Chapter 8

Masses and the Higgs Mechanism

The Z, like the $W^{\pm}$ must be very heavy. This much can be inferred immediately because a massless Z boson would give rise to a peculiar parity-violating long-range force. However, when the $SU(2) \times U(1)$ gauge structure for the EM interactions was first written down by Glashow in 1960, it was not clear how to give mass to the $W^{\pm}$ and the Z bosons without breaking the gauge symmetry explicitly. That adding mass terms for the $W^{\pm}$ and the Z bosons by hand breaks the symmetry is not a problem as such. Nice as the idea of local symmetry is, we, as physicists, would gladly sacrifice it if that enables us to explain some facts. The main problem is that addition of such terms renders the theory unrenormalizable. It leads to dimension 5 operators that make infinite contributions to the interaction Hamiltonian when one attempts to take higher order quantum corrections into account. The longitudinal component of the Z field, Z_L, then appears only in the mass term, and not in the kinetic energy term, thus acting as an auxiliary field. Such a Z_L propagator does not fall off with momentum, and we get infinities that do not cancel and cannot be swept out of the observable realm. But all is not lost. If we somehow preserve the gauge-invariance structure and give mass to $W^{\pm}$ and Z, we may be able to preserve renomalizability. This is what Weinberg and Salam accomplished by invoking the idea of *spontaneous symmetry breaking*. The result was not only a consistent theory, but the masses of the $W^{\pm}$ and Z were calculated in terms of parameters that had been measured before we had the capability to produce those particles on mass shell. The theory was quickly put to test by building a machine powerful enough to produce those particles if their masses were not too different from their predicted values. When it emerged with flying colors, it marked one of the towering achievements in theoretical physics.

The mechanism of spontaneous symmetry breaking also provides a way for fermions to get their masses, but those masses cannot be predicted - they appear as free parameters in the SM.

Essentially, it is conjectured that the "vacuum", defined as the ground state of nature, is not a complete void. Instead, it is permeated by a scalar (i.e. spin-0) field, called a Higgs field, that is a doublet in the $SU(2)$ space of weak isospin, and carries a non-zero $U(1)$ hypercharge, but is electrically neutral and a singlet in the $SU(3)$ space of color. The gauge bosons and fermions can interact with this field in a way so as to appear massive. This means that the $SU(2)$ and $U(1)$ quantum numbers of vacuum are non-zero, so those symmetries are effectively broken. The associated charges can appear from or disappear into the vacuum even though the corresponding currents are conserved. When the symmetry is broken in this way, i.e. the symmetry is valid for the Lagrangian but not for the ground state of the system, it is said to be a spontaneously broken symmetry.

8.1 Spontaneous Symmetry Breaking

As a simple example of spontaneous symmetry breaking, consider the theory with a single Hermitian scalar field and the Lagrangian

$$\mathcal{L}(\phi) = T - V = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \left(\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4\right), \tag{8.1}$$

where μ and λ are parameters of the potential. With a single field, there can be no continuous internal symmetry, but the above Lagrangian is invariant under the "reflection" $\phi \to -\phi$. By general quantum mechanical principles, the potential must have minima if it is to describe a physical system. This requires $\lambda > 0$. Given a minimum (which is the classical ground state of the system), we can follow the normal perturbative procedure in quantum mechanics to expand the fields around their values at the minimum and determine the excitations. In quantum field theory, it is conventional to call this ground state the vacuum, and the excitations are particles. Their masses are determined by the form of the Lagrangian near the vacuum. The Lagrangian in Eq. 8.1 is not the most general, but it is more general than it may appear. It can be shown that higher powers of ϕ would lead to infinities in physical quantities and must therefore be excluded.

If $\mu^2 > 0$, then we have the vacuum at $\phi = 0$, and the Lagrangian describes a scalar field with mass μ that has a quartic self coupling of strength $\frac{\lambda}{4}$. But, what if $\mu^2 < 0$? There is no physical reason to exclude such a possibility. Now we have a local maximum, instead of a minimum, at $\phi = 0$. We would not want to perturb around such a point where the free theory contains a tachyon, a particle with imaginary mass! Rewriting the potential as

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 + \frac{\mu^2}{\lambda}\right)^2, \tag{8.2}$$

