

Introduction to High Energy Physics
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Abstract

This is a first draft of what I hope will someday evolve into reasonable classroom material for an introductory course on particle physics. As of the latest revision, the writing is progressing with my second teaching of the course. The contents are still crude and incomplete. Comments/suggestions to dhiman@fnal.gov are welcome, but please do not share these notes with anyone without my explicit permission.

Contents

1	Special Relativity	3
1.1	The geometry of space-time	3
1.2	Relativistic kinematics	10
1.3	Maxwell's equations and the electromagnetic field	13
2	Symmetries, Groups, and Conservation Laws	15
2.1	Groups and Representations	15
2.2	The Group $SU(2)$	20
2.3	The Group $SU(3)$	22
2.4	Parity transformation	27
2.5	Charge Conjugation	28
2.6	Lagrangian Density, Field Equations, and Conserved Currents	29
3	Quantum Electrodynamics	34
3.1	The Dirac Equation	34
3.2	The γ matrices and trace theorems	36
3.3	Lagrangian and symmetries for a spin- $\frac{1}{2}$ field	37
3.4	Explicit plane-wave solutions	38
3.5	Bilinear Covariants	40
3.6	Massless Fermions	40
4	Decay Widths and Scattering Cross Sections	42
4.1	Physical meaning of decay width	43
4.2	Physical meaning of scattering cross section	44
4.3	Calculation of widths and cross sections	45
4.3.1	The Golden Rule for Decays	45
4.3.2	The Golden Rule for Scattering	46
4.4	Feynman rules for calculating the amplitude	46
5	Electromagnetic Scattering	48
5.1	Electron-Muon Scattering	49
5.2	e^+e^- annihilation to $\mu^+\mu^-$	53

6	Gauge Theories in Particle Physics	55
6.1	Gauge Invariance in Quantum Theory	55
6.2	Strong Isospin: an example of $SU(2)$	57
6.3	Non-Abelian Gauge Theories	59
6.4	Gauge Theories for Quarks and Leptons	60
7	The Electroweak theory and Quantum Chromodynamics	63
7.1	The Quark and Lepton states	64
7.2	The quark and lepton Lagrangian	65
7.2.1	The $U(1)$ terms	66
7.2.2	The $SU(2)$ terms	66
7.3	The Electroweak Neutral Current	67
7.4	The Charged Current	70
7.5	The Quarks and the QCD Lagrangian	71
7.6	The Fermion-Gauge Boson Lagrangian	72
8	Masses and the Higgs Mechanism	74
8.1	Spontaneous Symmetry Breaking	75
8.2	Spontaneous breaking of a continuous symmetry: the Goldstone Theorem	77
8.3	Complex scalar field - a Global Symmetry	78
8.4	The Abelian Higgs Mechanism	79
8.5	The Higgs Mechanism in the Standard Model	81
8.6	Fermion Masses	85
8.7	Comment on Vacuum Energy	87
9	Quark (and Lepton) Mixing	88
9.1	The Cabibbo Mixing of d and s Quarks	88
9.2	The c quark and the GIM Mechanism	89
9.3	Kobayashi-Maskawa Mixing in the Six-Quark Model	92
10	Interaction of Particles with Matter	94
10.1	Electromagnetic Interaction of Particles with Matter	95
10.1.1	Energy Loss by Ionization	95
10.1.2	Fluctuations in Ionization Energy Loss	97

Chapter 1

Special Relativity

In the far-reaching theory of *Special Relativity* of Einstein, the homogeneity and isotropy of the 3-dimensional space are generalized to include the time dimension as well. The space-time structure embodied in the theory provides the foundation on which all branches of modern physics are formulated.¹

1.1 The geometry of space-time

The basic tenet of the theory of relativity is that there is a fundamental symmetry between the three space dimensions and the time dimension, as manifested most directly in the constancy of the velocity of light in all coordinate frames. In order to formulate this theory mathematically, it is useful to introduce a set of convenient definitions and notations.

Definition 1.1 (Event): An *event*, characterised by the spatial coordinates $\{x^i; i = 1, 2, 3\}$ and the time t , will be denoted by $\{x^\mu; \mu = 0, 1, 2, 3\}$ where

$$x^{\mu=0} = ct, \quad x^{\mu=i} = x^i, \quad (1.1)$$

and c is the velocity of light in vacuum.² In particle physics, the natural units are chosen, whereby $c = 1$ (by definition). The convention is that Greek indices refer to space-time in general (hence range over 0 to 3), and Roman indices refer to 3-space only (hence range over 1 to 3). We shall use the notation \mathbf{x} to indicate a 3-vector.

Definition 1.2 (Coordinate Four-vector, Length of Vectors): Let x_1^μ and x_2^μ represent two events. The separation between the two events defines a *coordinate four-vector* $x_1^\mu = x_1^\mu - x_2^\mu$. The *length* $|x|$ of a 4-vector x is defined by

$$|x|^2 \equiv (x^0)^2 - (\mathbf{x})^2 = t^2 - \mathbf{x} \cdot \mathbf{x}. \quad (1.2)$$

¹Only in the theory of gravitation is it necessary to generalize the concepts of space-time beyond the special theory of relativity. The resulting theory of General Relativity is intimately related to the group of general coordinate transformations. We shall not venture into that theory in this course.

²In this notation, all four coordinates carry the same scale dimension - the dimension of length.

The coordinates x^μ of an event can be considered as a 4-vector if we understand it to mean the difference between the event and the event represented by the origin $(0, \mathbf{0})$. In this notation, the wave-front of a light signal sent out from the space origin at $t = 0$ will satisfy the simple equation, $|x| = 0$.

In terms of the *metric tensor* $g^{\mu\nu}$, the definition of the length of a vector x can be written as

$$|x|^2 = g_{\mu\nu}x^\mu x^\nu, \quad (1.3)$$

where the implicit summations extend over all 4 components.³

The metric tensor for the space-time vector space is called the *Minkowski metric*:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (1.4)$$

In contrast to the Euclidean metric $\delta_{\mu\nu}$, the Minkowski metric is not positive definite. This fact leads to important differences in the representations of the associated symmetry groups. The principle of special relativity stipulates that basic laws of physics are invariant with respect to translations in all 4 coordinates (homogeneity of space-time) and to all homogeneous linear transformations on the space-time coordinates which leave the length of 4-vectors invariant (isotropy of space-time).

Definition 1.3 (Homogeneous Lorentz Transformation): *Homogeneous Lorentz Transformations* are continuous linear transformations Λ on coordinate components given by

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad (1.5)$$

which preserve the length of 4-vectors, i.e.

$$|x|^2 = |x'|^2 \quad (1.6)$$

Combining Eqs. 1.3-1.6, one can formulate the condition on Lorentz transformations Λ without referring to any specific 4-vector as either

$$g_{\mu\nu}\Lambda^\mu{}_\lambda\Lambda^\nu{}_\sigma = g_{\lambda\sigma} \quad (1.7)$$

or

$$\Lambda^\mu{}_\lambda\Lambda^\nu{}_\sigma g^{\lambda\sigma} = g^{\mu\nu}, \quad (1.8)$$

where $g^{\mu\nu} = g_{\mu\nu}$. This result is an apparent generalization of rotation in 3-dimensional Euclidean space. Suppressing the indices in Eq. 1.7, we can write it (in the matrix form) as

$$\Lambda^{-1} = g\Lambda^T g^{-1}, \quad (1.9)$$

³This is known as the Einstein summation convention, whereby whenever an index appears twice in a product, once as a superscript, once as a subscript, the term is summed over all allowed values of that index. Since such an index does not represent any particular value, it is often called a “dummy” index or variable.

which is to be compared with $R^{-1} = R^T$ for rotation in the 3-dimensional Euclidean space.

Taking the determinant on both sides of Eq. 1.9, we obtain $(\det(\Lambda))^2 = 1$, hence $\det(\Lambda) = \pm 1$.

An example of a “large” Lorentz transformation with $\det(\Lambda) = -1$ is

$$\Lambda^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1.10)$$

which just flips the sign of the time coordinate, and is therefore known as time reversal:

$$t' = -t, \quad x' = x, \quad y' = y, \quad z' = z. \quad (1.11)$$

Another “large” Lorentz transformation is parity, or space, inversion:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (1.12)$$

so that

$$t' = t, \quad x' = -x, \quad y' = -y, \quad z' = -z. \quad (1.13)$$

It was once thought that the laws of physics have to be invariant under these transformations, until it was shown experimentally in the 1950’s that parity is violated in the weak interactions, specifically in the weak decays of the ^{60}Co nucleus and of the K^\pm mesons. Likewise, experiments in the 1960’s on the decays of K^0 mesons showed that time-reversal is violated (at least if very general properties of quantum mechanics and special relativity are assumed).

However, all experiments up to now are consistent with invariance of the laws of physics under Lorentz transformations that are continuously connected to the identity transformation. Such transformations are known as “proper” Lorentz transformations. So, for these, we must have

$$\det(\Lambda) = \Lambda^0{}_\mu \Lambda^1{}_\nu \Lambda^2{}_\lambda \Lambda^3{}_\sigma \varepsilon^{\mu\nu\lambda\sigma} = 1, \quad (1.14)$$

where $\varepsilon^{\mu\nu\lambda\sigma}$ is the 4-dimensional totally antisymmetric unit tensor with $\varepsilon^{0123} = 1$ (the Levi-Civita tensor).⁴ This condition can be rewritten as

$$\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \Lambda^\gamma{}_\lambda \Lambda^\delta{}_\sigma \varepsilon^{\mu\nu\lambda\sigma} = \varepsilon^{\alpha\beta\gamma\delta}, \quad (1.15)$$

⁴That is,

$$\varepsilon^{\mu\nu\lambda\sigma} = \begin{cases} +1 & \text{if } \mu\nu\lambda\sigma \text{ is an even permutation of } 0123, \\ -1 & \text{if } \mu\nu\lambda\sigma \text{ is an odd permutation of } 0123, \\ 0 & \text{otherwise.} \end{cases}$$

We also note that, setting $\lambda = \sigma = 0$ in Eq. 1.7, we obtain the condition

$$(\Lambda^0_0)^2 - \sum_i (\Lambda^i_0)^2 = 1. \quad (1.16)$$

This implies $(\Lambda^0_0)^2 \geq 1$, hence $\Lambda^0_0 \geq 1$ or $\Lambda^0_0 \leq -1$. The two solutions represent disjoint regions of the real axis for Λ^0_0 . Since $\Lambda^0_0 = 1$ for the identity transformation, continuity requires that all proper Lorentz transformations have

$$\Lambda^0_0 \geq 1. \quad (1.17)$$

Obviously, the other branch is associated with time reversal. To summarize, homogeneous proper Lorentz transformations are linear transformations of 4×4 matrices with $\Lambda^0_0 \geq 1$ that leave two special tensors, $g^{\mu\nu}$ and $\varepsilon^{\mu\nu\lambda\sigma}$ invariant.

A general homogeneous proper Lorentz transformation depends on 6 real parameters. This can be seen as follows: the 4×4 real matrix Λ has 16 elements, that are subject to 10 independent constraints represented by Eq. 1.7.

Rotations in the 3 spatial dimensions are examples of Lorentz transformations in this generalized sense. They are of the form

$$R^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & R^i{}_j & & \\ 0 & & & \end{pmatrix}, \quad (1.18)$$

where $R^i{}_j$ denotes ordinary 3×3 rotation matrices. For example, the counterclockwise rotation by an angle α in the x, y plane is represented by

$$R^i{}_j = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (1.19)$$

whence we have

$$\begin{aligned} t' &= t, \\ x' &= x \cos \alpha + y \sin \alpha, \\ y' &= -x \sin \alpha + y \cos \alpha, \\ z' &= z. \end{aligned} \quad (1.20)$$

Of greater interest to us are *special Lorentz transformations* which mix spatial coordinates with the time coordinate. The simplest of these is a *Lorentz boost* along a given coordinate axis, say the x -axis:

$$L_1^\mu{}_\nu = \begin{pmatrix} \cosh \eta & -\sinh \eta & 0 & 0 \\ -\sinh \eta & \cosh \eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1.21)$$

or,

$$\begin{aligned}
t' &= t \cosh \eta - x \sinh \eta, \\
x' &= -t \sinh \eta + x \cosh \eta, \\
y' &= y, \\
z' &= z.
\end{aligned}
\tag{1.22}$$

Physically, this corresponds to the transformation of a position vector from the unprimed frame to the primed frame, the latter moving with respect to the former along the x direction at the speed $\beta = \tanh \eta$.⁵ By defining

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}},
\tag{1.23}$$

we have

$$\sinh \eta = \beta \gamma, \quad \cosh \eta = \gamma.
\tag{1.24}$$

Thus,

$$\begin{aligned}
t' &= \gamma(t - \beta x), \\
x' &= \gamma(-\beta t + x), \\
y' &= y, \\
z' &= z.
\end{aligned}
\tag{1.25}$$

So, a Lorentz boost along the x -axis by the velocity β can be interpreted as a “rotation” in the t, x plane by the hyperbolic angle $\eta = \tanh^{-1}(\beta)$, called *rapidity*.⁶

A general Lorentz transformation can be written as the product of spatial rotations and Lorentz boosts.⁷

Definition 1.4 (Minkowski Space): The 4-dimensional space-time endowed with the Minkowski metric, Eq. 1.4, is called the *Minkowski space*. Any 4-component object a^μ , transforming under Lorentz transformations as the coordinate vector in Eq. 1.5 is said to be a *four-vector* or a *Lorentz vector*.

Definition 1.5 (Scalar Product): The *scalar product* of two 4-vectors a^μ and b^μ is defined as

$$a \cdot b \equiv g_{\mu\nu} a^\mu b^\nu = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}.
\tag{1.26}$$

Definition 1.6 (Covariant and Contravariant Components): By convention, the ordinary components of a Lorentz vector $\{a^\mu\}$ are referred to as the *contravariant components*. An alternative way to represent the same vector is by

⁵Of course, the two frames must coincide for $\beta = 0$.

⁶Note: $0 \leq |\beta| \leq 1$, $1 \leq \gamma$.

