Assignment: HW5 [40 points]

Assigned: 2009/11/10
Due: 2009/11/16

Solutions

P5.1 [10 points]
A simple plane pendulum has a mass m hanging at the end of a massless string of length ℓ in a field of constant gravitational acceleration $\vec{g}$. While the pendulum is in motion, the length of the string is changed at a constant rate $\dot{\ell} = v_0$. Find the Lagrangian and the Hamiltonian, determine whether or not $T+V$ and H are conserved, and comment on the physical interpretation of your results. This is a rather famous problem discussed by Einstein, Lorentz, and others at the 1911 Solvay Conference.
Hint: Since ℓ is not an independent generalized coordinate, but is constrained to be a simple linear function of time, the system only has one degree of freedom.

S5.1 Let θ be the angle between the string and $\vec{g}$. The kinetic and potential energies are

$$T = \frac{m}{2}(\dot{\ell}^2 + \ell^2\dot{\theta}^2) \qquad \text{and} \qquad V = -mg\ell\cos\theta. \tag{1}$$

So, the Lagrangian is

$$L = T - V = \frac{m}{2}(\dot{\ell}^2 + \ell^2\dot{\theta}^2) + mg\ell\cos\theta. \tag{2}$$

This is a valid expression for the Lagrangian even though ℓ is not an independent generalized coordinate, but is constrained to be a simple linear function of time, i.e.,

$$\ell = \ell_0 + v_0 t. \tag{3}$$

In the Hamiltonian $H = p_i\dot{q}_i - L$, the implied sum $p_i\dot{q}_i$ contributes only

$$p_\theta\dot{\theta} = \frac{\partial L}{\partial\dot{\theta}}\dot{\theta} = m\ell^2\dot{\theta}^2 = \frac{p_\theta^2}{m\ell^2}. \tag{4}$$

Hence,

$$\begin{aligned} H =& \frac{p_\theta^2}{m\ell^2} - \frac{m}{2}\left(v_0^2 + \frac{p_\theta^2}{m^2\ell^2}\right) - mg\ell\cos\theta \\ =& \frac{p_\theta^2}{2m\ell^2} - \frac{m}{2}v_0^2 - mg\ell\cos\theta \\ =& \frac{p_\theta^2}{2m(\ell_0+v_0t)^2} - \frac{m}{2}v_0^2 - mg(\ell_0+v_0t)\cos\theta \end{aligned} \tag{5}$$

It can be seen that $\dfrac{\partial H}{\partial t} \neq 0$. Hence, the Hamiltonian is not conserved. Moreover, $T+V = H + mv_0^2$. So, the mechanical energy $T+V$ is not conserved either. Physically, the nonconservation of both H and $T+V$ results from the energy flowing between the system and the external agent that drives the length of the string.

P5.2 $[2 + 4 + 4 = 10 \text{ points}]$

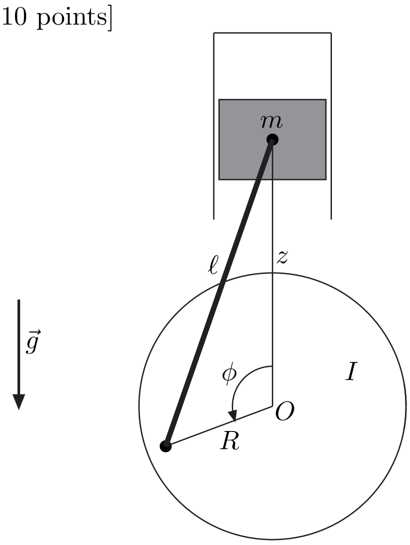

Figure 5.1

A lawn-mower engine contains a piston of mass m that moves along $\hat{z}$ in a field of constant gravitational acceleration $\vec{g} = g\hat{\mathbf{z}}$. The center of mass of the piston is connected to a flywheel of moment of intertia I at a distance R from its center by a rigid and massless rod of length ℓ, as shown in Fig. 5.1. The system has only one degree of freedom but two natural coordinates, ϕ and z.

(a) Express the Lagrangian in terms of $q_1 = z$, $q_2 = \phi$ and write the constraint equation that connects the two coordinates.

(b) From the above results, write down the two coupled equations of motion using the method of "undetermined multiplier"s. Then eliminate the undetermined multiplier to obtain a single equation of motion (it can still involve both coordinates).

(c) Find $p_\phi(z, \phi, \dot{\phi})$.

