric tensor is ε_{ijk}, which means the complex conjugate behaves as

$$\psi'^{*i} = \varepsilon_{ijk}\psi^j\psi^k. \tag{2.51}$$

Among the 8 generators of $SU(3)$, two can be diagonalized simultaneously.[5] In the fundamental representation of Eq. 2.44, they are already given by T_3 and T_8. Therefore, $SU(3)$ states are labeled by eigenvalues of T_3 and T_8. For a given simultaneous eigenstate, two eigenvalues define a point on a 2-dimensional (t_3, t_8) plane. The remaining generators combine to form the raising or lowering operators that shift one stae to another:

$$\begin{aligned} I_\pm &= \tfrac{1}{\sqrt{2}}(T_1 + iT_2), \\ V_\pm &= \tfrac{1}{\sqrt{2}}(T_4 + iT_5), \\ U_\pm &= \tfrac{1}{\sqrt{2}}(T_6 + iT_7). \end{aligned} \tag{2.52}$$

Each of these matrices has a single non-zero element, which is, of course, off-diagonal, so as to transform one (T_3, T_8) eigenstate to another. The following commutation relations follow:

$$\begin{aligned} &[T_3, I_\pm] = \pm I_\pm, && [T_8, I_\pm] = 0, \\ &[T_3, V_\pm] = \pm\tfrac{1}{2}V_\pm, && [T_8, V_\pm] = \tfrac{\sqrt{3}}{8}V_\pm, \\ &[T_3, U_\pm] = \mp\tfrac{1}{2}U_\pm, && [T_8, U_\pm] = \tfrac{\sqrt{3}}{8}U_\pm. \end{aligned} \tag{2.53}$$

These imply that $I_\pm$, $U_\pm$, and $V_\pm$ raise or lower the values of t_3 and t_8 by the coefficients on the right-hand sides. Therefore, they are expressed by 2-dimensional vectors, which point from the origin to one of the vertices of a regular hexagon.

In a fashion similar to the one demonstrated for $SU(2)$, it is possible to construct the $SU(3)$ representation. The simultaneous eigenvectors of T_3 and T_8 are

$$\psi^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \psi^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \psi^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.54}$$

We see that

$$\begin{aligned} T_3\psi^1 &= \tfrac{1}{2}\psi^1 \\ T_8\psi^1 &= \tfrac{\sqrt{3}}{6}\psi^1 \end{aligned} \quad \Rightarrow \quad \vec{\mu_1}^1 = |\tfrac{1}{2}, \tfrac{\sqrt{3}}{6}\rangle,$$
$$\begin{aligned} T_3\psi^2 &= -\tfrac{1}{2}\psi^2 \\ T_8\psi^2 &= \tfrac{\sqrt{3}}{6}\psi^2 \end{aligned} \quad \Rightarrow \quad \vec{\mu_2}^1 = |-\tfrac{1}{2}, \tfrac{\sqrt{3}}{6}\rangle, \tag{2.55}$$
$$\begin{aligned} T_3\psi^3 &= 0 \\ T_8\psi^3 &= -\tfrac{\sqrt{3}}{3}\psi^3 \end{aligned} \quad \Rightarrow \quad \vec{\mu_3}^1 = |0, -\tfrac{\sqrt{3}}{3}\rangle,$$

[5] Hence, $SU(3)$ has a *rank* 2.

where we have introduced three 2-dimensional vectors, called *weight vectors* $\vec{\mu}_i^1$, to represent the states ψ^i. The superscript 1 for the $\vec{\mu}$ is to distinguish it from another set tof weights $\vec{\mu}_i^2$ which will be introduced shortly. The weight vectors form a unit equilateral triangle centered at the origin of the t_3, t_8 plane.

The three states of the fundamental representation are related to each other through the raising and lowering oprators. For instance, it is easy to check in the 3-component vector form that

$$\psi^1 = V_+ \psi^3. \tag{2.56}$$

In terms of weight vectors, this is expressed as

$$\vec{\mu}_1^1 = \vec{\alpha}_1 + \vec{\mu}_3^1, \tag{2.57}$$

where the *root vector*

$$\vec{\alpha}_1 = |\frac{1}{2}, \frac{\sqrt{3}}{2}\rangle \tag{2.58}$$

relates two weights additively and increases a weight by the "unit $\vec{\alpha}_1$". Similarly, one can consider another root

$$\vec{\mu}_2^1 = \vec{\alpha}_2 + \vec{\mu}_2^1, \tag{2.59}$$

which raises a weight by another unit

$$\vec{\alpha}_2 = |\frac{1}{2}, -\frac{\sqrt{3}}{2}\rangle. \tag{2.60}$$

The root vectors $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are independent. In general, all weight vectors are related by

$$\vec{\mu}' = \vec{\mu} + l\vec{\alpha}_1 + m\vec{\alpha}_2, \tag{2.61}$$

where l and m are some integers.

Notice the correspondence between the root vectors and lowering and raising operators:

$$\vec{\alpha}_1 \sim V_+, \quad \vec{\alpha}_2 \sim U_-. \tag{2.62}$$

In principle, one could choose any two independent operators out of the six: $I_\pm$, $U_\pm$, $V_\pm$. In the particular choice above, the two roots are called *simple roots.*

In $SU(3)$ there is another fundamental representation which is the complex conjugate ψ'^{*i} (see Eq. 2.51). Complex conjugation of the commutation relations in Eq. 2.46 leads to

$$[-T_a{}^*, T_b{}^*] = i f_{abc} T_c{}^*, \tag{2.63}$$

implying $-T_a{}^* = -\frac{1}{2}\lambda_a{}^*$ can be another representation. The diagonal generators T_3 and T_8 are replaced simply by the negatives of the original ones, and, therefore, the weight vectors change their signs. In other words,

$$\begin{aligned}
\psi^{1*} \rightarrow \vec{\mu_1}^2 &= | -\tfrac{1}{2}, -\tfrac{\sqrt{3}}{6}\rangle, \\
\psi^{2*} \rightarrow \vec{\mu_2}^2 &= |\tfrac{1}{2}, -\tfrac{\sqrt{3}}{6}\rangle, \\
\psi^{3*} \rightarrow \vec{\mu_3}^2 &= |0, \tfrac{\sqrt{3}}{3}\rangle,
\end{aligned} \tag{2.64}$$

which form another triangle, rotated by π w.r.t. the first one. Notice that the new states represented by the new triangle are still connected by the same simple root vectors. In the $SU(3)$ of strong interactions, one representation represents the color states of a quark, while the other represents the color states of an antiquark, but the same gluons (the generator coefficients) mediate the transitions between the different states within each set.