⁷The set of all proper Lorentz transformations $\{\Lambda\}$ satisfying the conditions of Eqs. 1.7, 1.15 and 1.17 forms the *Proper Lorentz Group* or, in short, the *Lorentz Group*. The group consists of all *special “orthogonal”* 4×4 matrices - the quotation marks here call attention to the non-Euclidean signature of the invariant metric $g^{\mu\nu}$, $(1, -1, -1, -1)$. Thus, Λ -matrices for Lorentz boosts are not unitary like the rotation matrices.

its *covariant components* $\{a_\mu\}$ defined as

$$a_\mu \equiv g_{\mu\nu} a^\nu. \quad (1.27)$$

So, $a_0 = a^0$, and $a_i = -a^i$, $i = 1, 2, 3$. With these definitions, we can simplify the definition of the scalar product, Eq. 1.26 to

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu. \quad (1.28)$$

The covariant components of a 4-vector a transform under proper Lorentz transformations as

$$a_\mu \rightarrow a'_\mu = a_\nu (\Lambda^{-1})^\nu{}_\mu. \quad (1.29)$$

This result displays the transformation property of a_μ in the form which most explicitly indicates why $a^\mu a_\mu$ is an invariant. There is a natural covariant 4-vector, the 4-gradient ∂_μ defined as

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla} \right). \quad (1.30)$$

We can verify that

$$\partial_\mu \rightarrow \partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = (\Lambda^{-1})^\nu{}_\mu \frac{\partial}{\partial x^\nu}. \quad (1.31)$$

With respect to an arbitrarily chosen coordinate origin, space-time is divided into three distinct regions separated by the *light-cone* which is defined by the equation

$$\tau^2 \equiv x^\mu x_\mu = t^2 - \mathbf{x} \cdot \mathbf{x} = 0. \quad (1.32)$$

The *future* consists of all points with $\tau^2 > 0$ and $x^0 > 0$. These points can be reached by the “world line” of an event at the origin. The *past* consists of all points with $\tau^2 > 0$ and $x^0 < 0$. Events at any of these points can, in principle, evolve through the origin. By a suitable Lorentz transformation, the coordinates of any point in these two regions can be transformed into the form $(t', \mathbf{0})$; hence these coordinate vectors are said to be *time-like*. The region outside the light-cone are characterized by $\tau^2 < 0$. For any given point in this region, there exists some Lorentz transformation which transforms the components of the coordinate vector into the form $(0, \mathbf{x}')$. Hence these coordinate vectors are said to be *space-like* and the entire region is called the space-like region (with respect to the origin). No world-line from the space-like region can evolve through the origin and vice versa. When $\tau^2 = 0$, the point is said to be *light-like* since only world lines of a light signal (a photon) connect such points to the origin. By analogy to the coordinates, an arbitrary 4-vector is said to be time-like, space-like, or light-like depending on whether $a^\mu a_\mu$ is less than, greater than, or equal to 0.

For two events occurring at x^μ and $x^\mu + d^\mu$, the scalar product of their 4-vector coordinate difference with itself,

$$(d\tau)^2 \equiv d^\mu d_\mu = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \quad (1.33)$$

is called the *proper interval (squared)* between the two events. The proper interval is, of course, independent of the choice of Lorentz frame. If two events A and B have $(d\tau)^2 < 0$, then the separation between them is space-like (i.e. they are not within each other's light cone). Their time-ordering is frame-dependent and, therefore, they cannot be causally connected. Note, however, that both can still be within the light cone of, and therefore, causally connected to, a third event C if C is either in the absolute past, or in the absolute future, of both A and B .

For two frames with a relative Lorentz boost $\vec{\beta}$, Eq. 1.33 gives us the proper interval between the space origins at time dt . For convenience, let us choose the 4-coordinate origins to coincide, with x and x' axes oriented along the boost, so that the 3-velocity of the space origin of the frame moving along the positive x -axis of the other is $\frac{d\mathbf{x}}{dt} = (\beta, 0, 0)$. Then, dividing both sides of Eq. 1.33 by $(dt)^2$, we get

$$\frac{d\tau}{dt} = \sqrt{1 - \beta^2} = \frac{1}{\gamma}, \quad (1.34)$$

or,

$$dt = \gamma d\tau. \quad (1.35)$$

This result is sometimes referred to as *time dilation*. It is as a consequence of this, that high-energy muons (\Rightarrow traveling near the speed of light, as we shall soon see) created in collisions between high-energy cosmic ray particles (primarily protons) and atomic nuclei in Earth's upper atmosphere frequently traverse distances of $O(10 \text{ km})$ to reach the surface of earth even though the proper interval between the creation and decay of a muon is typically less than $2 \mu\text{s}$, during which even light can travel no more than 600 m. This is possible because, while a muon may live only $\sim 2 \mu\text{s}$ in its own rest frame, to an observer on earth, the time interval between the production and decay of a cosmic-ray muon appears much longer owing to the very high speed at which the particle is moving.⁸ This poses no contradiction to the observer in the muon's rest frame either. The $\sim 2 \mu\text{s}$ that he sees the muon before it decays is enough for it to travel to Earth because the thickness of the Earth's atmosphere, which is measured at, say 30 km, by a terrestrial observer, appears much less to him:

$$dx' = \frac{1}{\gamma} dx \quad (1.36)$$

This effect is sometimes referred to as *length contraction*.

In Newtonian mechanics, t is an external (and universal) parameter. Therefore, $\beta \equiv \mathbf{v} \equiv \frac{d\mathbf{x}}{dt}$ is a 3-vector, i.e. it transforms like \mathbf{x} in the 3-dimensional Euclidean space. Not so in the 4-dimensional Minkowski space, where t itself is

⁸The proper interval between the production and decay of an unstable particle X follows an exponential distribution: $N_X(t) = N_X(0)e^{-\frac{t}{\tau_X}}$, where $N_X(t)$ is the number of particles at time t . τ_X is a property, called the *lifetime*, of the particle. $\tau_\mu \approx 2.2 \mu\text{s}$.

a coordinate. Therefore, $v^\mu \equiv \frac{dx^\mu}{dt} = \frac{d^\mu}{d^0}$ is not a Lorentz 4-vector. However,

$$u^\mu \equiv \frac{d^\mu}{d\tau} = \gamma \frac{d^\mu}{dt} = \gamma v^\mu \quad (1.37)$$

is a Lorentz 4-vector, and is called the *relativistic velocity*. It can be easily shown that $u^\mu u_\mu = 1$.

Since $v^0 = 1$, $u^0 = \gamma$. Thus, expressed in terms of the relativistic 4-velocity u^μ (with $u^2 = u^3 = 0$), instead of the non-relativistic 3-velocity $\vec{\beta}$, the Lorentz transformation of Eq. 1.25 reduces to a simpler, more intuitive form:

$$\begin{aligned} t' &= u^0 t - u^1 x, \\ x' &= -u^1 t + u^0 x, \\ y' &= y, \\ z' &= z, \end{aligned} \quad (1.38)$$

which is reminiscent of the Galilean transformation, except for the fact that space and time coordinates now mix in a symmetric manner.

1.2 Relativistic kinematics

The momentum 4-vector of a particle is

$$p^\mu = (E, \mathbf{p}), \quad (1.39)$$

where E is the energy of the particle, and \mathbf{p} its Euclidean 3-momentum.

In free space (i.e. in the absence of any external field interacting with the particle), the momentum is all-kinetic.⁹ For a free particle with a non-zero mass m , the momentum is simply the product of its mass and velocity, just as in Newtonian mechanics:

$$p^\mu = m u^\mu \quad (1.40)$$

Thus,

$$p^\mu p_\mu = m^2. \quad (1.41)$$

Equation 1.41 holds for massless particles, such as photons, as well. Of course, for a given p^μ , $u^\mu \rightarrow \infty$ as $m \rightarrow 0$. For $m = 0$, u^0 and at least one u^i are undefined, but p^μ are finite. For a particle at rest, $u^0 = 1$, and we get $E = m(c^2 = 1)$, the equation that most people readily associate with Einstein.

In a closed system of particles, homogeneity of space-time ensures that the 4-momentum is conserved in any interaction (scattering, decay, annihilation):

$$\sum_i p^\mu = \sum_f p^\mu, \quad (1.42)$$

⁹When the total momentum is different from the kinetic momentum, the latter is sometimes denoted by π^μ .

where the summation on the LHS runs over all particles in the initial state (before an interaction) and the summation on the RHS runs over all particles in the final state (after an interaction). Equations 1.42 are by far the most important (and often the only) ingredient in the study of particle interactions, especially from the experimental perspective.

Consider a process $ab \rightarrow cd$. From the 4-momenta of the 4 particles, we can form 10 Lorentz-invariant scalar products: $p_a^\mu p_a^\mu, p_a^\mu p_b^\mu, p_a^\mu p_c^\mu$, etc. However, these are subject to the following 8 constraints: first, conservation of 4-momentum results in the 4 equations

$$p_a^\mu + p_b^\mu = p_c^\mu + p_d^\mu, \quad (1.43)$$

and second, $p_i^\mu p_i^\mu = m_i^2$ for $i = a, b, c, d$. Thus, there must be two independent variables that describe the process. In non-relativistic mechanics, they are usually chosen to be the energy and the scattering angle. In particle physics, frame-independent quantities prove to be more convenient. Following the above arguments it is natural to define the following Lorentz scalars that are quadratic in the momenta:

$$s = (p_a + p_b)^2 = (p_c + p_d)^2, \quad (1.44)$$

$$t = (p_c - p_a)^2 = (p_d - p_b)^2, \quad (1.45)$$

$$u = (p_d - p_a)^2 = (p_c - p_b)^2. \quad (1.46)$$

These are called *Mandelstam variables*.

Clearly, the Mandelstam variables are invariant under time reversal. Also, “crossing” of processes merely results in the interchange of the Mandelstam variables. It is customary to denote the main physical process, i.e. $ab \rightarrow cd$ in this case, as the s channel since \sqrt{s} is the total CM energy. In the cross process $a\bar{c} \rightarrow \bar{b}d$, the CM energy would be what we now have as t , hence it is called the t channel. Similarly, $a\bar{c} \rightarrow \bar{b}d$ would be the u channel.

Any two of the three Mandelstam variables completely determines the third. A little algebra leads to the relation

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \quad (1.47)$$

Although only two are independent, we define all three for the sake of symmetry. Since s, t, u are Lorentz scalars, they can be evaluated in any frame and used without change in any other. We shall deal mostly with symmetric colliders where the laboratory center-of-mass (CM) frame is stationary in the laboratory. By convention, we choose a coordinate system such that one of the particles is moving along the z -axis.¹⁰ Thus, we have for the colliding particles

$$p_a^\mu = (E_a, 0, 0, p) \quad (1.48)$$

$$p_b^\mu = (E_b, 0, 0, -p) \quad (1.49)$$

¹⁰It makes good practical sense for this to lie in a horizontal plane, resulting in the choice of the perpendicular horizontal direction for x and the vertical as y to form a right-handed rectangular coordinate system for a particle detector.

and for the particles emerging out of the collision,

$$p_c^\mu = (E_c, \mathbf{p}') \quad (1.50)$$

$$p_d^\mu = (E_d, -\mathbf{p}') \quad (1.51)$$

Let us take the liberty to rotate our frame about the z -axis so that \mathbf{p}' lies in the zx plane:

$$\mathbf{p} = (p' \sin \theta, 0, p' \cos \theta), \quad (1.52)$$

where θ is the scattering angle in the usual 3-dimensional sense.

Then, in the CM system

$$s = (E_a + E_b)^2 = \left(\sqrt{m_a^2 + p^2} + \sqrt{m_b^2 + p^2} \right)^2, \quad (1.53)$$

which can be solved for p ,

$$p^2 = \frac{(s - (m_a + m_b)^2)(s - (m_a - m_b)^2)}{4s}. \quad (1.54)$$

A little more algebra gives

$$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} \quad (1.55)$$

And similarly for E_b, E_c, E_d .

For the scattering angle, we have

$$\begin{aligned} t &= m_c^2 - m_a^2 - 2p_c^\mu p_a^\mu \\ &= m_c^2 - m_a^2 - 2E_c E_a - 2\mathbf{p}' \cdot \mathbf{p} \\ &= m_c^2 - m_a^2 - 2E_c E_a - 2p' p \cos \theta \end{aligned} \quad (1.56)$$

and

$$u = m_d^2 - m_a^2 - 2E_d E_a - 2p' p \cos \theta \quad (1.57)$$

For a process where the masses are negligible compared to the energies involved (as is most often the case in high-energy particle physics),

$$E_a = E_b = p = E_c = E_d = p' = \frac{\sqrt{s}}{2} \quad (1.58)$$

$$t = -\frac{s}{2}(1 - \cos \theta) \quad (1.59)$$

$$u = -\frac{s}{2}(1 + \cos \theta) \quad (1.60)$$

Like velocity, ordinary force, defined as the time derivative of momentum, is not a 4-vector. But the quantity

$$f^\mu \equiv \frac{dp^\mu}{d\tau} \quad (1.61)$$

is the force 4-vector in the Minkowski space.

Define 4 four-vectors which in a particular frame are given by the infinitesimal differentials

$$\begin{aligned}
 A^\mu &= (dt, 0, 0, 0), \\
 B^\mu &= (0, dx, 0, 0), \\
 C^\mu &= (0, 0, dy, 0), \\
 D^\mu &= (0, 0, 0, dz).
 \end{aligned}
 \tag{1.62}$$

Then the 4-dimensional volume element

$$d^4x \equiv dx^0 dx^1 dx^2 dx^3 = A^\mu B^\nu C^\lambda D^\sigma \varepsilon_{\mu\nu\lambda\sigma} \tag{1.63}$$

is Lorentz invariant (since the RHS has no uncontracted 4-vector index). It follows that if $F(x)$ is a Lorentz scalar function of x^μ , then the integral

$$I[F] = \int d^4x F(x) \tag{1.64}$$

is invariant under Lorentz transformations. Lagrangians in particle physics theories are defined in terms of such *action* integrals.

1.3 Maxwell's equations and the electromagnetic field

To formulate and solve Maxwell's equations in the framework of special relativity, we need to introduce two new 4-vectors: the charge-current density $J^\mu = (\rho, \mathbf{J})$, and the electromagnetic potential $A^\mu = (\phi, \mathbf{A})$.

