3.3 Lagrangian and symmetries for a spin-$\frac{1}{2}$ field

The Lagrangian for the free spin-$\frac{1}{2}$ field is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/_\mu - m)\psi. \tag{3.31}$$

The corresponding Hamiltonian density is

$$\mathcal{H} = \bar{\psi}(\vec{\gamma}\cdot\vec{p} + m)\psi. \tag{3.32}$$

The Lagrangian has a global gauge symmetry, since $\psi \to \psi' = e^{i\theta}\psi$ leaves the field equations unchanged. Thus we expect a conserved current density

$$J^\mu = e\bar{\psi}\gamma^\mu\psi, \tag{3.33}$$

which saisfies the continuity equation $\partial_\mu J^\mu = 0$.

What is the relativistic extension for the angular momentum? By commuting $\vec{x}\times\vec{p}$ with $H = \gamma^0(\vec{\gamma}\cdot\vec{p} + m)$, we can try to determine what must be added to the orbital angular momentum to make a conserved quantity. It turns out that the simplest extension of the nonrelativistic expression works. Defining the 4×4 matrices

$$\Sigma^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}, \tag{3.34}$$

$\vec{J} = \vec{x}\times\vec{p} + \frac{1}{2}\vec{\Sigma}$ satisfies $[H, \vec{J}] = 0$.

The plane wave fields we are using are eigenstates of $\vec{p}$. They are not eigenstates of $\vec{J}$ since $[\vec{J}, \vec{p}] \neq 0$ because $[\vec{x}, \vec{p}] \neq 0$, and that affects the orbital component of the total spin. If we can isolate the spin operator alone in an expression that commutes with H, then we can get a quantity that does commute with the momentum. Taking $\vec{J}\cdot\vec{p}$, the term $(\vec{x}\times\vec{p})\cdot\vec{p}$ vanishes, leaving $\frac{1}{2}\vec{\Sigma}\cdot\vec{p}$. This *helicity* operator now commutes with H and $\vec{p}$ and can therefore be simultaneously diagonalized. The resulting states are helicity states. Note that this works because the intrinsic spin operator is independent of position, as is the resulting spin-dependent factor in the amplitude of the plane wave solution. This argument holds just as well for particles for other values of the spin.

As an example of interaction that can be added to the free-particle Lagrangian, let us consider electromagnetism. In classical electrodynamics, the minimal substitution into the basic equations of motion of a particle with charge $-q$, caused by the presence of an electromagnetic field, is $p^\mu \to p^\mu + qA^\mu$. In the quantum case, the replacement $p^\mu \to i\partial^\mu$ leads to the modified Lagrangian

$$\mathcal{L} = \bar{\psi}\gamma^\mu(i\partial_\mu + qA_\mu)\psi - m\bar{\psi}\psi. \tag{3.35}$$

To include the photon field, we need to add the term $\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$. The local gauge transformation now is

$$\psi \to e^{iq\theta}\psi \quad \text{and} \quad A^\mu \to A^\mu + \partial_\mu\theta, \tag{3.36}$$

which leaves the $\mathcal{L}$ unchanged.

The function $\theta(x^\mu)$ is an arbitrary function of space-time. So, the field transformations must be carefully coordinated. The interaction term arises directly out of the modification to the free field Lagrangian if we add the local gauge symmetry requirement.[1] Thus, the interaction term in Eq. 3.35 is

$$\mathcal{L}_{\rm Int} = eA_\mu \bar{\psi}\gamma_\mu\psi. \tag{3.37}$$

Comparing this expression with Eq. 2.71 we see that the conserved electromagnetic current is still $J_\mu = e\bar{\psi}\gamma^\mu\psi$: the expression for the conserved current is not modified in the presence of an electromagnetic field (that is, when going from a global to a local gauge symmetry) for the Dirac field. This is a consequence of the derivative not explicitly appearing in the expression for the free particle current for spin-$\frac{1}{2}$.[2] For other spins, e.g. spin-0, the expression for the current is modified. This will bear on the treatment of symmetry breaking.

Other interactions can be introduced to the Lagrangian in an analogous way. For example, the interaction with a hypothetical scalar particle will add a term

$$\mathcal{L}_{\rm Int} = -g\phi\bar{\psi}\psi, \tag{3.38}$$

where ϕ is the scalar field. This interaction also leaves the expression for the conserved current unchanged provided ϕ is real and unchanged under the gauge transformations. Physically, $\phi \to \phi$ under a gauge transformation implies that ϕ carries no charge. Only particles of the ψ field, i.e. the fermions, are charged. Note also that g has to be a real number since $\mathcal{L}_{\rm Int}$ is hermitian.