(the additive constant is of no consequence) we see that there are two degenerate minima at $\phi = \pm\sqrt{\frac{-\mu^2}{\lambda}}$. The potential in a symmetric neighborhood of $\phi = 0$ that contains both minima looks like a smooth "W". Physically, the two minima are equivalent. We can choose either one to perturb around, so let us pick $\phi = \sqrt{\frac{-\mu^2}{\lambda}}$. For the ϕ field,

$$v = \sqrt{\frac{-\mu^2}{\lambda}} \tag{8.3}$$

is a *vacuum expectation value* (VEV). To determine the particle spectrum, let us rewrite the theory in terms of a field with zero VEV:

$$\eta(x) = \phi(x) - v. \tag{8.4}$$

We could equally well have chosen $\eta(x) = \phi(x) + v$, but the physics conclusions would not be affected since the theory is symmetric under $\phi \to -\phi$. But having made a particular choice of η, the potential is not symmetric about its minimum. The Langrangian is not invariant under $\eta \to -\eta$. The symmetry has been spontaneously broken by the choice of vacuum.

Substituting Eq. 8.4 into Eq. 8.1 we get

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu \eta \partial^\mu \eta) - \Bigg(\frac{1}{2}\mu^2(v^2 + 2\eta v + \eta^2) \\
&\qquad + \frac{1}{4}(v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4)\Bigg) \\
&= \frac{1}{2}(\partial_\mu \eta \partial^\mu \eta) - \Bigg(\frac{v^2}{2}(\mu^2 + \frac{1}{2}\lambda v^2) + \eta v(\mu^2 + \lambda v^2) \\
&\qquad + \frac{\eta^2}{2}(\mu^2 + 3\lambda v^2) + \lambda v \eta^3 + \frac{1}{4}\lambda\eta^4\Bigg).
\end{aligned} \tag{8.5}$$

The term linear in η vanishes (by Eq. 8.3), as it must near the minimum, and $\mathcal{L}$ simplifies to

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta \partial^\mu \eta) - \left(\lambda v^2 \eta^2 + \lambda v \eta^3 + \frac{1}{4}\lambda \eta^4\right) + \text{irrelevant constant.} \tag{8.6}$$

Now the term with η^2 has the correct sign so it can be interpreted as a mass term. This Lagrangian describes a scalar field η that appears as a particle of mass

$$m_\eta^2 = 2\lambda v^2 = -2\mu^2, \tag{8.7}$$

and with two interactions, a cubic one of strength λv and a quartic one of strength $\frac{\lambda}{4}$. Both of these depend on λ, which is a free parameter as far as we can tell, and are therefore interactions of undetermined strengths.

The two descriptions of the theory in terms of ϕ or η must be equivalent if the problem is exactly solvable. If we want a perturbative description, it is essential to perturb around the minimum to have a convergent description. The scalar particle described by the theory with $\mu^2 < 0$ is a real scalar, with a mass obtained by its self-interaction with other scalars, because at the minimum of the potential there is a non-vanishing VEV v.

Next we will repeat the analysis for increasingly complicated symmetries until we see what happens when the symmetry of $\mathcal{L}$ is the SM $SU(2)_L \times U(1)_Y$ invariance and when we have the combined Lagrangian of gauge bosons, fermions, and Higgs fields. At each stage surprising new features wil emerge.

8.2 Spontaneous breaking of a continuous symmetry: the Goldstone Theorem

Breakdown of continuous symmetries is slightly more subtle. Let us consider the fairly general situation described by the Lagrangian

$$\mathcal{L}(\phi) = T - V = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi), \tag{8.8}$$

where ϕ is some multiplet of spinless fields and $V(\phi)$ and thus $\mathcal{L}(\phi)$ is invariant under some symmetry group

$$\delta\phi = i\epsilon_a T^a \phi, \tag{8.9}$$

T^a being imaginary antisymmetric matrices (because ϕ are Hermitian).

As in the previous section, we want to perturb around a minimum of the potential $V(\phi)$. We expect the ϕ field to have a VEV, $\langle\phi\rangle$, which minimizes V. To simplify notations, we define

$$V_{j_1\dots j_n} = \frac{\partial^n}{\partial\phi_{j_1}\dots\partial\phi_{j_n}} V(\phi). \tag{8.10}$$

Then we can write the condition that λ be an extremum of $V(\phi)$ as

$$V_j(\lambda) = 0. \tag{8.11}$$

For V to have a minimum at λ, we must also have

$$V_{jk}(\lambda) \geq 0. \tag{8.12}$$

The second derivative matrix $V_{jk}(\lambda)$ is the mass-squared matrix. We can see this by expanding $V(\phi)$ in a Taylor series in the shifted fields $\eta = \phi - \lambda$ and noting that the mass term is $\frac{1}{2}V_{jk}(\lambda)\eta_j\eta_k$. Thus, Eq. 8.12 assures us that there are no tachyons in the free field case about which we are perturbing.