In the non-relativistic framework, Maxwell's equations are written as

$$\begin{aligned}
 \vec{\nabla} \cdot \mathbf{E} &= \rho, \\
 \vec{\nabla} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \mathbf{J}, \\
 \vec{\nabla} \cdot \mathbf{B} &= 0, \\
 \vec{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0.
 \end{aligned}
 \tag{1.65}$$

Adding the $\frac{\partial}{\partial t}$ of the first of the above equations to the divergence of the second, we get (since the divergence of a curl vanishes identically) the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \mathbf{J} = 0, \tag{1.66}$$

which guarantees the local conservation of electric charge. The solutions to Maxwell's equations in terms of the scalar and vector potentials are

$$\mathbf{E} = -\vec{\nabla}V - \frac{\partial \mathbf{A}}{\partial t}, \quad (1.67)$$

$$\mathbf{B} = \vec{\nabla} \times \mathbf{A}. \quad (1.68)$$

In 4-vector notation, the continuity equation Eq. 1.66 readily simplifies to

$$\partial_\mu J^\mu = 0. \quad (1.69)$$

The electric and magnetic fields can be written as components of the antisymmetric *electromagnetic field tensor*

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \quad (1.70)$$

So, unlike in the 3-dimensional Euclidean space, the electric and magnetic fields are not vectors, but components of a tensor:

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = -E_i, \quad (1.71)$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = \varepsilon_{ijk} B^k, \quad (1.72)$$

where ε_{ijk} is the Levi-Civita tensor in 3-dimensions.

Now the first two of Maxwell's equations become the relativistic wave equation

$$\partial_\mu F^{\mu\nu} = J^\nu, \quad (1.73)$$

or, equivalently,

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = J^\nu, \quad (1.74)$$

while the last two follow from the identity

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0. \quad (1.75)$$

The continuity equation Eq. 1.69 follows directly from Eq. 1.73: since the partial derivatives commute and $F^{\mu\nu}$ is antisymmetric, $\partial_\mu J^\mu = \partial_\mu \partial_\nu F^{\mu\nu} = 0$.

Note that $F^{\mu\nu}$ is explicitly invariant under a transformation

$$A^\mu \rightarrow A^\mu + \partial^\mu \lambda(x), \quad (1.76)$$

where $\lambda(x)$ is any scalar function of space-time. Since only the electromagnetic fields are physically manifested, any two 4-potentials that differ only by the 4-gradient of an arbitrary scalar function of space-time are equally valid for describing a physical process. This is a particular example of a general class of symmetry, called "gauge symmetry", that play a central role in the formulation of particle interactions within the standard model (electromagnetic, weak and strong interactions) and beyond.

Chapter 2

Symmetries, Groups, and Conservation Laws

The dynamical properties and interactions of a system of particles and fields are derived from the *principle of least action*, where the *action* is a 4-dimensional Lorentz-invariant integral of the corresponding Lagrangian density. The general theorem called *Noether's theorem* dictates that to every symmetry of the Lagrangian there is a conserved current. It is a key ingredient in the construction of theories in particle physics. Symmetries appear in many ways in the studies of particle interactions: gauged (local) and global symmetries, exact and approximate symmetries, explicitly realized and spontaneously broken symmetries. The branch of mathematics devoted to the study of symmetries is called *Group theory*. It will be useful to familiarize ourselves with some basic concepts of group theory.

2.1 Groups and Representations

Definitions A group is a set G on which a law of composition “ \cdot ” is defined with the following properties:

1. *Closure*: if x_1 and x_2 are in G , so is $x_1 \cdot x_2$;
2. *Identity*: there is an identity element e in G such that $x \cdot e = e \cdot x = x$ for any x in G ;
3. *Inverse*: for every x in G , there is an inverse element x^{-1} in G such that $x \cdot x^{-1} = x^{-1} \cdot x = e$;
4. *Associativity*: for every x_1, x_2 , and x_3 in G , $(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$.

A group is said to be commutative or *Abelian* if $x_1 \cdot x_2 = x_2 \cdot x_1$ for all x_1, x_2 in G . Otherwise, it is *non-Abelian*.

A group may have a finite or infinite number of elements. For example, the set of all real numbers form a continuous Abelian group with an infinite number of elements under the composition law of arithmetic addition. The set of all possible permutations of 3 labelled objects is an example of a discrete non-Abelian group with a finite number of elements:

$$\begin{aligned}
() &: (a, b, c) \rightarrow (a, b, c), \\
(12) &: (a, b, c) \rightarrow (b, a, c), \\
(23) &: (a, b, c) \rightarrow (a, c, b), \\
(31) &: (a, b, c) \rightarrow (c, b, a), \\
(123) &: (a, b, c) \rightarrow (c, a, b) \quad (\text{cyclic permutation}), \\
(321) &: (a, b, c) \rightarrow (b, c, a).
\end{aligned} \tag{2.1}$$

The permutation group is an example of a *transformation group* on a physical system. In quantum mechanics, a transformation of the system is associated with a unitary operator in the Hilbert space.¹ Thus, a transformation group of a quantum mechanical system is associated with a mapping of the group into a set of unitary operators. So, for each x in G there is a $D(x)$ which is a unitary (linear) operator. Furthermore, the mapping must preserve the composition law

$$D(x_1)D(x_2) = D(x_1 \cdot x_2) \tag{2.2}$$

for all x_1, x_2 in G . A mapping which satisfies Eq. 2.2 is called a *representation* of the group G .² For example, the mapping

$$D(x) = e^{-ipx}, \tag{2.3}$$

is a representation of the additive group of real numbers because

$$e^{-ipx_1}e^{-ipx_2} = e^{-ip(x_1+x_2)}. \tag{2.4}$$

The following mapping is a representation of the permutation group on 3 labelled objects:

$$\begin{aligned}
D() &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & D(12) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
D(23) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & D(31) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
D(123) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & D(321) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{2.5}$$

¹We will ignore the possibility of antiunitary operators, which are irrelevant in our context.

²Unitarity is not required in the definition of representation.

For example, the composition $(12) \cdot (23) = (123)$ is mapped into the matrix multiplication

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2.6)$$

Thus, in any representation of a group, the composition law is realized by multiplication of (finite- or infinite-dimensional) matrices that the group elements map into. Such a mapping is not necessarily one-to-one. When it is, we call it the *fundamental representation*.

Group theory makes it possible to determine many properties of any representation from the abstract properties of the group. It is convenient to view representations both as abstract linear operators and as matrices. The connection is as follows: let $|i\rangle$ be an orthonormal basis in the space on which $D(g)$ acts as a linear operator. then

$$D(g)_{ij} = \langle i | D(g) | j \rangle. \quad (2.7)$$

So,

$$D(g)|i\rangle = \sum_j |j\rangle \langle j | D(g) | i \rangle = \sum_j |j\rangle D(g)_{ji}. \quad (2.8)$$

Two representations are *equivalent* if they are related by a *similarity transformation*

$$D_2(x) = S D_1(x) S^{-1}, \quad (2.9)$$

with a fixed operator S for all x in G .

A representation is *reducible* if it is equivalent to a representation D' with block-diagonal form:

$$D'(x) = S D(x) S^{-1} = \begin{pmatrix} D'_1(x) & 0 \\ 0 & D'_2(x) \end{pmatrix}, \quad (2.10)$$

whence the vector space on which D' acts breaks up into two orthogonal subspaces, each of which is mapped into itself by all the operators in $D'(x)$. The representation D' is said to be the *direct sum* of D'_1 and D'_2 ,

$$D' = D'_1 \oplus D'_2. \quad (2.11)$$

A representation is *irreducible* if it is not reducible, that is if it cannot be put into a block-diagonal form by any similarity transformation. Any finite dimensional representation of a finite group is completely reducible into a direct sum of irreducible representations.

Group elements are rarely dealt with as abstract mathematical objects. Instead, a representation is used to obtain the composition table which is, in a sense, the group. For the groups of our interest (in the realm of quantum theories of particles and fields), all irreducible representations are equivalent to representations by unitary operators.

A *Lie group* is a group of unitary operators that are labeled by a set of continuous real parameters with a composition law that depends smoothly on the parameters. If the volume of the parameter space of a Lie group is finite, then it is called a *compact Lie group*. Any element of a compact Lie group can be obtained from the identity element by continuous changes in the parameters and can be expressed as $e^{i\alpha_a X_a}$, where α_a ($a = 1 \dots n$) are real parameters and X_a are linearly independent hermitian operators (a sum over the repeated index a is implied). The X_a are a basis of a vector space spanned by the linear combinations $\alpha_a X_a$, called the *generators* of the group. Any function of the generators that commutes with all generators of a Lie group is called a *Casimir operator* of that group.

Note that the space of the group generators is different from the space on which the generators act, which is some as yet unspecified Hilbert space. For the compact Lie groups, the space on which the generators act are finite dimensional, so the X_a can be expressed as finite hermitian matrices.

Generators have two nice features. First, since the generators form a vector space, unlike the group elements, they can be multiplied by numbers and added to obtain other generators. Second, they satisfy simple commutation relations which determine (almost) the full structure of the group. Consider the composition

$$e^{i\lambda X_b} e^{i\lambda X_a} e^{-i\lambda X_b} e^{-i\lambda X_a} = 1 + \lambda^2 [X_a, X_b] + \dots$$

Because of the properties of group composition, the result corresponds to another group element and can be written as $e^{i\beta_c X_c}$. As $\lambda \rightarrow 0$, we must have $\lambda^2 [X_a, X_b] \rightarrow i\beta_c X_c$. Writing $\beta_c = \lambda^2 f_{abc}$, we get

$$[X_a, X_b] = i f_{abc} X_c. \quad (2.12)$$

The constants f_{abc} are called the *structure constants* of the group. The structure constants reflect the group composition law. This can be seen as follows. It is always possible to define

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} \equiv e^{i\delta_c X_c}, \quad (2.13)$$

where δ_c is determined by α , β and f :

$$\delta_c = \alpha_c + \beta_c - \frac{1}{2} f_{abc} \alpha_a \beta_b + \dots \quad (2.14)$$

The generators also satisfy the *Jacobi identity*:

$$[X_a, [X_b, X_c]] + \text{cyclic permutations} = 0. \quad (2.15)$$

This is obvious for the representation, since then the X_a are just linear operators, but in fact it is true for the abstract group generators. In terms of the structure constants, the Jacobi identity becomes

$$f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0. \quad (2.16)$$

If we define a set of matrices T_a

$$(T_a)_{bc} \equiv -if_{abc}, \quad (2.17)$$

Then, after similar definitions for T_b and T_c , Eq. 2.16 can be rewritten as

$$[T_a, T_b] = if_{abc}T_c. \quad (2.18)$$

In other words, the structure constants themselves generate a representation of the algebra. The representation generated by the structure constants is called the *adjoint representation*. The *dimension* of a representation is the dimension of the vector space on which it acts. The dimension of the adjoint representation is just the number of generators, which is the number of real parameters necessary to describe a group element.

The generators and the commutation relations define the *Lie algebra* associated with the Lie group. Every representation of the group defines a representation of the algebra. The generators in the representation, when exponentiated, give the operators of the group representation. The definitions of equivalence, reducibility and irreducibility can be transferred unchanged from the group to the algebra.

Spacetime symmetries like rotations in an Euclidean space are particularly obvious examples of transformation groups. Other important transformation groups include the Lorentz group of special relativity and the Poincaré group (Lorentz boost plus translations and rotations). However, these are not compact groups. The nature of their representations is different from that of the groups which involve changes in particle identities, with no connection to the structure of space and time. These groups are associated with internal symmetries, and are the primary objects of our interest.

The structure constants depend on the choice of bases in the vector space of the generators. For the treatment of internal symmetries in this course, we will deal with unitary unimodular groups called $SU(n)$.³ They belong to a class called *compact semisimple* Lie groups, for which one can choose a basis such that

$$\text{Tr}(T_a T_b) = \lambda \delta_{ab} \quad (2.21)$$

for some positive real number λ . In this basis, the structure constants are completely antisymmetric, because one can write

$$f_{abc} = -i\lambda^{-1} \text{Tr}([T_a, T_b], T_c), \quad (2.22)$$

³The unitary group $U(n)$ is the subgroup consisting of those elements A of the general linear group $GL(n, C)$, represented by $n \times n$ complex matrices, such that $AA^\dagger = \mathbf{1}$. The special unitary group $SU(n)$ is that subgroup of $U(n)$ for which $\det A = 1$. The latter condition requires the generators to be traceless since

$$\text{for } \psi \rightarrow \psi' = U\psi \quad \text{with} \quad U = \exp\left(\frac{i}{2} \sum_a \alpha_a X_a\right), \quad (2.19)$$

$$\det U = \exp(\text{Tr}(\log U)) = \exp\left(\frac{i}{2} \text{Tr}(\alpha_a X_a)\right). \quad (2.20)$$

Since α_a are arbitrary numbers, $\det U = 1 \Rightarrow \text{Tr}(X_a) = 0$.

whence the antisymmetry of the RHS is ensured by the cyclic property of the trace. Also in this basis, the generators in the adjoint representation are hermitian matrices. In fact, it can be shown that for compact Lie groups (as for finite groups) any representation is equivalent to a representation by hermitian operators and all irreducible representations are finite hermitian matrices. The $SU(n)$ group has $n^2 - 1$ generators (one less than the $U(n)$), of which $n - 1$ can be simultaneously diagonalized.

2.2 The Group $SU(2)$

The simplest non-Abelian Lie algebra consists of three generators J_a ; $a = 1, 2, 3$, with $f_{abc} = \varepsilon_{abc}$, resulting in the commutation relations

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2. \quad (2.23)$$

This is the angular momentum algebra obeyed by the generators of the rotation group in 3 dimensions. They determine the properties of $SU(2)$, the unimodular unitary group that is the most frequently appearing symmetry in particle physics, as it describes not only spin, but also isospin symmetry, e.g. that between the proton and the neutron, and of the three charged states of the pion.