3.4 Explicit plane-wave solutions

Let us start with the simplest case of a free particle at rest. The Dirac equation then reduces to

$$\left(i\gamma_0\frac{\partial}{\partial t} - m\right)\psi = 0. \tag{3.39}$$

In this case, since γ_0 is diagonal, the equations do not mix the 4 components of ψ (of Eq. 3.4):

$$\begin{aligned} i\frac{\partial\psi_1}{\partial t} - m\psi_1 = 0, \qquad & i\frac{\partial\psi_2}{\partial t} - m\psi_2 = 0, \\ i\frac{\partial\psi_3}{\partial t} + m\psi_3 = 0, \qquad & i\frac{\partial\psi_4}{\partial t} + m\psi_4 = 0. \end{aligned} \tag{3.40}$$

Taking a solution of the form e^{-iEt}, we get $E = m$ for ψ_1 and ψ_2. However, for ψ_3 and ψ_4, this would give $E = -m$. This dilemma is resolved by taking a

[1] The Dirac equation (Eq. 3.17) does not have a local gauge symmetry.

[2] All terms involving space-time derivatives are modified because of the minimal substitution used to introduce the electromagnetic interaction.

solution of the form e^{iEt} and transfering the minus sign to t in order to keep E positive.[3] Consequently, ψ_3 and ψ_4 represent the antiparticle states.

Let us now try to tackle particles in motion by generalizing the solution to the form $e^{\pm ip^\mu x_\mu} = e^{\pm ip\cdot x}$ which reduce to $e^{\pm mt}$ as $\vec{p} \to 0$. Then we have

$$i\gamma_\mu \partial_\mu e^{\pm ip\cdot x} = \gamma_\mu p_\mu e^{\pm ip\cdot x} = \not{p} e^{\pm ip\cdot x} \tag{3.41}$$

with

$$\not{p} = \begin{pmatrix} E & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E \end{pmatrix}, \quad \text{where} \quad \vec{\sigma}\cdot\vec{p} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_3 \end{pmatrix}. \tag{3.42}$$

Explicitly, the four solutions are

$$\psi_1 = u_1 e^{-ip\cdot x}, \quad \psi_2 = u_2 e^{-ip\cdot x}, \quad \psi_3 = v_1 e^{ip\cdot x}, \quad \psi_4 = v_2 e^{ip\cdot x}. \tag{3.43}$$

For $\vec{p} = 0$, we've already found the four solutions, which can be written as $u(0) = \begin{pmatrix} \chi \\ 0 \end{pmatrix}$, $v(0) = \begin{pmatrix} 0 \\ \chi \end{pmatrix}$, where $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For the more general situation, we have

$$(\not{p} - m)u = 0, \quad \text{and} \quad (\not{p} + m)v = 0. \tag{3.44}$$

The general solution can be obtained by noting that $(\not{p} - m)(\not{p} + m) = p^2 - m^2 = 0$, so we can take

$$u(p) = (\not{p} + m)u(0), \quad v(p) = (\not{p} - m)v(0). \tag{3.45}$$

We can now choose $u(0)$ and $v(0)$ as above to get $u(p)$ and $v(p)$.

The normalization condition for the spinors is

$$\bar{\psi}\psi = 2m, \tag{3.46}$$

which is equivalent to requiring the current $\bar{\psi}\gamma_0\psi = \psi^\dagger\psi$ (the number of particles per unit volume) to be $2E$. Thus, for the particle at rest we have

$$u(0) = \begin{pmatrix} \chi \\ 0 \end{pmatrix}, \quad v(0) = \begin{pmatrix} 0 \\ \chi \end{pmatrix}. \tag{3.47}$$

These lead to the general results

$$u = \sqrt{E+m} \begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi \end{pmatrix}, \quad v = \sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi \\ \chi \end{pmatrix}. \tag{3.48}$$

$\vec{\sigma}\cdot\vec{p}$ is the operator proportional to the helicity of the two-component spinor χ. If $\vec{p} = p\hat{z}$, then taking $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ will yield two helicity eigenstates for u or v. For an arbitrary $\vec{p}$, we can rotate $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ using the 2×2 rotation operators to generate helicity eigenstates along $\vec{p}$.

[3] This applies to all fields irrespective of spin, as can be seen in the Klein-Gordon equation.