Now comes the interesting part, the behavior of the VEV λ under the transformations in Eq. 8.9. There are two cases. If

$$T_a\lambda = 0 \tag{8.13}$$

for all a, the symmetry is not broken. This is certainly what happens if $\lambda = 0$. But Eq. 8.13 is the more general statement that the vacuum doesn't carry the charge T_a, so the charge cannot disappear into the vacuum. In the second case,

$$T_a\lambda \neq 0 \quad \text{for some } a. \tag{8.14}$$

Then the charge T_a can disappear into the vacuum even though the associated current is conserved. This is spontaneous breaking of a continuous symmetry.

Often there are some generators of the original symmetry that are spontaneously broken while others are not. The set of generators satisfying Eq. 8.13 is closed under commutation (because $T_a\lambda = 0$ and $T_b\lambda = 0$ implies $[T_a, T_b]\lambda = 0$) and generates an unbroken subgroup of the original symmetry group.

Returning to the mass matrix, because V is invariant under the transformation in Eq. 8.9, we can write

$$V(\phi + \delta\phi) - V(\phi) = iV_k(\phi)\epsilon_a(T^a)_{kl}\phi_l = 0. \tag{8.15}$$

If we differentiate with respect to ϕ_j, we get (since ϵ^a are arbitrary),

$$V_{jk}(\phi)(T^a)_{kl}\phi_l + V_k(\phi)(T^a)_{kj} = 0. \tag{8.16}$$

Setting $\phi = \lambda$, we find that the second term drops out because of Eq. 8.11, and we obtain

$$V_{jk}(\phi)(T^a)_{kl}\phi_l = 0. \tag{8.17}$$

But $V_{jk}(\lambda)$ is the mass-squared matrix M^2_{jk} for the spinless fields, so we can rewrite the last equation in the matrix form as

$$M^2T^a\lambda = 0. \tag{8.18}$$

For T^a in the unbroken subgroup, this condition is trivially satisfied. But if $T^a\lambda \neq 0$, then it requires $T^a\lambda$ is an eigenvector of M^2 with zero eigenvalue. It corresponds to a massless boson field given by

$$\phi^T T^a \lambda. \tag{8.19}$$

This is called a *Goldstone boson* after J. Goldstone, who first established this connection between spontaneously broken continuous symmetries and massless particles. We will see a specific example in the next section.

8.3 Complex scalar field - a Global Symmetry

Suppose that ϕ is a complex scalar,

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{8.20}$$

and

$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2. \tag{8.21}$$

This is invariant under a global gauge transformation,

$$\phi \to \phi' = e^{i\chi}\phi, \tag{8.22}$$

so the symmetry of $\mathcal{L}$ is now a global $U(1)$ symmetry rather than a reflection as in Section 8.1. In terms of the real components, we have

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{\mu^2}{2}(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2. \tag{8.23}$$

In the ϕ_1, ϕ_2 plane, the potential clearly has a minimum at the origin if $\mu^2 > 0$, while for $\mu^2 < 0$, the minimum is along a circle of radius

$$\phi_1^2 + \phi_2^2 = \frac{\mu^2}{\lambda} = v^2. \tag{8.24}$$

The potential in a symmetric neighborhood of $\phi = 0$ that contains the minimum looks like a Mexican sombrero or the bottom of a wine bottle.

As before, to analyze the case with $\mu^2 < 0$ we have to expand around $\phi_1^2 + \phi_2^2 = v^2$. We could choose any point on the circle, but to proceed we have to choose some point, which will break the symmetry for the solutions. We pick, arbitrarily, the point $\phi_1 = v$, $\phi_2 = 0$, and write, with η and ρ real,

$$\phi = \frac{1}{\sqrt{2}}(v + \eta(x) + i\rho(x)). \tag{8.25}$$

Substituting this in Eq. 8.23, we again find that the Lagrangian can be written in a form that is readily interpreted in terms of particles and their interactions:

$$\begin{aligned}\mathcal{L} \quad = \quad & \frac{1}{2}(\partial_\mu \eta \partial^\mu \eta) + \frac{1}{2}(\partial_\mu \rho \partial^\mu \rho) + \mu^2 \eta^2 \\ & -\lambda v(\eta \rho^2 + \eta^3) - \frac{\lambda}{4}(\eta^4 + 2\eta^2\rho^2 + \rho^4) \\ & + \text{irrelevant constant.}\end{aligned} \tag{8.26}$$

The first two terms represent the normal kinetic energy. The term $+\mu^2\eta^2$ tells us that the η field corresponds to a physical particle of squared mass

$$m_\eta^2 = 2|\mu^2|. \tag{8.27}$$

Note that there's no equivalent term in ρ^2, implying that the particle associated with the field ρ has zero mass. It is the Goldstone boson of the theory. As we expected, since we chose a particular direction in the ϕ_1, ϕ_2 plane to associate with the vacuum, the gauge invariance is no longer present in Eq. 8.26.