The $SU(2)$ matrices are complex 2×2 matrices

$$U = \exp \left(i \sum_{k=1}^3 \phi^k J_k \right) = \begin{pmatrix} u^1_1 & u^1_2 \\ u^2_1 & u^2_2 \end{pmatrix} \quad (2.24)$$

with the constraints

$$U^\dagger = U^{-1}, \quad \det U = 1. \quad (2.25)$$

In the fundamental representation, the $SU(2)$ algebra is realized by

$$J_i = \frac{1}{2} \sigma_i, \quad (2.26)$$

where the σ_i , are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.27)$$

The operator U operates on a complex two-component spinor $\psi = (\psi^1, \psi^2)$, which transforms under $SU(2)$ as

$$\psi' = U\psi \quad \text{or,} \quad (\psi')^i = \sum_{j=1}^2 u^i_j \psi^j. \quad (2.28)$$

The metric tensor is the two-dimensional Levi-Civita tensor $\varepsilon_{ij} = \varepsilon^{ij}$. Using this metric, covariant spinors can be obtained from contravariant spinors and vice-versa:

$$\psi_i = \varepsilon_{ij} \psi^j, \quad \psi^i = \varepsilon^{ij} \psi_j. \quad (2.29)$$

The invariance of the inner product of two spinors (ψ^1, ψ^2) and (ϕ^1, ϕ^2)

$$\phi^{1*}\psi^1 + \phi^{2*}\psi^2 \equiv \phi^{i*}\psi^i, \quad (2.30)$$

implies that the contravariant complex conjugate ψ^* transforms the same way as the covariant ψ :

$$\psi^{i*} \sim \varepsilon_{ij}\psi^j = \psi_i. \quad (2.31)$$

This property is called the reality of $SU(2)$. It means that the complex conjugate ψ^{i*} does not introduce any new representation.

The basis for the fundamental representation of $SU(2)$ is conventionally chosen to be the eigenvalues of J_3 , that is, the column vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

describing a spin- $\frac{1}{2}$ particle of spin projection up and spin projection down along the 3-axis, respectively. The other two spin components combine to form raising and lowering operators

$$J^\pm \equiv \frac{1}{\sqrt{2}}(J_1 \pm iJ_2) \quad (2.32)$$

so called because when they act on an eigenstate of J_3 , they raise or lower the eigenvalue by one unit (up to the highest or down to the lowest possible value). This is easily seen from the commutation relations

$$\begin{aligned} [J_3, J^\pm] &= \pm J^\pm \\ [J^+, J^-] &= J_3 \end{aligned} \quad (2.33)$$

So, if

$$J_3|m\rangle = m|m\rangle, \quad (2.34)$$

then

$$J_3 J^\pm|m\rangle = J^\pm J_3|m\rangle \pm J^\pm|m\rangle = (m \pm 1)J^\pm|m\rangle. \quad (2.35)$$

Suppose that a set of $|m\rangle$ forms an M -dimensional representation. The eigenvalues m are called *weights*. Let j be the highest weight. Then, by definition,

$$J^+|j\rangle = 0. \quad (2.36)$$

applying the lowering operator to $|m\rangle$, we find

$$J^-|m\rangle = N_m|m-1\rangle, \quad (2.37)$$

where N_m is a normalization constant which is determined as follows. From Eq. 2.37, we find

$$\langle m-1|J^-|m\rangle = N_m \quad \Leftrightarrow \quad \langle m|J^+|m-1\rangle = N_m^*. \quad (2.38)$$

By suitably choosing the phase of N_m , we have

$$J^-|m\rangle = N_m|m-1\rangle; \quad J^+|m-1\rangle = N_m|m\rangle. \quad (2.39)$$

Taking the square of Eq. 2.37, we get

$$\begin{aligned} N_m^2 &= \langle m|J^+J^-|m\rangle \\ &= \langle m|J^-J^+|m\rangle + m \\ &= N_{m+1}^2 + m. \end{aligned} \quad (2.40)$$

Solving this recursion formula for N_m under the initial condition $N_j^2 = j$ we get

$$N_m = \sqrt{\frac{1}{2}(j+m)(j-m+1)}. \quad (2.41)$$

There are $2j$ coefficients that are non-zero and real for $-(j-1) \leq m \leq j$. From Eq. 2.37, N_m appears when a state $|m-1\rangle$ is created from $|m\rangle$ by applying J^- . Starting from $|j\rangle$, they are $|j-1\rangle, |j-2\rangle, \dots, |-j\rangle$. Adding to these the initial state $|j\rangle$, the total number of states is $M = 2j + 1$. This completes the M -dimensional representation of $SU(2)$, with j corresponding to the total spin and m to the 3rd component of the spin. In the above we have not used the properties of the only Casimir operator

$$J^2 = J_1^2 + J_2^2 + J_3^2 \quad (2.42)$$

for the rotation group. There is an alternative way to derive the same result by using the commutation relations

$$[J^2, J_i] = 0. \quad (2.43)$$

The method shown here can be extended to $SU(3)$.

2.3 The Group $SU(3)$

Another symmetry group that has many manifestations in particle physics is $SU(3)$, the group of 3×3 unitary unimodular matrices. Its generators are 3×3 hermitian traceless matrices.⁴ The standard basis in physics literature consists

⁴The tracelessness is a consequence of the condition that the determinant be 1.

of the 8 ($= 3^2 - 1$) Gell-Mann λ matrices:

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{2.44}$$

The generators are

$$T_a = \frac{1}{2} \lambda_a, \tag{2.45}$$

normalized by Eq. 2.21 and satisfying the commutation relations

$$[T_a, T_b] = i f_{abc} T_c. \tag{2.46}$$

Clearly, T_1 , T_2 , and T_3 generate a $SU(2)$ subgroup of $SU(3)$. It is called the *isospin subgroup*, because in the physical application of uds (quark) flavor $SU(3)$, it represents isospin.

The structure constants of $SU(3)$ in the λ_i basis of Eq. 2.44 are fully antisymmetric under any pairwise interchange of indices, and the non-vanishing values are permutations of

$$\begin{aligned}
f_{123} &= 1, \\
f_{458} &= f_{678} = \frac{\sqrt{3}}{2}, \\
f_{147} &= f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}.
\end{aligned} \tag{2.47}$$

Just as in $SU(2)$, the fundamental representation of $SU(3)$ is based on the transformation

$$\psi' = U\psi \quad \text{or,} \quad \psi'^i = \sum_{j=1}^3 u^i_j \psi^j, \tag{2.48}$$

but with u^i_j as the components of the 3×3 special unitary matrix

$$U = e^{i\alpha_a T^a} \tag{2.49}$$

However, unlike the $SU(2)$ case, the $SU(3)$ representation (ψ^i) is not real, i.e. the complex conjugate transforms as

$$\psi'^{*i} = u^i_j{}^* \psi'^{*j} \tag{2.50}$$

which is independent of Eq. 2.49. This is because the metric tensor is ε_{ijk} , which means the complex conjugate behaves as

$$\psi^{i*} = \varepsilon_{ijk} \psi^j \psi^k. \quad (2.51)$$

Among the 8 generators of $SU(3)$, two can be diagonalized simultaneously.⁵ In the fundamental representation of Eq. 2.44, they are already given by T_3 and T_8 . Therefore, $SU(3)$ states are labeled by eigenvalues of T_3 and T_8 . For a given simultaneous eigenstate, two eigenvalues define a point on a 2-dimensional (t_3, t_8) plane. The remaining generators combine to form the raising or lowering operators that shift one state to another:

$$\begin{aligned} I_{\pm} &= \frac{1}{\sqrt{2}}(T_1 \pm iT_2), \\ V_{\pm} &= \frac{1}{\sqrt{2}}(T_4 \pm iT_5), \\ U_{\pm} &= \frac{1}{\sqrt{2}}(T_6 \pm iT_7). \end{aligned} \quad (2.52)$$

Each of these matrices has a single non-zero element, which is, of course, off-diagonal, so as to transform one (T_3, T_8) eigenstate to another. The following commutation relations follow:

$$\begin{aligned} [T_3, I_{\pm}] &= \pm I_{\pm}, & [T_8, I_{\pm}] &= 0, \\ [T_3, V_{\pm}] &= \pm \frac{1}{2}V_{\pm}, & [T_8, V_{\pm}] &= \pm \frac{\sqrt{3}}{2}V_{\pm}, \\ [T_3, U_{\pm}] &= \mp \frac{1}{2}U_{\pm}, & [T_8, U_{\pm}] &= \pm \frac{\sqrt{3}}{2}U_{\pm}. \end{aligned} \quad (2.53)$$

These imply that I_{\pm} , U_{\pm} , and V_{\pm} raise or lower the values of t_3 and t_8 by the coefficients on the right-hand sides. Therefore, they are expressed by 2-dimensional vectors, which point from the origin to one of the vertices of a regular hexagon.

In a fashion similar to the one demonstrated for $SU(2)$, it is possible to construct the $SU(3)$ representation. The simultaneous eigenvectors of T_3 and T_8 are

$$\psi^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \psi^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (2.54)$$

We see that

$$\begin{aligned} T_3 \psi^1 &= \frac{1}{2} \psi^1 & \Rightarrow \vec{\mu}_1^1 &= \left| \frac{1}{2}, \frac{\sqrt{3}}{6} \right\rangle, \\ T_8 \psi^1 &= \frac{\sqrt{3}}{6} \psi^1 & & \\ T_3 \psi^2 &= -\frac{1}{2} \psi^2 & \Rightarrow \vec{\mu}_2^1 &= \left| -\frac{1}{2}, \frac{\sqrt{3}}{6} \right\rangle, \\ T_8 \psi^2 &= \frac{\sqrt{3}}{6} \psi^2 & & \\ T_3 \psi^3 &= 0 & \Rightarrow \vec{\mu}_3^1 &= \left| 0, -\frac{\sqrt{3}}{3} \right\rangle, \\ T_8 \psi^3 &= -\frac{\sqrt{3}}{3} \psi^3 & & \end{aligned} \quad (2.55)$$

⁵Hence, $SU(3)$ has a rank 2.

where we have introduced three 2-dimensional vectors, called *weight vectors* $\vec{\mu}_i^1$, to represent the states ψ^i . The superscript 1 for the $\vec{\mu}$ is to distinguish it from another set of weights $\vec{\mu}_i^2$ which will be introduced shortly. The weight vectors form a unit equilateral triangle centered at the origin of the t_3, t_8 plane.

The three states of the fundamental representation are related to each other through the raising and lowering operators. For instance, it is easy to check in the 3-component vector form that

$$\psi^1 = V_+ \psi^3. \quad (2.56)$$

In terms of weight vectors, this is expressed as

$$\vec{\mu}_1^1 = \vec{\alpha}_1 + \vec{\mu}_3^1, \quad (2.57)$$

where the *root vector*

$$\vec{\alpha}_1 = \left| \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \quad (2.58)$$

relates two weights additively and increases a weight by the “unit $\vec{\alpha}_1$ ”. Similarly, one can consider another root

$$\vec{\mu}_2^1 = \vec{\alpha}_2 + \vec{\mu}_3^1, \quad (2.59)$$

which raises a weight by another unit

$$\vec{\alpha}_2 = \left| \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle. \quad (2.60)$$

The root vectors $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are independent. In general, all weight vectors are related by

$$\vec{\mu}' = \vec{\mu} + l\vec{\alpha}_1 + m\vec{\alpha}_2, \quad (2.61)$$

where l and m are some integers.

Notice the correspondence between the root vectors and lowering and raising operators:

$$\vec{\alpha}_1 \sim V_+, \quad \vec{\alpha}_2 \sim U_-. \quad (2.62)$$

In principle, one could choose any two independent operators out of the six: I_\pm , U_\pm , V_\pm . In the particular choice above, the two roots are called *simple roots*.

In $SU(3)$ there is another fundamental representation which is the complex conjugate ψ'^{*i} (see Eq. 2.51). Complex conjugation of the commutation relations in Eq. 2.46 leads to

$$[-T_a^*, T_b^*] = if_{abc} T_c^*, \quad (2.63)$$

implying $-T_a^* = -\frac{1}{2}\lambda_a^*$ can be another representation. The diagonal generators T_3 and T_8 are replaced simply by the negatives of the original ones, and, therefore, the weight vectors change their signs. In other words,

$$\begin{aligned} \psi^{1*} \rightarrow \vec{\mu}_1^2 &= \left| -\frac{1}{2}, -\frac{\sqrt{3}}{6} \right\rangle, \\ \psi^{2*} \rightarrow \vec{\mu}_2^2 &= \left| \frac{1}{2}, -\frac{\sqrt{3}}{6} \right\rangle, \\ \psi^{3*} \rightarrow \vec{\mu}_3^2 &= \left| 0, \frac{\sqrt{3}}{3} \right\rangle, \end{aligned} \quad (2.64)$$

which form another triangle, rotated by π w.r.t. the first one. Notice that the new states represented by the new triangle are still connected by the same simple root vectors. In the $SU(3)$ of strong interactions, one representation represents the color states of a quark, while the other represents the color states of an antiquark, but the same gluons (the generator coefficients) mediate the transitions between the different states within each set.

2.4 Parity transformation

An extremely simple group is one that has only two elements: $\{e, P\}$. Obviously, $P^{-1} = P$, so $P^2 = e$, with e represented by the unit $n \times n$ matrix in an n -dimensional representation. Thus, if $|\psi\rangle$ is an eigenstate of P , then

$$P|\psi\rangle = \pm|\psi\rangle \quad (2.65)$$

We could represent P as a phase-changing operator $P = e^{in\pi}$, where n is an integer. Such a phase would be additive for a composite system. However, the common practice is just to keep track of the sign, which then becomes a multiplicative quantum number.

The most familiar example of a multiplicative quantum number is *parity*, or *space inversion*, given by the Lorentz transformation in Eq. 1.12. Since both \mathbf{x} and \mathbf{p} change sign under a parity transformation, $\mathbf{J} = \mathbf{r} \times \mathbf{p}$ does not. Since $[P, p] \neq 0$, momentum eigenstates of particles in motion are not eigenstates of P , but one would expect stationary systems described as eigenstates of H and J^2 , J_3 to be. Indeed, it had been known that parity is conserved in electromagnetic and strong interactions. However, to everyone's surprise, it was found in the 1950's that parity is violated (i.e. $[P, H] \neq 0$) in weak interactions. Indeed, it is a maximal violation: in the limit of a massless neutrino, which can only be produced in weak interactions, its spin 3-vector always points opposite to its momentum 3-vector, i.e. in a *left-handed helicity* state.⁶

It is useful to know how fields transform under parity. Since the single particle states are obtained by applying the field operator to the vacuum, this tells us the parity transformation properties of the state. For integral spin, if the transformation of the field is the same as that of a spatial tensor of the same rank (e.g. a scalar or a vector), then the field is said to have *natural* spin-parity. If the field transforms with an extra minus sign (e.g. a pseudoscalar or a pseudovector), then it is said to have *unnatural* spin-parity. The photon has natural spin-parity since the polarization vectors are ordinary 4-vectors. For spin- $\frac{1}{2}$ there is no analogy to space tensors, but we can (and will, in the next chapter) examine how the current behaves under parity transformation. We state, without proof for now, that we must assign opposite intrinsic parities to the fermions and antifermions. Thus, in the massless limit, all antineutrinos produced in weak interactions are right-handed.

