Chapter 4

Decay Widths and Scattering Cross Sections

We are now ready to calculate the rates of some simple scattering and decay processes. The former is expressed in terms of *cross section*, σ, which is a measure of the probability of a specific scattering process under some given set of initial and final conditions, such as momenta and spin polarization. The latter is expressed in terms of *lifetime*, τ, or, equivalently, *decay width*, Γ ($\propto \frac{1}{\tau}$), which is a measure of the probability of a specific decay process occuring within a given amount of time in the parent particle's rest frame. The calculation involves two steps:

1. Calculate the *amplitude*, $\mathcal{M}$, of the process. It is also often referred to as the *matrix element*, and denoted by $\mathcal{M}_{fi}$, to indicate that in a matrix representation of the transformation process, with the initial and final states as bases, this is the element that connects a particular final state f to a given initial state i. A process can be a combination of subprocesses, in which case, the total amplitude is the sum of the subprocess amplitudes.[1] Each simple (sub)process is represented by a unique *Feynman diagram*. Its amplitude is a point function in the phase space of all the particles involved, including any intermediate *propagator*, and depends on the nature of the *coupling* at each vertex (of the diagram). For a given diagram, the amplitude can be obtained by following the *Feynman rules* for combining the elements - a factor for each external line (representing a free particle in the initial or final state), one for each internal line (representing a virtual *propagator* particle), and one for each vertex point where the lines meet.

2. Integrate the amplitude over the allowed phase space to get the σ or Γ, as the case may be. The integral can be constructed, easily in principle, by following *Fermi's golden rule*, although its evaluation can be extremely

[1] An amplitude is a Lorentz scalar, but generally complex-valued.

challenging except in the simplest of cases such as those we will encounter in this course.

This chapter we describe the above rules and use them to calculate the decay rates and cross sections for some simple (and sometimes hypothetical) processes in quantum electrodynamics (QED).

4.1 Physical meaning of decay width

One of the most important charateristics of a particle is its lifetime. It depends, of course, on the available *decay modes or channels*, which are subject to conservation laws for appropriate quantum numbers, coupling strength of the decay process, and kinematic constraints. The lifetime of an individual particle cannot be predicted, but a statistical distribution can be specified for a large sample. Equivalently, one can express it in terms of the *decay rate*, Γ, which is the *probability per unit time* that a given particle will decay.

The probability that a single unstable entity will cease to exist as such after an interval is proportional to that interval. The constant of proportionality is called the decay rate. For complex unstable entities such as stars, living organisms, businesses, economies etc., any two are rarely "identical", and each evolves in its own complex manner with time. Their decay rates depend on their constitution, age, and external factors, making it very difficult to estimate their lifetimes, even on average. Fortunately, that is not the case with elementary particles. Thus, for an ensemble of $N \to \infty$ identical particles, the change in the number after a time dt is

$$dN = -\Gamma N dt. \tag{4.1}$$

So, the expected number surviving after time t is

$$N(t) = N(0)e^{-\Gamma t}. \tag{4.2}$$

The time after which the ensemble is expected to shrink to $\frac{1}{e}$ of its original size is called the *lifetime*:

$$\tau = \frac{1}{\Gamma}. \tag{4.3}$$

If multiple decay modes are available, as is often the case, then one can associate a decay rate for each mode, and the total rate, will be the sum of the rates of the individual modes.

$$\Gamma_{\text{total}} = \sum_{i=1}^{n} \Gamma_i. \tag{4.4}$$

The particle's lifetime is them given by

$$\tau = \frac{1}{\Gamma_{\text{total}}}. \tag{4.5}$$

In such cases, we are often interested in the *branching fractions*, i.e. the probabilities of the decay by individual modes. The branching fraction of mode i is

$$B_i = \frac{\Gamma_i}{\Gamma_{\text{total}}}. \tag{4.6}$$

Since the dimension of Γ is the inverse of time, in our system of *natural units*, it has the same dimension as mass (or energy). When the mass of an elementary particle is measured, the total rate shows up as the irreducible "width" of the shape of the distribution.[2] Hence the name *decay width*.

4.2 Physical meaning of scattering cross section

Consider the $2 \to n$ scattering process

$$ab \to cd \dots \tag{4.7}$$

The system of incoming particles labeled a, b constitute the initial state $|i\rangle$, and that of the outgoing particles labeled c, $d, \dots$ constitute the final state $|f\rangle$.[3] If a packet of a particles is made to pass head-on through a packet of b particles so that the overlap area is A, and the number of particles swept by that overlap area in the two packets are N_a and N_b repectively, then the number of scatterings, N_S is directly proportional to N_a and N_b, and inversely to A. The overall constant of proportionality is called the cross section, σ:

$$N_S = \sigma \frac{N_a N_b}{A}. \tag{4.8}$$

Thus, the cross section must have the same dimension as area. Cross sections in contemporary HEP experiments are typically measured in units of nanobarn (nb) to femtobarn (fb), where a *barn* is defined as

$$1b = 10^{-24}\ \text{cm}^2 = 2.568\ \text{GeV}^{-2}. \tag{4.9}$$

As for decays, one is often more interested in various *differential (or exclusive) cross sections*, σ_i rather than the *total (or inclusive) cross section*, σ_{total}:

$$\sigma_{\text{total}} = \sum_{i=1}^{n} \sigma_i \tag{4.10}$$

For example, the total cross section of proton-antiproton collisions at a center-of-mass energy ($\sqrt{s}$), as in Tevatron Run 2, is huge,

$$\sigma(p\bar{p} \to X) \approx 75\ \text{mb}, \tag{4.11}$$

[2] As opposed to the *statistical and systematic uncertainties* in the measurement, which can be reduced, in principle, to zero by building an infinitely precise and accurate measuring device and collecting an infinite amount of data with it.