S5.2 (a) Let us take the origin to be at the center of the flywheel.

$$T = \frac{m}{2}\dot{z}^2 + \frac{I}{2}\dot{\phi}^2; \quad V = mgz \quad \Rightarrow \quad L = T - V = \frac{m}{2}\dot{z}^2 + \frac{I}{2}\dot{\phi}^2 - mgz. \tag{6}$$

The constraint equation is

$$z^2 + R^2 - 2Rz\cos\phi = \ell^2. \tag{7}$$

(b) A constraint equation of the form $C = 0$ leads to Lagrange equation(s) of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} - \lambda\frac{\partial C}{\partial q_i} = 0, \tag{8}$$

where $\lambda(\vec{q}, t)$ is the undetermined multiplier. Thus, in our case, the equations of motion are

$$\frac{d}{dt}(m\dot{z}) = -mg - \lambda(2z - 2R\cos\phi) \tag{9}$$

and

$$\frac{d}{dt}\left(I\dot{\phi}\right) = -2\lambda Rz\sin\phi. \tag{10}$$

Eliminating λ from Eqs. 9 and 10 we find that z and ϕ satisfy the differential equation

$$m\ddot{z} + \left(\frac{\cot\phi}{z} - \frac{\csc\phi}{R}\right) I\ddot{\phi} + mg = 0. \tag{11}$$

(c) Differentiating the constraint equation w.r.t. t we get

$$\begin{aligned} &\dot{z}(z - R\cos\phi) + Rz\dot{\phi}\sin\phi = 0, \\ \text{or,}\qquad &\dot{z} = -\frac{Rz\sin\phi}{z - R\cos\phi}\dot{\phi}. \end{aligned} \tag{12}$$

Differentiating further w.r.t. $\dot{\phi}$ we get

$$\begin{aligned} &\frac{\partial\dot{z}}{\partial\dot{\phi}}(z - R\cos\phi) + Rz\sin\phi = 0, \\ \text{or,}\qquad &\frac{\partial\dot{z}}{\partial\dot{\phi}} = -\frac{Rz\sin\phi}{z - R\cos\phi}. \end{aligned} \tag{13}$$

Thus, using Eqs. 6, 12, and 13,

$$p_\phi(z,\phi,\dot{\phi}) \equiv \frac{\partial L}{\partial\dot{\phi}} = I\dot{\phi} + m\dot{z}\frac{\partial\dot{z}}{\partial\dot{\phi}} = \left[I + m\left(\frac{Rz\sin\phi}{z - R\cos\phi}\right)^2\right]\dot{\phi}. \tag{14}$$

P5.3 [$4+4+2 = 10$ points]

Liouville's Theorem gives information about the statistical properties of systems containing a very large number of particles. The theorem can be expressed as $\dot{D} = 0$, where D is the phase space density of possible systems in that region of phase space. There is no equivalent theorem that can be expressed in terms of quantities in configuration space. Thus, problems in statistical mechanics are important examples where the Hamiltonian approach offers a solution while the Lagrangian approach does not.

Now consider the example of a beam of identical charged particles with momentum P, produced by an accelerator. Suppose that in the plane perpendicular to the incident direction, the beam initially has a uniformly populated circular cross section of radius r_1 in configuration space, and a uniformly populated circular cross section of radius p_1 in momentum space. A pair of quadrupole magnets with appropriate relative orientation can focus a beam of charged particles, i.e., can reduce the transverse radius from r_1 to r_2, where $r_2 < r_1$.

(a) What does Liouville's theorem tell us about the consequences of this focusing operation?

(b) Suppose that the beam pipe has an internal radius R. At what maximum distance downstream from the focus must another focusing element be located in order to avoid some of the beam particles scraping the pipe?

(c) A collimator (an absorber with a hole in it) could also be used to produce a beam with radius r_2. Contrast the consequences of using a focusing element vis-a-vis a collimator.

S5.3 (a) The focusing element (the magnetic field of the quadrupoles) does not alter the number of particles in the beam. In the context of this problem, Liouville's theorem then tells us that the phase space volume V occupied by the beam is not changed. In this problem, two dimensions of the general 6d phase space are irrelevant: the spatial and momentum dimensions along the beam direction. Therefore, we can denote the phase space volume of the beam by

$$V \propto \pi r^2 \pi p^2 = k r^2 p^2, \tag{15}$$

where r and p are the radii of the beam in the transverse configuration and momentum directions, respectively, and k is a constant. Thus, by Liouville's theorem,

$$p_2 = p_1 \frac{r_1}{r_2}, \tag{16}$$

i.e., squeezing of the cross section of the beam into a narrower span of configuration space necessarily makes the cross section of the beam in momentum space swell up.