One of the clearest manifestation of the maximal parity violation can be seen in the decay of charged pions, which is a weak process. Even though energy-wise a larger phase space is open for the decay $\pi^- \rightarrow e^- \bar{\nu}_e$ than $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$, the former is exceedingly rare: $B(\pi^- \rightarrow e^- \bar{\nu}_e) = 1.23 \times 10^{-4}$ vs $B(\pi^- \rightarrow \mu^- \bar{\nu}_\mu) = 0.999877$ (No other 2-body decay is kinematically feasible)! This is explained by the fact that for spin (=0 for pions) to be conserved in the process, the ℓ^- and the $\bar{\nu}_\ell$ must have the same helicity. This is harder to achieve in the $e^- \bar{\nu}_e$ channel since the electron, being ~ 200 times lighter than the muon, gets a much higher boost in the π^- rest frame. Consequently, the phase space is severely squeezed in the

⁶The helicity of a massive particle is not absolute - it can be flipped by a Lorentz boost.

$e^- \bar{\nu}_e$ channel because the boost needed for a frame to observe the e^- and the $\bar{\nu}_e$ in the opposite helicity states is much larger than for the μ^- and the $\bar{\nu}_\mu$.

2.5 Charge Conjugation

Charge conjugation is an operation that converts a particle to its antiparticle:

$$C|\psi\rangle = |\bar{\psi}\rangle, \quad (2.66)$$

resulting in the inversion of *all internal quantum numbers*, i.e. electric charge, isospin, color, lepton number, baryon number, \dots , without alternating mass, momentum, and spin. This is another example of a group with only two elements, with

$$C^2|\psi\rangle = |\psi\rangle, \quad (2.67)$$

so the eigenvalues of C are ± 1 . Unlike P , however, only particles that are their own antiparticles are eigenstates of C .

Classical electrodynamics is invariant under charge conjugation. The potentials and fields all change their signs so as to leave the forces unaffected. Since the field changes its sign, its quantum, the photon, has a charge conjugation eigenvalue of -1 . In general, a fermion-antifermion system with orbital angular momentum l and total spin s constitutes an eigenstate of C with eigenvalue $(-1)^{l+s}$. This is the basis of classification of mesons, which are quark-antiquark bound states, in terms of J^{PC} .

Like P , C is a multiplicative quantum number that is conserved in the strong and electromagnetic interactions, but not in weak interactions. This can be readily seen as a consequence of parity violation in weak processes: applying C to any process involving a (massless left-handed) neutrino will result in one with a left-handed antineutrino, which does not exist. It was once argued by some that charge conjugation should be considered an integral part of a more general definition of “parity” amounting to CP by our definitions, which would be conserved in weak interactions. But a superbly reasoned prediction followed by a landmark experiment on mixing of the neutral kaon with its own antiparticle ($\Delta S = 2$) established CP violation in weak processes, albeit extremely mild compared to C or P violations separately. Subsequent studies of semileptonic decays of K_L^0 , the longer-lived (near-symmetric) admixture of the two pseudoscalar CP eigenstates, showed even more dramatic evidence of CP violation through a slight imbalance in the decay fractions to $\pi^+ e^- \bar{\nu}_e$ and $\pi^- e^+ \nu_e$. This is a process that makes an absolute distinction between matter and antimatter, and provides an unambiguous, convention-free definition of positive charge: it is the charge carried by the lepton preferentially produced in the decay of K_L^0 .⁷

⁷However, the level of CP violation within the standard model seems to fall far short of explaining the observed degree of preponderance of matter over antimatter in today’s universe.

2.6 Lagrangian Density, Field Equations, and Conserved Currents

A particle theory is set up by defining the dynamical variables $\phi_j(x^\mu)$ that are functions of space-time. These *fields* are described by the Lagrangian density, which is a function of the fields and their first derivatives only

$$\mathcal{L} = \mathcal{L}(\phi_j, \partial_\mu \phi_j), \quad (2.68)$$

where $j = 1, 2, \dots, n$ label the fields and/or different components of a field. The variational principle contends that the action integral

$$S = \int \mathcal{L}(\phi(x)) d^4x \quad (2.69)$$

is stationary with respect to any changes in ϕ that vanish on the boundary. Then, in a manner analogous to the one in classical mechanics where the Lagrangian is a function of space-time directly, it can be shown that the variational principle leads to the equations of motion

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)} - \frac{\partial \mathcal{L}}{\partial \phi_j} = 0. \quad (2.70)$$

The elementary particles of a theory appear as the solutions of the field equations resulting from the associated Lagrangian.⁸ For example, in quantum electrodynamics, the photon is the quantum of the electromagnetic field, represented by the vector potential A^μ . The electron is represented by the fermion field ψ . The Lagrangian contains the fundamental interactions of the theory. For electrodynamics, that is the $J_\mu A^\mu$ term in the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu. \quad (2.71)$$

(*Exercise: show that Maxwell's equations follow from this Lagrangian.*)

It is the potential energy parts of the Lagrangian that specify the theory. The kinetic energy parts are general and only depend on the spins of the particles. The potential energy terms specify the forces. These terms are collectively called the *interaction Lagrangian*.

Consider an infinitesimal change $\delta\phi_j$ in fields ϕ_j that is a symmetry of \mathcal{L} in the the sense that

$$\mathcal{L}(\phi_j + \delta\phi_j) = \mathcal{L}(\phi_j). \quad (2.72)$$

⁸Composite objects may appear as bound states of the elementary particles.

We can then write using Eq. 2.70,

$$\begin{aligned}
0 &= \delta\mathcal{L}(\phi) = \mathcal{L}(\phi_j + \delta\phi_j) - \mathcal{L}(\phi_j) = \frac{\partial\mathcal{L}}{\partial\phi_j}\delta\phi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta(\partial_\mu\phi_j) \\
&= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\right)\delta\phi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\partial_\mu\delta\phi_j \\
&= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j\right).
\end{aligned} \tag{2.73}$$

Thus the conserved current is

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j. \tag{2.74}$$

We will be particularly interested in transformations of the ϕ fields that are homogeneous, linear, and unitary. Such a symmetry takes the form

$$\delta\phi = i\epsilon_a T^a \phi, \tag{2.75}$$

where the T^a are a set of $n \times n$ Hermitian matrices acting on the space of the ϕ 's and the ϵ_a are infinitesimal parameters. Equation 2.75 is the infinitesimal form of a unitary transformation

$$\phi \rightarrow \phi' = \exp(i\epsilon_a T^a)\phi \simeq (1 + i\epsilon_a T^a)\phi. \tag{2.76}$$

Let us consider for the moment only *global symmetries*, in which the parameters ϵ_a are independent of x , but otherwise arbitrary. Then the following is true:

$$\delta(\partial^\mu\phi) = i\epsilon_a T^a \partial^\mu\phi, \tag{2.77}$$

which looks just like Eq. 2.75. We say $\partial^\mu\phi$ transforms (under the symmetry operation) like ϕ .

In this case, the conserved currents take a particularly simple form. Taking out the infinitesimal parameters, we can write the conserved currents as

$$J_\mu^a = -i \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)} T^a \phi. \tag{2.78}$$

So far we have not put the quantum into the quantum field theory. When we do, the fields ϕ_j and their conjugate momenta

$$\Pi_j = \frac{\partial\mathcal{L}}{\partial(\partial^0\phi_j)} \tag{2.79}$$

satisfy the *equal time (anti-)commutation* relations for (fermion) boson fields:

$$\begin{aligned}
[\phi_j(x), \phi_k(y)]_\pm^{\text{ET}} &= [\Pi_j(x), \Pi_k(y)]_\pm^{\text{ET}} = 0 \\
[\phi_j(x), \Pi_k(y)]_\pm^{\text{ET}} &= i\delta_{jk}\delta^{(3)}(\mathbf{x} - \mathbf{y}),
\end{aligned} \tag{2.80}$$

where “ET” stands for equal time, $x^0 = y^0$; and $[A, B]_{\pm} = AB \pm BA$.

We will deal with situations where the matrices are organized into an algebra that closes under commutation relations of Eq. 2.18. The time components of the currents are related to charges

$$Q^a = \int d^3x J_0^a(x), \quad (2.81)$$

which satisfy Eq. 2.18. It follows after some algebra that

$$[Q^a, Q^b] = i f^{abc} Q^c. \quad (2.82)$$

The existence of these charges and their associated currents is an important consequence of the symmetry. Note that nothing in the derivation requires the interpretation to be in terms of electric charge. Indeed, particles have a number of charges, of which at least some can be related to conserved currents. Let us examine a few examples.

Example 1

The simplest example is that of a real scalar (i.e., a spinless) field ϕ of mass m , described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_{\mu}\phi\partial^{\mu}\phi - m^2\phi^2), \quad (2.83)$$

which leads to the Klein-Gordon wave equation

$$\partial^{\mu}\partial_{\mu}\phi = m^2\phi. \quad (2.84)$$

Identifying the free-particle 4-momentum with the 4-gradient in space-time $p^{\mu} = \partial^{\mu} = \frac{\partial}{\partial x_{\mu}}$, we see that this is merely the relativistic rendition of the Schrödinger equation. The solution (the field amplitude), except for a normalization constant, is given by

$$\phi(x^{\mu}) = e^{-ip^{\mu}x_{\mu}}. \quad (2.85)$$

Interactions of the field with other particles requires the introduction of a *source* term into the field equations. The simplest example modifies the field equations as follows:

$$(\partial^{\mu}\partial_{\mu} - m^2)\phi = \rho. \quad (2.86)$$

Since ϕ is a Lorentz scalar, ρ must be so as well. If the source is localized, static and unperturbed by the interactions (a very heavy particle at rest would be a good approximation), then choosing the origin at the source,

$$\rho = g\delta^3(\mathbf{x}). \quad (2.87)$$

Since ρ is not time dependent, Eq. 2.86 becomes

$$(-\nabla^2 - m^2)\phi = g\delta^3(\mathbf{x}). \quad (2.88)$$

Writing

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{\phi}(\mathbf{k}) \quad (2.89)$$

and the inverse Fourier transform

$$\tilde{\phi}(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \phi(\mathbf{x}) \quad (2.90)$$

yields (since $\nabla^2 \rightarrow -\mathbf{k}^2$)

$$(\mathbf{k}^2 + m^2)\tilde{\phi}(\mathbf{k}) = \frac{g}{2\pi^{\frac{3}{2}}}. \quad (2.91)$$

Substituting this into Eq. 2.89, we get

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + m^2}. \quad (2.92)$$

Evaluation of the integral yields the time-independent solution

$$\phi(r) = \frac{g}{4\pi} \frac{e^{-mr}}{r}, \quad (2.93)$$

which is called the *Yukawa potential*. We see that the strength of the potential at a given point is determined by the *coupling constant* g , and m . A large value of m gives a short range of interaction. This is the reason, in fact, of the weakness of the weak interactions. It is an example of a general result that high-mass physics is hard to see at low energies, since it corresponds only to phenomena at very short distances.

If we had removed the constraint that the source be time-independent, the denominator in the integrand on the RHS of Eq. 2.92 would simply be modified to $m^2 - k^2$, where $k^2 = k^\mu k_\mu$. Indeed, such a denominator appears as a propagator whenever a particle is exchanged in an interaction. This conforms to the general interpretation, in a quantum field theory, that all interactions are due to the exchange of field quanta. The concepts of force and of interaction are used interchangeably. Usually the matrix elements are written in the momentum space. Then from Eq. 2.92, the momentum space quantity representing the exchanged particle of mass m is

$$\frac{1}{k^2 - m^2}. \quad (2.94)$$

This is called a *propagator*, which will show up whenever we write the matrix element. The complete propagator also has a phase factor and a numerator that depends on the spin of the exchanged particle, but for most calculation these can be considered as technical details that do not affect the qualitative results.

Example 2

Some interesting physics emerges if we consider a system of two real scalar fields, ϕ_1 and ϕ_2 , that have the same mass m . Then we can expect from Eq. 2.83 that

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1 \partial^\mu \phi_1 - m^2 \phi_1^2) + \frac{1}{2}(\partial_\mu \phi_2 \partial^\mu \phi_2 - m^2 \phi_2^2). \quad (2.95)$$

We can combine ϕ_1 and ϕ_2 into a single complex scalar field ϕ by writing

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2). \quad (2.96)$$

Then

$$\phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \quad (2.97)$$

So, the Lagrangian can be rewritten as

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \quad (2.98)$$

Note that ϕ has the same mass m as ϕ_1 and ϕ_2 , and ϕ and ϕ^* are normalized to the same total amplitude as ϕ_1 and ϕ_2 are.

The current density is

$$J^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*), \quad (2.99)$$

satisfying the continuity equation

$$\partial_\mu J^\mu = 0. \quad (2.100)$$

The field (wave) equations are

$$(\partial^\mu \partial_\mu - m^2)\phi = (\partial^\mu \partial_\mu - m^2)\phi^* = 0. \quad (2.101)$$

Example 3

If there were an Abelian vector (spin-1) field B^μ , like the electromagnetic field, but massive, the Lagrangian given in Eq. 2.71 would acquire an additional term

$$\frac{1}{2}m^2 B^\mu B_\mu, \quad (2.102)$$

to accommodate a mass term will in the wave equation. If we see a term $B^\mu B_\mu$ appear in a Lagrangian, we can identify its coefficient as $\frac{m^2}{2}$. Such a term explicitly violates gauge symmetry. Thus, the gauge symmetry forbids a non-zero mass for the photon.

Chapter 3

Quantum Electrodynamics

We now turn to spin- $\frac{1}{2}$ particles. Let us study the electron as a specific example. The electron is a spin- $\frac{1}{2}$ particle, which implies that each momentum state has two possible helicities, $\lambda = +\frac{1}{2}$ or $\lambda = -\frac{1}{2}$. The states in the particle rest frame can be determined using the spin- $\frac{1}{2}$ representation of the rotation group, $SU(2)$.

We can describe the two spin choices in terms of the base states:

$$\chi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.1)$$

These states, called spinors, correspond to spins $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively, along a chosen space axis, which we take to be the 3-axis (z).