Physical interpretation of the massless boson is not difficult. Excitations in the radial direction requires moving up in the potential (away from the minimum), and a mass term arises from the resistance against that effort. Along the circle, the potential does not vary, so there is no resistance to motion along the circle. Thus, excitation along the circle amounts to creation of a massless boson. The Goldstone phenomenon is widespread in physics. We have encountered a simple example. The $U(1)$ symmetry is broken because we had to choose a particular point on the circle to perturb around. The presence and the particular form of the remaining (interaction) terms in Eq. 8.26 carry the link to the original unbroken symmetry, but not in an obvious way.

8.4 The Abelian Higgs Mechanism

Having worked out the breaking of a global gauge symmetry, let us now try a local one, i.e. let us consider a Lagrangian that is invariant under local gauge transformations. We know from our earlier discussions that such an invariance requires the introduction of a massless vector field, say A_μ, and we know that we should write the Lagrangian in terms of the covariant derivative,

$$\partial_\mu \to D_\mu = \partial_\mu - igA_\mu. \tag{8.28}$$

The gauge field transforms as

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{g}\partial_\mu \chi(x) \tag{8.29}$$

and ϕ as

$$\phi(x) \to \phi'(x) = e^{i\chi(x)}\phi(x). \tag{8.30}$$

The Lagrangian is then

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{8.31}$$

For $\mu^2 > 0$ this describes the interaction of a charged scalar particle (with $g = e$) of mass μ with the electromagnetic field A_μ, for example. Note that there is no mass term for A_μ. The kinetic energy terms for the vector field are contained in $F_{\mu\nu}F^{\mu\nu}$ and we shall carry them along, but they do not play a role in the spontaneous breakdown of the symmetry. As in the previous sections, we are primarily interested in the scenario where $\mu^2 < 0$. Note that this Lagrangian contains four independent degrees of freedom: the two components ϕ_1 and ϕ_2 of the scalar field, and the two transverse polarization states of the massless vector boson (as expected, if A_μ represents a photon). We could proceed as before. The algebra gets increasingly complicated, however, so in order to simplify the analysis, let us use what we have already learned.

In general, ϕ can be written in the form $\phi(x) = \eta(x)e^{i\rho(x)}$, where η, ρ are real, so we can rewrite ϕ as

$$\phi(x) = \frac{1}{\sqrt{2}}(v + h(x)), \tag{8.32}$$

with h real, having used a transformation as in Eq. 8.30, knowing that if necessary, we could find a χ to accomplish that. We could not have done this in the previous section, since the Lagrangian there was only invariant under a global symmetry, not a local one. Now we substitute this in $\mathcal{L}$. Since the original choice of the field A_μ was not fixed by physics, we do not bother to carry its transformation (Eq. 8.29) along. So

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left((\partial_\mu - igA_\mu)(v+h)\right)\left((\partial^\mu + igA^\mu)(v+h)\right) \\
&\qquad\qquad -\frac{\mu^2}{2}(v+h)^2 - \frac{\lambda}{4}(v+h)^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&= \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{2}g^2v^2A_\mu A^\mu - \lambda v^2h^2 - \lambda v h^3 - \frac{\lambda}{4}h^4 \\
&\qquad\qquad -g^2 v h A_\mu A^\mu + \frac{1}{2}g^2h^2A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.
\end{aligned} \tag{8.33}$$

The surprising result is the second term on the RHS: we now have a mass term for the gauge boson! But since we started with a gauge invariant theory and only made algebraic transformations, we expect the resulting theory to be gauge invariant as well. The gauge boson mass is the square root of the coefficient of $\frac{1}{2}A_\mu A^\mu$,

$$M_A = gv, \tag{8.34}$$

which is non-zero only when the gauge symmetry is spontaneously broken as a result of the Higgs field acquiring a non-zero VEV, v. So, the theory is only gauge invariant in a restricted sense. As before, the Lagrangian is gauge invariant, but the vacuum is not, because we had to choose a particular direction in the ϕ_1, ϕ_2 plane and the minimum of the potential along that direction to label "the vacuum".