(b) As a result of focusing down to a radius r_2, the beam will diverge at an angle α such that

$$\tan\alpha = \frac{p_2}{P} = \frac{p_1}{P}\frac{r_1}{r_2}. \tag{17}$$

At a distance d downstream, the maximum spatial radius of the beam will grow to

$$r(d) = r_2 + d\tan\alpha. \tag{18}$$

We are asked to find $d_{\max}$ such that $r(d_{\max}) = R$. Thus,

$$d_{\max} = (R - r_2)\frac{P}{p_1}\frac{r_2}{r_1}. \tag{19}$$

(c) If loss of particles is not a consideration, the spatial cross section of a particle beam can be made smaller by passing it through a collimator, without any obvious effect on the properties of the beam in momentum space. In terms of the notation above, such collimation obviously leads to $V_2 < V_1$. Liouville's theorem is not violated, however, since the number of particles also decreases in a way so as to leave the phase space density D unaltered.

P5.4 $[3 + 3 + 4 = 10$ points$]$

For a three-particle system with masses m_i, coordinates $\vec{r}_i$, and canonical momenta $\vec{p}_i$ $(i = 1, 2, 3)$, we introdce the following (Jacobian) coordinates:

$$\begin{aligned}
\vec{\rho}_1 &\equiv \vec{r}_2 - \vec{r}_1 \quad \text{(the coordinate of particle 2 relative to 1)}, \\
\vec{\rho}_2 &\equiv \vec{r}_3 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \quad \text{(the coordinate of particle 3 relative to the c.m. of 1 and 2)}, \\
\vec{\rho}_3 &\equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3} \quad \text{(the c.m. of the three particles)}, \\
\vec{\pi}_1 &\equiv \frac{m_1\vec{p}_2 - m_2\vec{p}_1}{m_1 + m_2}, \\
\vec{\pi}_2 &\equiv \frac{(m_1 + m_2)\vec{p}_3 - m_3(\vec{p}_1 + \vec{p}_2)}{m_1 + m_2 + m_3}, \\
\vec{\pi}_3 &\equiv \vec{p}_1 + \vec{p}_2 + \vec{p}_3.
\end{aligned} \tag{20}$$

Assume that the canonical momenta $\vec{p}_i$ are the same as the kinetic momenta (i.e., $\vec{p}_i = m_i\dot{\vec{r}}_i$).

(a) What are the physical interpretatiton of the momenta $\vec{\pi}_i$?

(b) How could we define such (Jacobian) coordinates and momenta for any arbitrary number of particles?

(c) Show that the transformation

$$\{\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{p}_1, \vec{p}_2, \vec{p}_3\} \to \{\vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_3, \vec{\pi}_1, \vec{\pi}_2, \vec{\pi}_3\} \tag{21}$$

is canonical.

Hint: Use reduced masses.

<u>S5.4</u> (a) The reduced mass of the two-body system of the mass points 1 and 2 is

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2}. \tag{22}$$

The reduced mass of the mass point 3 and the center-of-mass of the mass points 1 and 2 is

$$\mu_2 = \frac{(m_1 + m_2)m_3}{m_1 + m_2 + m_3}. \tag{23}$$

Then, we have

$$\vec{\pi}_1 = \mu_1\dot{\vec{\rho}}_1; \qquad \vec{\pi}_2 = \mu_2\dot{\vec{\rho}}_2. \tag{24}$$

This explains the meaning of these two momenta.

(b) Let us defifne

$$M_j \equiv \sum_{i=1}^{j} m_i, \tag{25}$$

so M_j is the total mass of the first j particles considered. Then we can write

$$\begin{aligned}
\vec{\rho}_j &= \vec{r}_{j+1} - \frac{1}{M_j}\sum_{i=1}^{j} m_i \vec{r}_i, \qquad (j = 1, 2, \ldots, N-1), \\
\vec{\rho}_N &= \frac{1}{M_N}\sum_{i=1}^{N} m_i \vec{r}_i, \\
\vec{\pi}_j &= \frac{1}{M_{j+1}}\sum_{i=1}^{j}\left(M_j \vec{p}_{j+1} - m_{j+1}\sum_{i=1}^{j}\vec{p}_i\right), \qquad (j = 1, 2, \ldots, N-1), \\
\vec{\pi}_N &= \sum_{i=1}^{N}\vec{p}_i.
\end{aligned} \tag{26}$$

(c) From the Poisson bracket relation $\{q_{i,a}, p_{j,b}\} = \delta_{ij}\delta_{ab}$, were a, b denote the particle index and i, j the cartesian coordinate indices, it follows that $\{\rho_{i,a}, \pi_{j,b}\} = \delta_{ij}\delta_{ab}$. For instance,

$$\begin{aligned}
\{\rho_{1,1}, \pi_{1,1}\} &= \left(\frac{m_1}{m_1+m_2} + \frac{m_2}{m_1+m_2}\right) = 1, \\
\{\rho_{2,1}, \pi_{1,1}\} &= \left(\frac{m_3}{m_1+m_2+m_3} - \frac{m_3}{m_1+m_2+m_3}\right) = 0,
\end{aligned} \tag{27}$$

etc.