The spin operator in the fermion rest frame is given in the basis above by

$$\vec{S} = \frac{\vec{\sigma}}{2}, \quad (3.2)$$

where $\vec{\sigma}$ is the Pauli spin matrix whose components are given by Eq. 2.27. In addition, we now define the identity matrix as the *0th component* of the spin matrix. matrix:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.3)$$

3.1 The Dirac Equation

Dirac's primary objective in deriving the field equations for fermions was to linearize the Klein-Gordon equation (Eq. 2.84) which, being quadratic in E , opened doors to solutions with negative energy that needed to be explained. Originally, Dirac handled the problem of preventing all fermions from falling into negative energy states without a lower bound by postulating that all such states are already full. This made for the possibility of an electron in a negative energy state making an occasional transition to a positive energy state, which would create a *hole* in the *sea* of negative energy state. Dirac called these "hole"s

positrons. Experimental confirmation of the existence of positrons is counted among the greatest triumphs in theoretical physics. Later, Feynman came up with an alternative interpretation of positrons as electrons traveling backward in time. This led to great simplification of the theory, which came to be known as quantum electrodynamics. So, to modify the Klein-Gordon equation to describe spin- $\frac{1}{2}$ particles, each energy two (+ve and -ve) energy states in its solution must be allowed two spin states. That is, the general wave function will have $2 \times 2 = 4$ components:

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad (3.4)$$

The linear equation should then take the form

$$H\psi = i\frac{\partial}{\partial t}\psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi = (\vec{\alpha} \cdot i\nabla + \beta m)\psi, \quad (3.5)$$

where β and α_i ($i = 1, 2, 3$) are 4×4 matrices. They can be determined by comparing Eq. 2.84 with the H^2 expressed in terms of the RHS of Eq. 3.5:

$$\frac{\partial^2}{\partial t^2} = (-\alpha^j \alpha^k \partial_j \partial_k - im(\alpha^j \beta + \beta \alpha^j) \partial_j + \beta^2 m^2) \psi \quad (3.6)$$

Since the partial derivatives commute, we can write

$$\alpha^j \alpha^k \partial_j \partial_k = \frac{1}{2}(\alpha^j \alpha^k + \alpha^k \alpha^j) \partial_j \partial_k \quad (3.7)$$

Then, for Eq. 3.6 to be consistent with Eq. 2.84 we must have

$$\beta^2 = 1, \quad (3.8)$$

$$\{\alpha^j, \beta\} = \alpha^j \beta + \beta \alpha^j = 0, \quad (3.9)$$

$$\{\alpha^j, \alpha^k\} = \alpha^j \alpha^k + \alpha^k \alpha^j = 2\delta^{jk}, \quad (3.10)$$

The solution to these can be written in terms of the Pauli matrices:

$$\beta = \gamma^0 \equiv \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}. \quad (3.11)$$

Note that the representation is not unique. The one above is known as the Dirac-Pauli representation. Another possibility, known as the Weyl- or chiral representation is

$$\beta = \gamma^0 \equiv \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \quad \alpha^j = \begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}. \quad (3.12)$$

Most of the formulae are independent of the representation. We will use the Pauli-Dirac representation.

Equation 3.5 is known as the Dirac equation and the 4-component wave function, a Dirac spinor.

3.2 The γ matrices and trace theorems

The Dirac equation can be written in a simpler form by multiplying it on the left by β and defining

$$\gamma^\mu = (\beta, \beta\vec{\alpha}), \quad (3.13)$$

or, explicitly,

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}. \quad (3.14)$$

These are known as the Dirac γ matrices. The result is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \quad (3.15)$$

It is useful to define the *Feynman slash* notation:

$$\gamma^\mu a_\mu = \not{a}, \quad (3.16)$$

so the Dirac equation takes the compact form

$$(i\not{\partial} - m)\psi = 0. \quad (3.17)$$

In practice, one almost never needs to know the explicit forms of the γ matrices. The following relations satisfied by them suffice for most calculations:

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0, \quad (\Rightarrow \gamma^{0\dagger} = \gamma^0, \quad \gamma^{j\dagger} = -\gamma^j), \quad (3.18)$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \quad (\Rightarrow \gamma^\mu \gamma_\mu = 4), \quad (3.19)$$

$$\gamma^\mu \not{a} \gamma_\mu = -2\not{a} \quad (3.20)$$

$$\gamma^\mu \not{a} \not{b} \gamma_\mu = 4a \cdot b \quad (3.21)$$

$$\gamma^\mu \not{a} \not{b} \not{c} \gamma_\mu = -2\not{a} \not{b} \not{c} \quad (3.22)$$

For reasons that will become clear soon, it is useful to define

$$\gamma^5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3. \quad (3.23)$$

The following trace theorems often come in handy:

$$\text{The trace of an odd number of } \gamma^\mu \text{'s vanish.} \quad (3.24)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}. \quad (3.25)$$

$$\text{Tr}(\not{a} \not{b}) = 4a \cdot b, \quad (3.26)$$

$$\text{Tr}(\not{a} \not{b} \not{c} \not{d}) = 4((a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)), \quad (3.27)$$

$$\text{Tr}(\gamma^5) = 0, \quad (3.28)$$

$$\text{Tr}(\gamma^5 \not{a} \not{b}) = 0, \quad (3.29)$$

$$\text{Tr}(\gamma^5 \not{a} \not{b} \not{c} \not{d}) = 4i\varepsilon_{\mu\nu\rho\sigma} a^\mu b^\nu c^\rho d^\sigma, \quad (3.30)$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita tensor in 4 dimensions.

3.3 Lagrangian and symmetries for a spin- $\frac{1}{2}$ field

The Lagrangian for the free spin- $\frac{1}{2}$ field is

$$\mathcal{L} = \bar{\psi}(i\partial_\mu - m)\psi. \quad (3.31)$$

The corresponding Hamiltonian density is

$$\mathcal{H} = \bar{\psi}(\vec{\gamma} \cdot \vec{p} + m)\psi. \quad (3.32)$$

The Lagrangian has a global gauge symmetry, since $\psi \rightarrow \psi' = e^{i\theta}\psi$ leaves the field equations unchanged. Thus we expect a conserved current density

$$J^\mu = e\bar{\psi}\gamma^\mu\psi, \quad (3.33)$$

which satisfies the continuity equation $\partial_\mu J^\mu = 0$.

What is the relativistic extension for the angular momentum? By commuting $\vec{x} \times \vec{p}$ with $H = \gamma^0(\vec{\gamma} \cdot \vec{p} + m)$, we can try to determine what must be added to the orbital angular momentum to make a conserved quantity. It turns out that the simplest extension of the nonrelativistic expression works. Defining the 4×4 matrices

$$\Sigma^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}, \quad (3.34)$$

$\vec{J} = \vec{x} \times \vec{p} + \frac{1}{2}\vec{\Sigma}$ satisfies $[H, \vec{J}] = 0$.

The plane wave fields we are using are eigenstates of \vec{p} . They are not eigenstates of \vec{J} since $[\vec{J}, \vec{p}] \neq 0$ because $[\vec{x}, \vec{p}] \neq 0$, and that affects the orbital component of the total spin. If we can isolate the spin operator alone in an expression that commutes with H , then we can get a quantity that does commute with the momentum. Taking $\vec{J} \cdot \vec{p}$, the term $(\vec{x} \times \vec{p}) \cdot \vec{p}$ vanishes, leaving $\frac{1}{2}\vec{\Sigma} \cdot \vec{p}$. This *helicity* operator now commutes with H and \vec{p} and can therefore be simultaneously diagonalized. The resulting states are helicity states. Note that this works because the intrinsic spin operator is independent of position, as is the resulting spin-dependent factor in the amplitude of the plane wave solution. This argument holds just as well for particles for other values of the spin.

As an example of interaction that can be added to the free-particle Lagrangian, let us consider electromagnetism. In classical electrodynamics, the minimal substitution into the basic equations of motion of a particle with charge $-q$, caused by the presence of an electromagnetic field, is $p^\mu \rightarrow p^\mu + qA^\mu$. In the quantum case, the replacement $p^\mu \rightarrow i\partial^\mu$ leads to the modified Lagrangian

$$\mathcal{L} = \bar{\psi}\gamma^\mu(i\partial_\mu + qA_\mu)\psi - m\bar{\psi}\psi. \quad (3.35)$$

To include the photon field, we need to add the term $\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$. The local gauge transformation now is

$$\psi \rightarrow e^{iq\theta}\psi \quad \text{and} \quad A^\mu \rightarrow A^\mu + \partial_\mu\theta, \quad (3.36)$$

which leaves the \mathcal{L} unchanged.

The function $\theta(x^\mu)$ is an arbitrary function of space-time. So, the field transformations must be carefully coordinated. The interaction term arises directly out of the modification to the free field Lagrangian if we add the local gauge symmetry requirement.¹ Thus, the interaction term in Eq. 3.35 is

$$\mathcal{L}_{\text{Int}} = eA_\mu\bar{\psi}\gamma^\mu\psi. \quad (3.37)$$

Comparing this expression with Eq. 2.71 we see that the conserved electromagnetic current is still $J_\mu = e\bar{\psi}\gamma^\mu\psi$: the expression for the conserved current is not modified in the presence of an electromagnetic field (that is, when going from a global to a local gauge symmetry) for the Dirac field. This is a consequence of the derivative not explicitly appearing in the expression for the free particle current for spin- $\frac{1}{2}$.² For other spins, e.g. spin-0, the expression for the current is modified. This will bear on the treatment of symmetry breaking.

Other interactions can be introduced to the Lagrangian in an analogous way. For example, the interaction with a hypothetical scalar particle will add a term

$$\mathcal{L}_{\text{Int}} = -g\phi\bar{\psi}\psi, \quad (3.38)$$

where ϕ is the scalar field. This interaction also leaves the expression for the conserved current unchanged provided ϕ is real and unchanged under the gauge transformations. Physically, $\phi \rightarrow \phi$ under a gauge transformation implies that ϕ carries no charge. Only particles of the ψ field, i.e. the fermions, are charged. Note also that g has to be a real number since \mathcal{L}_{Int} is hermitian.

3.4 Explicit plane-wave solutions

Let us start with the simplest case of a free particle at rest. The Dirac equation then reduces to

$$\left(i\gamma_0\frac{\partial}{\partial t} - m\right)\psi = 0. \quad (3.39)$$

In this case, since γ_0 is diagonal, the equations do not mix the 4 components of ψ (of Eq. 3.4):

$$\begin{aligned} i\frac{\partial\psi_1}{\partial t} - m\psi_1 &= 0, & i\frac{\partial\psi_2}{\partial t} - m\psi_2 &= 0, \\ i\frac{\partial\psi_3}{\partial t} + m\psi_3 &= 0, & i\frac{\partial\psi_4}{\partial t} + m\psi_4 &= 0. \end{aligned} \quad (3.40)$$

Taking a solution of the form e^{-iEt} , we get $E = m$ for ψ_1 and ψ_2 . However, for ψ_3 and ψ_4 , this would give $E = -m$. This dilemma is resolved by taking a

¹The Dirac equation (Eq. 3.17) does not have a local gauge symmetry.

²All terms involving space-time derivatives are modified because of the minimal substitution used to introduce the electromagnetic interaction.

solution of the form e^{iEt} and transferring the minus sign to t in order to keep E positive.³ Consequently, ψ_3 and ψ_4 represent the antiparticle states.

Let us now try to tackle particles in motion by generalizing the solution to the form $e^{\pm ip^\mu x_\mu} = e^{\pm ip \cdot x}$ which reduce to $e^{\pm mt}$ as $\vec{p} \rightarrow 0$. Then we have

$$i\gamma_\mu \partial_\mu e^{\pm ip \cdot x} = \gamma_\mu p_\mu e^{\pm ip \cdot x} = \not{p} e^{\pm ip \cdot x} \quad (3.41)$$

with

$$\not{p} = \begin{pmatrix} E & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -E \end{pmatrix}, \quad \text{where} \quad \vec{\sigma} \cdot \vec{p} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_3 \end{pmatrix}. \quad (3.42)$$

Explicitly, the four solutions are

$$\psi_1 = u_1 e^{-ip \cdot x}, \quad \psi_2 = u_2 e^{-ip \cdot x}, \quad \psi_3 = v_1 e^{ip \cdot x}, \quad \psi_4 = v_2 e^{ip \cdot x}. \quad (3.43)$$

For $\vec{p} = 0$, we've already found the four solutions, which can be written as $u(0) = \begin{pmatrix} \chi \\ 0 \end{pmatrix}$, $v(0) = \begin{pmatrix} 0 \\ \chi \end{pmatrix}$, where $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For the more general situation, we have

$$(\not{p} - m)u = 0, \quad \text{and} \quad (\not{p} + m)v = 0. \quad (3.44)$$

The general solution can be obtained by noting that $(\not{p} - m)(\not{p} + m) = p^2 - m^2 = 0$, so we can take

$$u(p) = (\not{p} + m)u(0), \quad v(p) = (\not{p} - m)v(0). \quad (3.45)$$

We can now choose $u(0)$ and $v(0)$ as above to get $u(p)$ and $v(p)$.

The normalization condition for the spinors is

$$\bar{\psi}\psi = 2m, \quad (3.46)$$

which is equivalent to requiring the current $\bar{\psi}\gamma_0\psi = \psi^\dagger\psi$ (the number of particles per unit volume) to be $2E$. Thus, for the particle at rest we have

$$u(0) = \begin{pmatrix} \chi \\ 0 \end{pmatrix}, \quad v(0) = \begin{pmatrix} 0 \\ \chi \end{pmatrix}. \quad (3.47)$$

These lead to the general results

$$u = \sqrt{E+m} \begin{pmatrix} \chi \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi \end{pmatrix}, \quad v = \sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi \\ \chi \end{pmatrix}. \quad (3.48)$$

$\vec{\sigma} \cdot \vec{p}$ is the operator proportional to the helicity of the two-component spinor χ . If $\vec{p} = p\hat{z}$, then taking $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ will yield two helicity eigenstates for u or v . For an arbitrary \vec{p} , we can rotate $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ using the 2×2 rotation operators to generate helicity eigenstates along \vec{p} .

³This applies to all fields irrespective of spin, as can be seen in the Klein-Gordon equation.

Bilinear operator O	Transformation property of $\psi O \psi$	No. of operators
1	Scalar	1
γ^μ	Vector	4
$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	Antisymmetric tensor	6
$\gamma^5 \gamma^\mu$	Pseudovector	4
γ^5	Pseudoscalar	1

Table 3.1: Hermitian bilinear operators for the spin- $\frac{1}{2}$ field.