Bilinear operator O	Transformation property of $\psi O \psi$	No. of operators
1	Scalar	1
γ^μ	Vector	4
$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	Antisymmetric tensor	6
$\gamma^5\gamma^\mu$	Pseudovector	4
γ^5	Pseudoscalar	1

Table 3.1: Hermitian bilinear operators for the spin-$\frac{1}{2}$ field.

3.5 Bilinear Covariants

Operators O for which $\bar{\psi} O \psi$ is hermitian and has well-defined properties under Lorentz transformations are of special interest since these are legitimate candidates to appear in $\mathcal{L}$ for terms involving only fields and no derivatives. We have encountered $O = 1$ (scalar) and $O = \gamma^\mu$ already, but there are others. In the Dirac-Pauli representation,

$$\gamma^5 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.49}$$

while in general,

$$\gamma^{5\dagger} = \gamma^5, \quad , \{\gamma^5, \gamma^\mu\} = 0, \quad \text{and} \quad (\gamma^5)^2 = 1. \tag{3.50}$$

The full list of hermitian operators for the spin-$\frac{1}{2}$ field is given in Table 3.1

3.6 Massless Fermions

In the (highly relativistic) limit $E \gg m$, we can put $m = 0$ in the Dirac equation to get two decoupled equations for the two-component spinors. For χ^+ along $\vec{p}$ and χ^- opposite $\vec{p}$,

$$E\chi^- = -\vec{\sigma} \cdot \vec{p}\, \chi^- \quad \text{and} \quad E\chi^+ = +\vec{\sigma} \cdot \vec{p}\, \chi^+, \tag{3.51}$$

so the first one represents a left-handed neutrino of energy E and momentum $\vec{p}$. Then the helicity states have simple representations:

$$u^\pm = \sqrt{E} \begin{pmatrix} \chi^\pm \\ \pm\chi^\pm \end{pmatrix} \quad \text{and} \quad v^\pm = \sqrt{E} \begin{pmatrix} \pm\chi^\pm \\ \chi^\pm \end{pmatrix}. \tag{3.52}$$

In this limit, the operators

$$P_R \equiv \frac{1}{2}(1 + \gamma^5) \quad \text{and} \quad P_L \equiv \frac{1}{2}(1 - \gamma^5) \tag{3.53}$$

act as right- and left-hand projection operators, so that

$$P_{R,L}u^{\pm} = u^{\pm}, \quad P_{R,L}v^{\pm} = v^{\pm}, \quad P_{R,L}u^{\mp} = 0, \quad P_{R,L}v^{\mp} = 0. \tag{3.54}$$

P_L and P_R satisfy the usual relations between projection operators:

$$P_i^2 = P_i, \quad P_L + P_R = 1, \quad P_L P_R = 0. \tag{3.55}$$

The γ^5 is thus called the *chirality* operator. It is diagonal in the Weyl representation. Consequently, in the Weyl representation, helicity is diagonalized in the extreme relativistic limit.

The chirality operator is very useful for keeping the Dirac spinor notation when writing a $\mathcal{L}$ that differentiates between right- and left-handed fermions. A good example is the *charged current* weak interaction between a charged lepton ℓ and its $SU(2)$ partner neutrino ν_ℓ, where, in contrast to the V (vector) form of the electromagnetic current, we have the $V - A$ form of the weak current:

$$J^\mu = \bar{\psi}_\ell \gamma^\mu \frac{1}{2}(1 - \gamma^5)\psi_{\nu_\ell}. \tag{3.56}$$

This ensures that parity is maximally violated in such intereactions, because

$$\frac{1}{2}(1 - \gamma^5)u_\nu = \begin{pmatrix} \sigma_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \chi^- \\ -\chi^- \end{pmatrix} = \begin{pmatrix} \chi^- \\ 0 \end{pmatrix}. \tag{3.57}$$

So, only the ν_L (and $\bar{\nu}_R$) are projected out: only the left-handed neutrinos and right-handed antineutrinos couple to their charged counterparts by weak interactions. Of course, if the neutrino mass is not strictly zero, then it is possible to perform a Lorentz transformation to change a ν_L to a a ν_R.[4]

[4]Even if neutrinos are not exactly massless, it is possible to ensure that the weak interactions couple only to ν_L and $\bar{\nu}_R$ by requiring that the neutrinos be their own antiparticles. Such neutrinos are known as *Majorana neutrinos*. Such neutrinos are best treated on a basis of *Majorana spinors*, which are structured differently than Dirac spinors.