The spectrum now consists of a single real Higgs boson h, that has a mass $\sqrt{2\lambda}v$, various self interactions and cubic and quartic interactions with the gauge field A_μ. Since the massive boson has three spin states (corresponding to $J_3 = 1$, 0, or -1 in its rest frame), the number of independent fields is still four.

What has happened here is that the Goldstone boson of the previous section has become the longitudinal polarization state of the gauge boson. This can be seen a little more explicitly if the calculation of this section is carried out without the simplifying step of Eq. 8.32, but using Eq. 8.25 instead. Then the mass appears for the gauge vector boson, and a term $A_\mu \partial^\mu \rho$, which apparently allows A_μ to turn into ρ as it propagates. When such cross terms appear, one can go to eigenstates by a diagonalization, which can be accomplished here by a gauge transformation, and which eliminates ρ from the Lagrangian. This phenomenon is sometimes referred to as the gauge boson having "eaten" the Goldstone boson.

The mechanism we have just studied is called the *Higgs mechanism*. Technically it is well understood, but at a physical level its meaning is not yet fully grasped in particle physics. In some sense, the longitudinal polarization state of the gauge boson (which must exist if it is to be massive in a Lorentz invariant theory where it is possible to go to its rest frame) is the Goldstone boson that would have appeared as a physical particle if the theory were not a local gauge theory. There is also a neutral spin-0 boson left over that apparently should exist as a physical particle; it is called the Higgs boson. Note that the gauge boson mass is fixed if g^2 and v are known, but the mass of the Higgs boson h depends on the unknown parameter λ. In the next section we will add the last bit of complexity needed to fully incorporate the Higgs mechanism into the SM.

8.5 The Higgs Mechanism in the Standard Model

Now we are ready to work out the Higgs mechanism in the SM by adding one further degree of complexity. In the last section, we saw how the Higgs field can break a local symmetry, but it was an Abelian symmetry and the Higgs field carried no charge: all of its quantum numbers, other than 4-momentum, were zero. In the SM, the Higgs field is assigned to a $SU(2)$ doublet. We can choose

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{8.35}$$

where the superscripts denote the electric charge. ϕ^+ and ϕ^0 are each complex fields:

$$\begin{aligned} \phi^+ &= \tfrac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \\ \phi^0 &= \tfrac{1}{\sqrt{2}}(\phi_3 + i\phi_4). \end{aligned} \tag{8.36}$$

In the $SU(2)$ space ϕ^+ and ϕ^0 are related by a rotation, like a spin-up and a spin-down state, or the left-handed ν_e to the left-handed electron. The scalar part of the Lagrangian has the same form as in last section

$$\mathcal{L}_\phi = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2, \tag{8.37}$$

but now ϕ is a column vector and $\phi^\dagger$ a row, each with 2 complex components, so

$$\phi^\dagger \phi = \begin{pmatrix} \phi^{+*} & \phi^{0*} \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \phi^{+*}\phi^+ + \phi^{0*}\phi^0 = \frac{\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2}{2}. \tag{8.38}$$

As before, we study the potential

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2. \tag{8.39}$$

$V(\phi)$ is invariant under the local gauge transformation

$$\phi(x) \to \phi'(x) = e^{i\vec{\alpha}(x)\cdot\frac{\vec{\sigma}}{2}}\phi(x), \tag{8.40}$$

where σ_i are the Pauli matrices and α_i are parameters. Proceeding as before, $V(\phi)$ has a minimum for $\mu^2 < 0$ at

$$\phi^\dagger \phi = \frac{-\mu^2}{2\lambda} = \frac{v^2}{2}. \tag{8.41}$$

From Eq. 8.38, we see that there are many ways to satisfy Eq. 8.41.

So, once again, we must choose a direction, this time in $SU(2)$ space, and expand around the minimum. We make the following choice and call it the vacuum (in the sense that it is the ground state of free space), ϕ_0:

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{8.42}$$

i.e. $\phi_3 = v$, $\phi_1 = \phi_2 = \phi_4 = 0$.

To study the spectrum by perturbing around the vacuum, we write

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \tag{8.43}$$

and we will look for the equations satisfied by H. We can make this simple choice because for an arbitrary $\phi(x)$ we could apply a gauge transformation, $\phi \to \phi' = \exp\left(\frac{i\vec{\sigma}\cdot\vec{\theta}(x)}{v}\right)\phi$, and rotate $\phi(x)$ into the form of Eq. 8.43. This amounts to "gauging away" three fields, which is consistent with what the Goldstone theorem. The original symmetry, from Eq. 8.41,

$$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = \text{invariant} \tag{8.44}$$

is an $O(4)$ symmetry. By choosing a direction, we have broken three continuous symmetries, so three massless bosons, and three fields, are gauged away. We shall soon see that these three are just what are needed to endow the $W^\pm$ and Z bosons with longitudinal polarization, and, therefore, mass.