3.5 Bilinear Covariants

Operators O for which $\bar{\psi} O \psi$ is hermitian and has well-defined properties under Lorentz transformations are of special interest since these are legitimate candidates to appear in \mathcal{L} for terms involving only fields and no derivatives. We have encountered $O = 1$ (scalar) and $O = \gamma^\mu$ already, but there are others. In the Dirac-Pauli representation,

$$\gamma^5 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (3.49)$$

while in general,

$$\gamma^{5\dagger} = \gamma^5, \quad \{\gamma^5, \gamma^\mu\} = 0, \quad \text{and} \quad (\gamma^5)^2 = 1. \quad (3.50)$$

The full list of hermitian operators for the spin- $\frac{1}{2}$ field is given in Table 3.1

3.6 Massless Fermions

In the (highly relativistic) limit $E \gg m$, we can put $m = 0$ in the Dirac equation to get two decoupled equations for the two-component spinors. For χ^+ along \vec{p} and χ^- opposite \vec{p} ,

$$E\chi^- = -\vec{\sigma} \cdot \vec{p} \chi^- \quad \text{and} \quad E\chi^+ = +\vec{\sigma} \cdot \vec{p} \chi^+, \quad (3.51)$$

so the first one represents a left-handed neutrino of energy E and momentum \vec{p} . Then the helicity states have simple representations:

$$u^\pm = \sqrt{E} \begin{pmatrix} \chi^\pm \\ \pm \chi^\pm \end{pmatrix} \quad \text{and} \quad v^\pm = \sqrt{E} \begin{pmatrix} \pm \chi^\pm \\ \chi^\pm \end{pmatrix}. \quad (3.52)$$

In this limit, the operators

$$P_R \equiv \frac{1}{2}(1 + \gamma^5) \quad \text{and} \quad P_L \equiv \frac{1}{2}(1 - \gamma^5) \quad (3.53)$$

act as right- and left-hand projection operators, so that

$$P_{R,L}u^\pm = u^\pm, \quad P_{R,L}v^\pm = v^\pm, \quad P_{R,L}u^\mp = 0, \quad P_{R,L}v^\mp = 0. \quad (3.54)$$

P_L and P_R satisfy the usual relations between projection operators:

$$P_i^2 = P_i, \quad P_L + P_R = 1, \quad P_L P_R = 0. \quad (3.55)$$

The γ^5 is thus called the *chirality* operator. It is diagonal in the Weyl representation. Consequently, in the Weyl representation, helicity is diagonalized in the extreme relativistic limit.

The chirality operator is very useful for keeping the Dirac spinor notation when writing a \mathcal{L} that differentiates between right- and left-handed fermions. A good example is the *charged current* weak interaction between a charged lepton ℓ and its $SU(2)$ partner neutrino ν_ℓ , where, in contrast to the V (vector) form of the electromagnetic current, we have the $V - A$ form of the weak current:

$$J^\mu = \bar{\psi}_\ell \gamma^\mu \frac{1}{2}(1 - \gamma^5)\psi_{\nu_\ell}. \quad (3.56)$$

This ensures that parity is maximally violated in such interactions, because

$$\frac{1}{2}(1 - \gamma^5)u_\nu = \begin{pmatrix} \sigma_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \chi^- \\ -\chi^- \end{pmatrix} = \begin{pmatrix} \chi^- \\ 0 \end{pmatrix}. \quad (3.57)$$

So, only the ν_L (and $\bar{\nu}_R$) are projected out: only the left-handed neutrinos and right-handed antineutrinos couple to their charged counterparts by weak interactions. Of course, if the neutrino mass is not strictly zero, then it is possible to perform a Lorentz transformation to change a ν_L to a ν_R .⁴

⁴Even if neutrinos are not exactly massless, it is possible to ensure that the weak interactions couple only to ν_L and $\bar{\nu}_R$ by requiring that the neutrinos be their own antiparticles. Such neutrinos are known as *Majorana neutrinos*. Such neutrinos are best treated on a basis of *Majorana spinors*, which are structured differently than Dirac spinors.

Chapter 4

Decay Widths and Scattering Cross Sections

We are now ready to calculate the rates of some simple scattering and decay processes. The former is expressed in terms of *cross section*, σ , which is a measure of the probability of a specific scattering process under some given set of initial and final conditions, such as momenta and spin polarization. The latter is expressed in terms of *lifetime*, τ , or, equivalently, *decay width*, Γ ($\propto \frac{1}{\tau}$), which is a measure of the probability of a specific decay process occurring within a given amount of time in the parent particle's rest frame. The calculation involves two steps:

1. Calculate the *amplitude*, \mathcal{M} , of the process. It is also often referred to as the *matrix element*, and denoted by \mathcal{M}_{fi} , to indicate that in a matrix representation of the transformation process, with the initial and final states as bases, this is the element that connects a particular final state f to a given initial state i . A process can be a combination of subprocesses, in which case, the total amplitude is the sum of the subprocess amplitudes.¹ Each simple (sub)process is represented by a unique *Feynman diagram*. Its amplitude is a point function in the phase space of all the particles involved, including any intermediate *propagator*, and depends on the nature of the *coupling* at each vertex (of the diagram). For a given diagram, the amplitude can be obtained by following the *Feynman rules* for combining the elements - a factor for each external line (representing a free particle in the initial or final state), one for each internal line (representing a virtual *propagator* particle), and one for each vertex point where the lines meet.
2. Integrate the amplitude over the allowed phase space to get the σ or Γ , as the case may be. The integral can be constructed, easily in principle, by following *Fermi's golden rule*, although its evaluation can be extremely

¹An amplitude is a Lorentz scalar, but generally complex-valued.

challenging except in the simplest of cases such as those we will encounter in this course.

This chapter describes the above rules and use them to calculate the decay rates and cross sections for some simple (and sometimes hypothetical) processes in quantum electrodynamics (QED).

4.1 Physical meaning of decay width

One of the most important characteristics of a particle is its lifetime. It depends, of course, on the available *decay modes or channels*, which are subject to conservation laws for appropriate quantum numbers, coupling strength of the decay process, and kinematic constraints. The lifetime of an individual particle cannot be predicted, but a statistical distribution can be specified for a large sample. Equivalently, one can express it in terms of the *decay rate*, Γ , which is the *probability per unit time* that a given particle will decay.

The probability that a single unstable entity will cease to exist as such after an interval is proportional to that interval. The constant of proportionality is called the decay rate. For complex unstable entities such as stars, living organisms, businesses, economies etc., any two are rarely “identical”, and each evolves in its own complex manner with time. Their decay rates depend on their constitution, age, and external factors, making it very difficult to estimate their lifetimes, even on average. Fortunately, that is not the case with elementary particles. Thus, for an ensemble of $N \rightarrow \infty$ identical particles, the change in the number after a time dt is

$$dN = -\Gamma N dt. \quad (4.1)$$

So, the expected number surviving after time t is

$$N(t) = N(0)e^{-\Gamma t}. \quad (4.2)$$

The time after which the ensemble is expected to shrink to $\frac{1}{e}$ of its original size is called the *lifetime*:

$$\tau = \frac{1}{\Gamma}. \quad (4.3)$$

If multiple decay modes are available, as is often the case, then one can associate a decay rate for each mode, and the total rate, will be the sum of the rates of the individual modes.

$$\Gamma_{\text{total}} = \sum_{i=1}^n \Gamma_i. \quad (4.4)$$

The particle’s lifetime is then given by

$$\tau = \frac{1}{\Gamma_{\text{total}}}. \quad (4.5)$$

In such cases, we are often interested in the *branching fractions*, i.e. the probabilities of the decay by individual modes. The branching fraction of mode i is

$$B_i = \frac{\Gamma_i}{\Gamma_{\text{total}}}. \quad (4.6)$$

Since the dimension of Γ is the inverse of time, in our system of *natural units*, it has the same dimension as mass (or energy). When the mass of an elementary particle is measured, the total rate shows up as the irreducible “width” of the shape of the distribution.² Hence the name *decay width*.

4.2 Physical meaning of scattering cross section

Consider the $2 \rightarrow n$ scattering process

$$ab \rightarrow cd \dots \quad (4.7)$$

The system of incoming particles labeled a, b constitute the initial state $|i\rangle$, and that of the outgoing particles labeled c, d, \dots constitute the final state $|f\rangle$.³ If a packet of a particles is made to pass head-on through a packet of b particles so that the overlap area is A , and the number of particles swept by that overlap area in the two packets are N_a and N_b respectively, then the number of scatterings, N_S is directly proportional to N_a and N_b , and inversely to A . The overall constant of proportionality is called the cross section, σ :

$$N_S = \sigma \frac{N_a N_b}{A}. \quad (4.8)$$

Thus, the cross section must have the same dimension as area. Cross sections in contemporary HEP experiments are typically measured in units of nanobarn (nb) to femtobarn (fb), where a *barn* is defined as

$$1b = 10^{-24} \text{ cm}^2 = 2.568 \text{ GeV}^{-2}. \quad (4.9)$$

As for decays, one is often more interested in various *differential (or exclusive) cross sections*, σ_i rather than the *total (or inclusive) cross section*, σ_{total} :

$$\sigma_{\text{total}} = \sum_{i=1}^n \sigma_i \quad (4.10)$$

For example, the total cross section of proton-antiproton collisions at a center-of-mass energy (\sqrt{s}), as in Tevatron Run 2, is huge,

$$\sigma(p\bar{p} \rightarrow X) \approx 75 \text{ mb}, \quad (4.11)$$

where X represents “anything”, but that for the most highly sought-after processes are small (duh!), e.g.

$$\sigma(p\bar{p} \rightarrow t\bar{t}X) \approx 7.5 \text{ pb}. \quad (4.12)$$

²As opposed to the *statistical and systematic uncertainties* in the measurement, which can be reduced, in principle, to zero by building an infinitely precise and accurate measuring device and collecting an infinite amount of data with it.

³Different labels are not intended to mean that the particles are necessarily different.

4.3 Calculation of widths and cross sections

The matrix element between the initial state $|i\rangle$ and the final state $|f\rangle$ is called the S matrix:

$$S_{fi} = (2\pi)^4 \delta^4(p_f - p_i) \mathcal{M}_{fi}, \quad (4.13)$$

where p_i is the total initial momentum, p_f the total final momentum, and the 4-dimensional δ function expresses the conservation of 4-momentum (E, \vec{p}) . The quantity \mathcal{M}_{fi} , called the (reduced) matrix element or amplitude of the process, contains the non-trivial physics of the problem, including spins and couplings. It is usually calculated by perturbative approximation.

The probability of the transition from $|i\rangle$ to $|f\rangle$ is given by

$$P_{i \rightarrow f} = \frac{S_{fi}}{\langle f|f\rangle \langle i|i\rangle}. \quad (4.14)$$

The rate of the transition is determined by Fermi's Golden Rule:⁴

$$\text{transition rate} = 2\pi |\mathcal{M}_{fi}|^2 \rightarrow (\text{phase space}). \quad (4.15)$$

4.3.1 The Golden Rule for Decays

For an n -body decay

$$i \rightarrow f_k; \quad k = 1, \dots, n \quad (4.16)$$

the differential decay rate is given by

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2m_i} \left(\prod_{k=1}^n \frac{d^3 \vec{p}_k}{(2\pi)^3 2E_k} \right) \times (2\pi)^4 \delta^4 \left(p_i - \sum_{k=1}^n p_k \right), \quad (4.17)$$

where p_k is the 4-momentum of the k th particle, and S is a product of statistical factors: $\frac{1}{m!}$ for each group of m identical particles in the final state.

Usually we are not interested in specific momenta of the decay products. So, the total decay rate is obtained by integrating Eq. 4.17. For a general 2-body decay, the total width is given by

$$\Gamma = \frac{S|\vec{p}|}{8\pi m_i} |\mathcal{M}|^2, \quad (4.18)$$

where $|\vec{p}|$ is the magnitude of the momentum of either outgoing particle in the parent's rest frame (this is fully determined by the masses of the 3 particles involved in the process), and \mathcal{M} is evaluated at the momenta required by the conservation laws.

⁴Derivation of this rule is outside the scope of this course, but can be found in any standard text of quantum field theory.

4.3.2 The Golden Rule for Scattering

Just as for the decay rate, for a $2 \rightarrow n$ scattering process

$$ij \rightarrow f_k; \quad k = 1, \dots, n \quad (4.19)$$

the differential cross section is given by

$$d\sigma = |\mathcal{M}|^2 \frac{S}{4\sqrt{(p_i \cdot p_j)^2 - (m_i m_j)^2}} \left(\prod_{k=1}^n \frac{d^3 \vec{p}_k}{(2\pi)^3 2E_k} \right) \times (2\pi)^4 \delta^4 \left(p_i + p_j - \sum_{k=1}^n p_k \right). \quad (4.20)$$

For a $2 \rightarrow 2$ process in the CM frame, this leads to

$$d\sigma = \frac{S}{64\pi^2 E_{\text{CM}}^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} |\mathcal{M}|^2 d\Omega, \quad (4.21)$$

where $|\vec{p}_f|$ is the magnitude of the momentum of either outgoing particle, $|\vec{p}_i|$ is the magnitude of the momentum of either incoming particle, and

$$d\Omega = \sin\theta d\theta d\phi \quad (4.22)$$

is the solid-angle element in which the final state particles scatter.

4.4 Feynman rules for calculating the amplitude

In the previous sections, the formulae for decay rates and scattering cross sections are given in terms of the amplitude \mathcal{M}_{fi} . Here we give the recipe to calculate $-i\mathcal{M}_{fi}$ for a given Feynman diagram for tree-level processes in QED:⁵

1. External lines:

- (a) For an incoming electron, positron, or photon, associate a factor u , \bar{v} , or e_μ , respectively.
- (b) For an outgoing electron, positron, or photon, associate a factor \bar{u} , v , or e_μ^* , respectively.

2. **Vertices:** For each vertex, include a factor of $ie\gamma^\mu$ for an electron or $-ie\gamma^\mu$ for a positron. Care must be exercised to get the overall sign for fermions correct.

3. Internal lines:

- (a) For an electron or a positron connecting two vertices, include a term

$$i \left(\frac{\not{p} + m}{p^2 - m^2 + i\varepsilon} \right). \quad (4.23)$$

⁵Only the vertex factor is different for QCD and weak interactions. We shall encounter them in due course.

