Chapter 3

Quantum Electrodynamics

We now turn to spin-$\frac{1}{2}$ particles. Let us study the electron as a specific example. The electron is a spin-$\frac{1}{2}$ particle, which implies that each momentum state has two possible helicities, $\lambda = +\frac{1}{2}$ or $\lambda = -\frac{1}{2}$. The states in the particle rest frame can be determined using the spin-$\frac{1}{2}$ representation of the rotation group, $SU(2)$.

We can describe the two spin choices in terms of the base states:

$$\chi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.1}$$

These states, called spinors, correspond to spins $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively, along a chosen space axis, which we take to be the 3-axis (z).

The spin operator in the fermion rest frame is given in the basis above by

$$\vec{S} = \frac{\vec{\sigma}}{2}, \tag{3.2}$$

where $\vec{\sigma}$ is the Pauli spin matrix whose components are given by Eq. 2.27. In addition, we now define the identity matrix as the *0th component* of the spin matrix. matrix:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.3}$$

3.1 The Dirac Equation

Dirac's primary objective in deriving the field equations for fermions was to linearize the Klein-Gordon equation (Eq. 2.84) which, being quadratic in E, opened doors to solutions with negative energy that needed to be explained. Originally, Dirac handled the problem of preventing all fermions from falling into negative energy states without a lower bound by postulating that all such states are already full. This made for the possibility of an electron in a negative energy state making an occassional transition to a positive energy state, which would create a *hole* in the *sea* of negative energy state. Dirac called these "hole"s *positrons*. Experimental confirmation of the existence of positrons is counted among the greatest triumphs in theoretical physics. Later, Feynman came up with an alternative interpretation of positrons as electrons traveling backward in time. This led to great simplification of the theory, which came to be known as quantum electrodynamics. So, to modify the Klein-Gordon equation to describe spin-$\frac{1}{2}$ particles, each energy two (+ve and -ve) energy states in its solution must be allowed two spin states. That is, the general wave function will have $2 \times 2 = 4$ components:

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{3.4}$$

The linear equation should then take the form

$$H\psi = i\frac{\partial}{\partial t}\psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi = (\vec{\alpha} \cdot i\nabla + \beta m)\psi, \tag{3.5}$$

where β and α_i $(i = 1, 2, 3)$ are 4×4 matrices. They can be determined by comparing Eq. 2.84 with the H^2 expressed in terms of the RHS of Eq. 3.5:

$$\frac{\partial^2}{\partial t^2} = \left(-\alpha^j\alpha^k\partial_j\partial_k - im(\alpha^j\beta + \beta\alpha^j)\partial_j + \beta^2 m^2\right)\psi \tag{3.6}$$

Since the partial derivatives commute, we can write

$$\alpha^j\alpha^k\partial_j\partial_k = \frac{1}{2}(\alpha^j\alpha^k + \alpha^k\alpha^j)\partial_j\partial_k \tag{3.7}$$

Then, for Eq. 3.6 to be consistent with Eq. 2.84 we must have

$$\beta^2 = 1, \tag{3.8}$$

$$\{\alpha^j, \beta\} = \alpha^j\beta + \beta\alpha^j = 0, \tag{3.9}$$

$$\{\alpha^j, \alpha^k\} = \alpha^j\alpha^k + \alpha^k\alpha^j = 2\delta^{jk}, \tag{3.10}$$

The solution to these can be wrirtten in terms of the Pauli matrices:

$$\beta = \gamma^0 \equiv \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \qquad \alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}. \tag{3.11}$$

Note that the representation is not unique. The one above is known as the Dirac-Pauli representation. Another possibility, known as the Weyl- or chiral representation is

$$\beta = \gamma^0 \equiv \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \qquad \alpha^j = \begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}. \tag{3.12}$$

Most of the formulae are independent of the representation. We will use the Pauli-Dirac representation.

Equation 3.5 is known as the Dirac equation and the 4-component wave function, a Dirac spinor.

3.2 The γ matrices and trace theorems

The Dirac equation can be written in a simpler form by multiplying it on the left by β and defining

$$\gamma^\mu = (\beta, \beta\vec{\alpha}), \tag{3.13}$$

or, explicitly,

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}. \tag{3.14}$$

These are known as the Dirac γ matrices. The result is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \tag{3.15}$$

It is useful to define the *Feynman slash* notation:

$$\gamma^\mu a_\mu = \not{a}, \tag{3.16}$$

so the Dirac equation takes the compact form

$$(i\not{\partial} - m)\psi = 0. \tag{3.17}$$

In practice, one almost never needs to know the explicit forms of the γ matrices. The following relations satisfied by them suffice for most calculations:

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0, \qquad (\Rightarrow \gamma^{0\dagger} = \gamma^0, \quad \gamma^{j\dagger} = -\gamma^j), \tag{3.18}$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \qquad (\Rightarrow \gamma^\mu\gamma_\mu = 4), \tag{3.19}$$

$$\gamma^\mu \not{a} \gamma_\mu = -2\not{a} \tag{3.20}$$

$$\gamma^\mu \not{a}\not{b} \gamma_\mu = 4a \cdot b \tag{3.21}$$

$$\gamma^\mu \not{a}\not{b}\not{c} \gamma_\mu = -2\not{c}\not{b}\not{a} \tag{3.22}$$

For reasons that will become clear soon, it is useful to define

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{3.23}$$

The following trace theorems often come in handy:

$$\text{The trace of an odd number of } \gamma^\mu\text{'s vanish.} \tag{3.24}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}. \tag{3.25}$$

$$\mathrm{Tr}(\not{a}\not{b}) = 4a \cdot b, \tag{3.26}$$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4((a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c), \tag{3.27}$$

$$\mathrm{Tr}(\gamma^5) = 0, \tag{3.28}$$

$$\mathrm{Tr}(\gamma^5 \not{a}\not{b}) = 0, \tag{3.29}$$

$$\mathrm{Tr}(\gamma^5 \not{a}\not{b}\not{c}\not{d}) = 4i\varepsilon_{\mu\nu\rho\sigma} a^\mu b^\nu c^\rho d^\sigma, \tag{3.30}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita tensor in 4 dimensions.