Before we write the covariant derivative and complete the calculation, let us examine a bit further what is happening. The electric charge Q, the weak isospin eigenvalue T_3, and the $U(1)$ hypercharge Y (for the Higgs field) are related by

$$Q = T_3 + \frac{Y}{2} \tag{8.45}$$

The electric charge assignment of Eq. 8.35 corresponds to $Y_H = 1$. The choice that only the neutral component ϕ^0 gets a VEV is very important, since whatever quantum numbers ϕ carries can vanish into the vacuum. If ϕ^+ had a non-zero VEV, then electric charge would not be conserved, contrary to observation.

If the vacuum ϕ_0 is invariant under some subgroups of the original $SU(2) \times U(1)$, any gauge bosons associated with that subgroup will still be massless. Since the Higgs field $\phi(x)$ is a doublet, but only one component gets a VEV, clearly the $SU(2)$ symmetry is broken. Since $Y_H \neq 0$, the $U(1)$ symmetry is broken. However, if we operate with the electric charge operator on ϕ_0, we get

$$Q\phi_0 = \left(T_3 + \frac{Y}{2}\right)\phi_0 = 0, \tag{8.46}$$

so ϕ_0 (i.e. the vacuum) is invariant under a transformation

$$\phi_0 \to {\phi_0}' = e^{i\alpha(x)Q}\phi_0 = \phi_0. \tag{8.47}$$

This is also a $U(1)$ transformation, so the vacuum is invariant under a certain $U(1)_Q$ whose generator is a particular linear combination of the generators of the original $SU(2)_L$ and $U(1)_Y$. Of course, this is the $U(1)$ of electromagnetism, and the gauge boson that remains massless is the photon (because it does not acquire a longitudinal polarization by "eating" a Goldstone boson). The presence of a massless gauge boson was a necessary consequence of electric charge conservation, which forces us to choose a vacuum that is electrically neutral.

Finally, let us carry out the algebra to see the consequences of the Higgs mechanism. For the full Lagrangian to be invariant under the transformation in Eq. 8.40, we have to replace ∂_μ by the covariant derivative D_μ where

$$D^\mu = \partial^\mu - ig_1\frac{Y}{2}B^\mu - ig_2\frac{\sigma_i}{2}W_i^\mu \tag{8.48}$$

(this is the same as Eq. 6.27 except that we have dropped the color $SU(3)$ term since that symmetry is not broken) and the gauge fields B_μ and $\vec{W}_\mu$ transform as in Chapter 6. Then when ϕ gets a VEV, proceeding as in the earlier sections, the Lagrangian contains extra terms

$$\phi^\dagger\left(ig_1\frac{Y}{2}B^\mu + ig_2\frac{\sigma_i}{2}W_i^\mu\right)^\dagger\left(ig_1\frac{Y}{2}B^\mu + ig_2\frac{\sigma_i}{2}W_i^\mu\right)\phi. \tag{8.49}$$

Putting $Y = 1$, writing the 2×2 matrices explicitly as in Chapter 7, and using Eq. 8.43, we get the contribution to $\mathcal{L}$,

$$\begin{aligned}\frac{1}{8}&\left|\begin{pmatrix} g_1B_\mu + g_2W_\mu^3 & g_2(W_\mu^1 - iW_\mu^2) \\ g_2(W_\mu^1 + iW_\mu^2) & g_1B_\mu - g_2W_\mu^3\end{pmatrix}\begin{pmatrix}0 \\ v\end{pmatrix}\right|^2 \\ &= \frac{1}{8}v^2g_2^2\left((W_\mu^1)^2 + (W_\mu^2)^2\right) + \frac{1}{8}v^2(g_1B_\mu - g_2W_\mu^3)^2.\end{aligned} \tag{8.50}$$