[3] Different labels are not intended to mean that the particles are necessarily different.

where X represents "anything", but that for the most highly sought-after processes are small (duh!), e.g.

$$\sigma(p\bar{p} \to t\bar{t}X) \approx 7.5 \text{ pb}. \tag{4.12}$$

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4.3 Calculation of widths and cross sections

The matrix element between the initial state $|i\rangle$ and the final state $|f\rangle$ is called the S matrix:

$$S_{fi} = (2\pi)^4\delta^4(p_f - p_i)\mathcal{M}_{fi}, \tag{4.13}$$

where p_i is the total initial momentum, p_f the total final momentum, and the 4-dimensional δ funtion expresses the conservation of 4-momentum $(E, \vec{p})$. The quantity $\mathcal{M}_{fi}$, called the (reduced) matrix element or amplitude of the process, contains the non-trivial physics of the problem, including spins and couplings. It is usually calculated by perturbative approximation.

The probability of the transition from $|i\rangle$ to $|f\rangle$ is given by

$$P_{i\to f} = \frac{S_{fi}}{\langle f|f\rangle\langle i|i\rangle}. \tag{4.14}$$

The rate of the transition is determined by Fermi's Golden Rule:[4]

$$\text{transition rate} = 2\pi|\mathcal{M}_{\{\rangle}|^2 \times (\text{phase space}). \tag{4.15}$$

4.3.1 The Golden Rule for Decays

For an n-body decay

$$i \to f_k; \qquad k = 1, \ldots, n \tag{4.16}$$

the differential decay rate is given by

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2m_i}\left(\prod_{k=1}^{n} \frac{d^3\vec{p}_k}{(2\pi)^3 2E_k}\right) \times (2\pi)^4\delta^4\left(p_i - \sum_{k=1}^{n} p_k\right), \tag{4.17}$$

where p_k is the 4-momentum of the kth particle, and S is a product of statistical factors: $\frac{1}{m!}$ for each group of m identical particles in the final state.

Usually we are not interested in specific momenta of the decay products. So, the total decay rate is obtained by integrating Eq. **??**. For a general 2-body decay, the total width is given by

$$\Gamma = \frac{S|\vec{p}|}{8\pi m_i}|\mathcal{M}|^2, \tag{4.18}$$

[4] Derivation of this rule is outside the scope of this course, but can be found in any standard text of quantum field theory.

where $|\vec{p}|$ is the magnitude of the momentum of either outgoing particle in the parent's rest frame (this is fully determined by the masses of the 3 particles involved in the process), and $\mathcal{M}$ is evaluated at the momenta required by the conservation laws.

4.3.2 The Golden Rule for Scattering

Just as for the decay rate, for a $2 \to n$ scattering process

$$ij \to f_k; \qquad k = 1, \ldots, n \tag{4.19}$$

the differential cross section is given by

$$d\sigma = |\mathcal{M}|^2 \frac{S}{4\sqrt{(p_i \cdot p_j)^2 - (m_i m_j)^2}} \left(\prod_{k=1}^{n} \frac{d^3\vec{p}_k}{(2\pi)^3 2E_k} \right) \times (2\pi)^4 \delta^4 \left(p_i + p_j - \sum_{k=1}^{n} p_k \right). \tag{4.20}$$

For a 2×2 process in the CM frame, this leads to

$$d\sigma = \frac{S}{64\pi^2 E_{\text{CM}}^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} |\mathcal{M}|^2 d\Omega, \tag{4.21}$$

where $|\vec{p}_f|$ is the magnitude of the momentum of either outgoing particle, $|\vec{p}_i|$ is the magnitude of the momentum of either incoming particle, and

$$d\Omega = \sin\theta d\theta d\phi \tag{4.22}$$

is the solid-angle element in which the final state particles scatter.

4.4 Feynman rules for calculating the amplitude

In the previous sections, the formulae for deay rates and scattering cross sections are given in terms of the amplitude $\mathcal{M}_{fi}$. Here we give the recipe to calculate $-i\mathcal{M}_{fi}$ for a given Feynman diagram for tree-level processes in QED:[5]

1. **External lines:**
   (a) For an incoming electron, positron, or photon, associate a factor u, $\bar{v}$, or e_μ, respectively.
   (b) For an outgoing electron, positron, or photon, associate a factor $\bar{u}$, v, or e^*_μ, respectively.
2. **Vertices:** For each vertex, include a factor of $ie\gamma^\mu$ for an electron or $-ie\gamma^\mu$ for a positron. Care must be exercised to get the overall sign for fermions correct.

[5] Only the vertex factor is different for QCD and weak interactions. We shall encounter them in due course.

3. **Internal lines:**

   (a) For an electron or a positron connecting two vertices, include a term

$$i\left(\frac{\not{p}+m}{p^2-m^2+i\varepsilon}\right). \tag{4.23}$$

   (b) For an photon connecting two vertices, include a term

$$\frac{ig_{\mu\nu}}{q^2+i\varepsilon}. \tag{4.24}$$

   (c) Integrate over all undetermined internal momenta.