The first term can be rewritten as

$$\left(\frac{1}{2}vg_2\right)^2 W_\mu^+W^{-\mu} \tag{8.51}$$

carefully keeping track of the $\sqrt{2}$ factors as in Eq. 7.9. For a charged boson, the expected mass term in a Lagrangian would be $m^2W^+W^-$, so we can conclude that the $W^\pm$ has indeed acquired a mass

$$M_W = \frac{1}{2}vg. \tag{8.52}$$

The second term in Eq. 8.50 is not diagonal. So, we have to define new eigenvalues to find the particles with definite mass (B and W^3 are the neutral states with diagonal weak hyercharge and weak isospin interactions). In fact, we already have the answer in hand, because the linear combination of B and W^3 appearing in Eq. 8.50 is just the combination we have called Z_μ (see Sec. 7.3 and note the choice of $Y_L = -1$). We expect mass terms for Z_μ and the photon A_μ. For a neutral field there is a factor of $\frac{1}{2}$ relative to the charged ones, so mass terms $\frac{1}{2}({M_Z}^2Z_\mu Z^\mu + {M_\gamma}^2A_\mu A^\mu)$ should appear. From Eq. 8.50 and the normalization of Z in Eq. 7.15, we can conclude that

$$M_Z = \frac{1}{2}v\sqrt{g_1^2 + g_2^2} \tag{8.53}$$

and

$$M_\gamma = 0. \tag{8.54}$$

It's good to see that the photon remains massless since no $A_\mu A^\mu$ term appears. Using the identities of Chapter 7, we can also write

$$\frac{M_W}{M_Z} = \cos\theta_W. \tag{8.55}$$

Since B and W^3 mix, the neutral state is not degenerate in mass with the charged ones, unless $\theta_W = 0$. Once θ_W is measured, SM fixes either of the *Intermediate Vector Boson* masses in terms of the other, and the result has been verified by experiments.

It's useful to define

$$\rho \equiv \frac{M_W}{M_Z \cos\theta_W}. \tag{8.56}$$

The SM predicts $\rho = 1$. In fact, it can be shown that $\rho = 1$ is guaranteed even if additional Higgs doublets are present. Any deviation from $\rho = 1$ would be an important signal of new physics.

8.6 Fermion Masses

Now that we have available the Higgs field in a $SU(2)$ doublet, it is possible to write a $SU(2)$-invariant interaction of fermions with the Higgs field. For example, we can add the following interaction term to the Lagrangian for the first-generation leptons:

$$\mathcal{L}_{\text{int}} = g_e \left(\bar{L}\phi e_R^- + \phi^\dagger \bar{e}_R^- L \right). \tag{8.57}$$

Since $L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$ and $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$, $\bar{L}\phi = \bar{\nu}_{eL}\phi^+ \bar{e}_L^- \phi^0$ is a $SU(2)$ invariant. Multiplying by the singlet e_R^- does not change the $SU(2)$ invariance. The second term is the Hermitian conjugate of the first. The coupling g_e is arbitrary; neither the presence of such terms nor g_e follows from a gauge principle.

Following the previous sections, we can calculate the observable effects of adding this term by replacing

$$\phi \to \begin{pmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{pmatrix}, \tag{8.58}$$

where v is the Higgs VEV, and H is the neutral, physical Higgs particle. Substituting it into Eq. 8.57 gives

$$\mathcal{L}_{\text{int}} = \frac{g_e v}{\sqrt{2}} \left(\bar{L}\phi e_R^- + \phi^\dagger \bar{e}_R^- L \right) + \frac{g_e}{\sqrt{2}} \left(\bar{L}\phi e_R^- + \phi^\dagger \bar{e}_R^- L \right) H. \tag{8.59}$$

The first term in Eq. 8.59 has exactly the same form as expected for a fermion of mass

$$m_e = \frac{g_e v}{\sqrt{2}}. \tag{8.60}$$

Thus the theory can now accommodate a non-zero electron mass. It should be noted here that it is only through such mass terms that a fermion can make a transition from a right-handed helicity state to a left-handed one (or vice versa). The amplitude is proportional to the fermion mass. This explains why $\dfrac{B(\pi^- \to e\bar{\nu}_e)}{B(\pi^- \to \mu\bar{\nu}_\mu)} \approx 1.23 \times 10^{-4}$.

Sicne g_e is arbitrary, the value of the electron mass has not been calculated. Rather, we can invert Eq. 8.60, so

$$g_e = \frac{\sqrt{2}m_e}{v}. \tag{8.61}$$

The second term in Eq. 8.59 says that the theory has an electron-Higgs vertex of strength $\frac{g_e}{\sqrt{2}} = \frac{m_e}{v}$, which determines the probability for an electron or positron to radiate a Higgs boson, or for a Higgs boson to decay into e^+e^-. We can eliminate g_e from the Lagrangian and rewrite it as

$$\mathcal{L}_{\rm int} = m_e\bar{e}e + \frac{m_e}{v}\bar{e}eH. \tag{8.62}$$

Note that no mass term appears for the neutrino since we have assumed that the theory contains no right-handed neutrino state ν_R, so a term analogous to Eq. 8.57 cannot be written that will subsequently lead to a mass term $\bar{\nu_R}\nu_L$. This implies that the neutrinos do not interact with H. If there were a ν_R, it would be hard to observe; since it would have $T_3 = 0$ and $Q = 0$, it would not couple to $W^\pm$, Z^0, or γ. However, a neutrino of non-zero mass, and therefore a ν_R, can be accommodated in the SM.

For quarks, this leads to another subtlety (one that would also have occured for leptons had ν_R existed). If $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$ is a $SU(2)$ doublet, then so is

$$\psi_c = -i\sigma_2\psi^* = \begin{pmatrix} -b^* \\ a^* \end{pmatrix}. \tag{8.63}$$

Then we can also write terms of $\mathcal{L}_{\rm int}$ using

$$\phi_c = \begin{pmatrix} -\phi^{0*} \\ \phi^- \end{pmatrix}, \tag{8.64}$$

which becomes, after invoking the Higgs mechanism,

$$\phi_c \to \begin{pmatrix} -\frac{v+H}{2} \\ 0 \end{pmatrix}. \tag{8.65}$$

ϕ has hypercharge $Y = +1$, ϕ_c has $Y = -1$, and still satisfies for each state, $Q = T_3 + \dfrac{Y}{2}$.

Then, for quarks, we have

$$\mathcal{L}_{\rm int} = g_d\bar{Q}_L\phi d_R + g_u\bar{Q}_L\phi_c u_R + \text{ Hermitian conjugate.} \tag{8.66}$$

Substituting from Eqs. 8.61 and 8.63,and for Q_L, gives

$$\mathcal{L}_{\rm int} = m_d\bar{d}d + m_u\bar{u}u + \frac{m_d}{v}\bar{d}dH + \frac{m_u}{v}\bar{u}uH, \tag{8.67}$$

where g_d and g_u have been eliminated in favor of their masses, following the same steps that led to Eq. 8.61. Again, the quark masses can be included in the description, but since g_d and g_u are arbitrary parameters, not related to each other or to g_e, the masses have to be measured. The last two terms of Eq. 8.67 describe the interaction of d and u quarks with H^0.

The entire procedure of this section can be repeated for the second and third generations, giving further pieces of $\mathcal{L}_{\rm int}$ which come from taking Eq. 8.62 with $e \to \mu, \tau$ and Eq. 8.67 with $u \to c, t$ and $d \to s, b$. Since the Higgs-fermion coupling strength is proportional to the latter's mass, it couples most strongly to the heaviest fermions.

8.7 Comment on Vacuum Energy

The Higgs mechanism contributes to an important problem when cosmological considerations are introduced. We found the Higgs VEV $v = \sqrt{-\mu^2/\lambda}$. Putting $\phi = v$ in Eq. 8.39, we can calculate the Higgs potential at its minimum:

$$V(\phi = v) = V_0 = \frac{\lambda v^4}{2}. \tag{8.68}$$

$v = 2M_W/g_2 \approx 246$ GeV $\Rightarrow V_0 \approx 2{\times}10^9\lambda$ GeV^4, and 1 GeV$^3 \approx 1.3 \times 10^{41}$ cm^{-3}. So, $V_0 \approx 2.6 \times 10^{50}\lambda$ GeV/cm^3. This is apparently the contribution of spontaneous symmetry breaking to the vacuum energy density of the universe.

But, from astrophysics, it is known that the energy density of luminous matter in the universe is about one proton per cubic meter on the average, and that the total density of matter is no more than $\sim$100 times this number. Thus, empirically, the total energy density is less than about 10^{-4} GeV/cm^3.

To compare this number with V_0, we need the value of λ, the arbitrary Higgs self-coupling in the Higgs potential. While λ is not known, if it were eventually to be determined by some fundamental argument, such as a gauge principle, presumably $\lambda >\sim 1/10$. Combining these, we find $V_0 \approx 2 \times 10^{49}$ GeV/cm^3, larger than the experimentally observed value by a factor of 10^{54}!

Technically, this is not a contradiction, because we can always add a constant to the potential in theories without gravity and cancel V_0, but to do so involves tuning the constant to a part in 10^{54}, which is hardly satisfactory. this is (essentially) what is referred to as the problem of the cosmological constant. If gravity is included, terms of the kind we are considering will contribute to the energy-momentum tensor and, through Einstein's equations, dramatically affect the geometry of space-time. This is another clue that in spite of the remarkable descriptive power of the SM, it is a theory that is incomplete at the fundamental level.