(b) For an photon connecting two vertices, include a term

$$\frac{i g_{\mu\nu}}{q^2 + i\varepsilon}. \quad (4.24)$$

(c) Integrate over all undetermined internal momenta.

Chapter 5

Electromagnetic Scattering

In this chapter, we will apply the methods discussed in the preceding chapters to compute cross sections of some simple electromagnetic processes involving charged leptons (and photons, of course). We will restrict ourselves to initial and final states involving no more than two particles, and to the lowest order in perturbative calculations, (i.e., to tree-level Feynman diagrams only). The coupling at each vertex is proportional to the electric charge e of the fermion (we will only deal with charged leptons in this chapter). At low energies, this is manifested in terms of the fine structure constant

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137}. \quad (5.1)$$

The value of α depends on the energy at which it is measured. The above value corresponds to the $E \rightarrow m_\ell$ limit. Moreover, up to energies a few GeV below M_W ($=80$ GeV) weak interactions can be safely ignored. Strong interactions are irrelevant in this context since leptons carry no color charge. Each higher order diagram will make a contribution proportional to α to the matrix element, which will subsequently have to be squared to get the cross section. Since leading-order diagrams for $2 \rightarrow 2$ processes involve 2 vertices, our calculations will actually hold good to a few parts per 10^{-4} level at low energies.

Before looking at some examples of electromagnetic scattering between two fermions, let us recall the conserved current density in Eq. 3.33. Putting in the (free) fermion wave functions given by Eq. 3.43 in the initial and final states, we see that the *transition current* at each vertex is

$$J_{fi}^\mu = -e\bar{\psi}_f\gamma^\mu\psi_i = -e\bar{u}_f\gamma^\mu u_i \exp(i(p_f - p_i) \cdot x), \quad (5.2)$$

where u_i and \bar{u}_f are the fermion spinors in the initial and final states, respectively. Such a factor at each end of a photon propagator is exactly what one would get by following the Feynman rules summarized in the last chapter.

It is instructive to note that if we had a scalar (i.e. spin-0) charged particle instead of the spin- $\frac{1}{2}$ fermion, then the transition current could be obtained

by simply dropping the spinors and the γ^μ from Eq. 5.2, while retaining the normalization:

$$J_{fi}^\mu = -e(p_f + p_i)^\mu \exp(i(p_f - p_i) \cdot x). \quad (5.3)$$

Operating on the electromagnetic vector field A^μ , such a current would give the interaction only of the electric charge and the photon. The difference between the two transition currents is due to the magnetic moment of the fermion. Indeed, the lepton-photon interaction at a vertex can be expressed in the form

$$\bar{u}_f \gamma^\mu u_i = \frac{1}{2m} \bar{u}_f ((p_f + p_i)^\mu + i\sigma^{\mu\nu} (p_f - p_i)_\nu) u_i, \quad (5.4)$$

known as the *Gordon decomposition* (into charge and magnetic moment parts).

5.1 Electron-Muon Scattering

Consider the process $e^- \mu^- \rightarrow e^- \mu^-$ shown in Fig. 5.1.

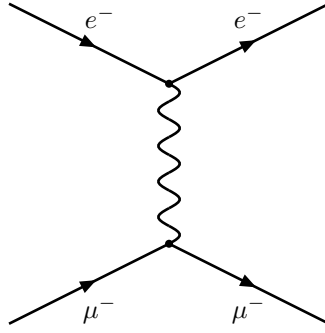


Figure 5.1: The Feynman diagram for $e^- \mu^- \rightarrow e^- \mu^-$.

The matrix element representing the exchange of a single photon between the electron and the muon currents is given, as prescribed by the Feynman rules, by

$$\mathcal{M}_{fi} = -J_{e^-}^\alpha \frac{g_{\alpha\beta}}{q^2 + i\varepsilon} J_{\mu^-}^\beta = -e^2 \frac{\bar{u}(p_f^{e^-}) \gamma^\alpha u(p_i^{e^-}) \bar{u}(p_f^{\mu^-}) \gamma_\alpha u(p_i^{\mu^-})}{q^2 + i\varepsilon}, \quad (5.5)$$

where the photon momentum q^μ is determined by the momentum conservation condition

$$q = p_i^{e^-} - p_f^{e^-} = p_f^{\mu^-} - p_i^{\mu^-}. \quad (5.6)$$

Note that switching the e^- and the μ^- in the above expression is of no consequence since the direction of the photon-mediated momentum transfer is irrelevant and will have to be integrated over to get the cross section anyway.

For the cross section, we'll need $|\mathcal{M}_{fi}|^2$, which can be written as

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4} (L_{e^-})^{\alpha\beta} (L_{\mu^-})_{\alpha\beta} \quad (5.7)$$

in terms of the tensors

$$(L_{\ell^-})_{\alpha\beta} \equiv \frac{1}{e^2} (J_{\ell^-})_{\alpha} (J_{\ell^-})_{\beta}^*. \quad (5.8)$$

Evaluation of these tensors are actually much easier than it may appear at this point. A general method exists to facilitate such calculations, and it does not depend on any specific representation of the γ matrices or spinors.

Our task is to evaluate the matrix element $\bar{u}(p_2)O_{\alpha}u(p_1)$, where O_{α} is a 4×4 matrix, made of momenta and γ matrices, in the spinor space. Here α represents a collection of Lorentz indices: no index represents a scalar, one a vector, and so on. Squaring the matrix element we get a tensor, such as those in Eq. 5.8, which will make up a part of $|\mathcal{M}_{fi}|^2$, as in Eq. 5.7. Since the complex conjugate and Hermitian conjugate are the same for a complex number, the tensor is

$$\begin{aligned} (\bar{u}(p_2)O_{\alpha}u(p_1))(\bar{u}(p_2)O_{\beta}u(p_1))^* &= (\bar{u}(p_2)O_{\alpha}u(p_1))(u^{\dagger}(p_1)O_{\beta}^{\dagger}\gamma_0u(p_2)) \\ &= (\bar{u}(p_2)O_{\alpha}u(p_1))(\bar{u}(p_1)\gamma_0O_{\beta}^{\dagger}\gamma_0u(p_2)). \end{aligned} \quad (5.9)$$

We define $\bar{O}_{\beta} \equiv \gamma_0 O^{\dagger} \gamma_0$.¹ Then, expressing the matrix multiplications in terms of the element indices, we get

$$\begin{aligned} (\bar{u}(p_2)O_{\alpha}u(p_1))(\bar{u}(p_2)O_{\beta}u(p_1))^* &= \bar{u}(p_2)_i (O_{\alpha})_{ij} u(p_1)_j (\bar{u}(p_1)_k (\bar{O}_{\beta})_{kl} \gamma_0 u(p_2))_l \\ &= (u(p_2)_l \bar{u}(p_2)_i) (O_{\alpha})_{ij} (u(p_1)_j \bar{u}(p_1)_k) (\bar{O}_{\beta})_{kl} \end{aligned} \quad (5.10)$$

² If we define a matrix whose m, n component is $u(p_1)_m \bar{u}(p_1)_n$, the above expression is just the trace of a matrix that is the product of 4 matrices. Thus, we can write the squared matrix element of Eq. 5.9, without explicit use of the indices, as

$$(\bar{u}(p_2)O_{\alpha}u(p_1))(\bar{u}(p_2)O_{\beta}u(p_1))^* = \text{Tr}(u(p_2)\bar{u}(p_2)O_{\alpha}u(p_1)\bar{u}(p_1)\bar{O}_{\beta}). \quad (5.11)$$

So, with simple expressions for the 4×4 matrix $u(p)\bar{u}(p)$, our calculations are reduced to taking traces using the relations listed in Sec. 3.2.

Now we are ready to handle the summing of spinors in the initial and final states. Commonly, the incoming beams are unpolarized and the spin polarization of the outgoing particles are undetermined. In such a case, we must take

¹For $O_{\beta} = \gamma_{\mu}, \gamma_{\mu}\gamma_5, i\gamma_5, \sigma_{\mu\nu}, \bar{O}_{\beta} = O_{\beta}$.

²Einstein summation is implied, but the Latin indices label the components of the 4-component Dirac spinor, and hence run from 1 to 4. They have nothing to do with the components of a Lorentz 4-vector in the Minkowski space. X^i and X_i are one and the same.

the average of the helicities in the initial state and sum over those in the final state, which appear only in the 4×4 matrices constructed from the spinors. The matrices O_α, \bar{O}_β do not depend on helicity. The sums can be derived from the spinors in Eq. 3.48, which simply yield

$$\sum_{\lambda=1,2} u(p, \lambda) \bar{u}(p, \lambda) = \not{p} + m \quad (5.12)$$

for fermions, and

$$\sum_{\lambda=1,2} v(p, \lambda) \bar{v}(p, \lambda) = \not{p} - m \quad (5.13)$$

for antifermions, where the helicity λ is now indicated explicitly in the spinor. Thus, after summing over the helicities of all the spin- $\frac{1}{2}$ particles in Eq. 5.11, we get

$$(\bar{u}(p_2) O_\alpha u(p_1)) (\bar{u}(p_2) O_\beta u(p_1))^* = \text{Tr}((\not{p}_2 + m_2) O_\alpha (\not{p}_1 + m_1)). \quad (5.14)$$

For each antiparticle, each $\not{p} + m$ is replaced by $\not{p} - m$. These expressions appear repeatedly in calculations involving spin- $\frac{1}{2}$ particles in the initial and final states of a process. They embody the linearity requirement of quantum mechanics, where \mathcal{M}_{fi} must contain a single power of each spinor leading to a product of the form $u(p) \bar{u}(p)$ in the square of the matrix element.

If the helicity of a spin- $\frac{1}{2}$ particle is fixed in the initial state or determined in the final state, then the matrix element will be a function of its helicity λ . One could use explicit spinors to calculate this, but an alternative form has been derived that can be used simply in trace calculations. Consider the spin- $\frac{1}{2}$ particle in its rest frame. Define a spin 3-vector in this frame as $\vec{s} = \chi^\dagger \vec{\sigma} \chi$, where χ is normalized so that $\chi^\dagger \chi = 1$. We can construct $s^\mu = (0, \vec{s})$, which transforms as a Lorentz 4-vector. So, $p \cdot s = 0$ is a Lorentz scalar. Using the spinors in Eq. 3.48, we can show that

$$\begin{aligned} u(p, \lambda) \bar{u}(p, \lambda) &= (\not{p} + m) \frac{(1 + \gamma_5 \not{\hat{e}}(\lambda))}{2} \\ v(p, \lambda) \bar{v}(p, \lambda) &= (\not{p} - m) \frac{(1 + \gamma_5 \not{\hat{e}}(\lambda))}{2}. \end{aligned} \quad (5.15)$$

Defining $\hat{e}(\lambda)$ as a unit vector along \vec{p} for positive helicity and opposite \vec{p} for negative helicity,

$$s = \left(\frac{\hat{e}(\lambda) \cdot \vec{p}}{m}, \frac{\hat{e}(\lambda) E}{m} \right) \quad (5.16)$$

gives $s(\lambda)$ for the two helicity choices. These spin-dependent expressions become indispensable in the studies of weak interactions, where parity violation makes helicity selection a quintessential feature.

Returning to the calculation of $e^- \mu^-$ scattering cross section with unpolarized incoming beams, and no discrimination on helicities of the outgoing particles, we take the average over the initial helicities (which amounts to summing

and dividing by 2) and sum over the final ones to get

$$(L_{\ell^-})_{\alpha\beta} = \frac{1}{2} \text{Tr} \left((\not{p}_f^{\ell^-} + m_\ell) \gamma_\alpha (\not{p}_i^{\ell^-} + m_\ell) \gamma_\beta \right), \quad (5.17)$$

where $\ell = e, \mu$.

The expression on the RHS has two terms with an even number of γ matrices, which can be evaluated by the trace formulae of Sec. 3.2:

$$\begin{aligned} (L_{\ell^-})_{\alpha\beta} &= \frac{1}{2} \left(\text{Tr}(\not{p}_f^{\ell^-} \gamma_\alpha \not{p}_i^{\ell^-} \gamma_\beta) + m_\ell^2 \text{Tr}(\gamma_\alpha \gamma_\beta) \right) \\ &= 2 \left((p_f^{\ell^-})_\alpha (p_i^{\ell^-})_\beta + (p_f^{\ell^-})_\beta (p_i^{\ell^-})_\alpha - (p_f^{\ell^-} \cdot p_i^{\ell^-} - m_\ell^2) g_{\alpha\beta} \right). \end{aligned} \quad (5.18)$$

Multiplying the tensors for the electron and the muon, we get finally the unpolarized result for $|\mathcal{M}_{fi}|^2$, summed over final spins:

$$\begin{aligned} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_{fi}|^2 &= \frac{8e^4}{q^4} \left((p_f^{e^-} \cdot p_f^{\mu^-}) (p_i^{e^-} \cdot p_i^{\mu^-}) + (p_f^{e^-} \cdot p_i^{\mu^-}) (p_i^{e^-} \cdot p_f^{\mu^-}) \right. \\ &\quad \left. - m_e^2 (p_i^{\mu^-} \cdot p_f^{\mu^-}) - m_\mu^2 (p_i^{e^-} \cdot p_f^{e^-}) + 2m_e^2 m_\mu^2 \right). \end{aligned} \quad (5.19)$$

This final result is a Lorentz invariant that is symmetric under the exchange $e^- \leftrightarrow \mu^-$.

In the extreme relativistic limit, the terms involving the particle masses can be neglected. This leads to the approximation

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 \approx \frac{8e^4}{(p_i^{e^-} - p_f^{e^-})^4} \left((p_f^{e^-} \cdot p_f^{\mu^-}) (p_i^{e^-} \cdot p_i^{\mu^-}) + (p_f^{e^-} \cdot p_i^{\mu^-}) (p_i^{e^-} \cdot p_f^{\mu^-}) \right). \quad (5.20)$$

Also, in this limit, the Mandelstam variables of Eqs. 1.44, 1.45, and 1.46 become

$$\begin{aligned} s &= (p_i^{e^-} + p_i^{\mu^-})^2 = (p_f^{e^-} + p_f^{\mu^-})^2 \approx 2p_i^{e^-} \cdot p_i^{\mu^-} \approx 2p_f^{e^-} \cdot p_f^{\mu^-}, \\ t &= (p_f^{e^-} - p_i^{e^-})^2 = (p_f^{\mu^-} - p_i^{\mu^-})^2 \approx -2p_i^{e^-} \cdot p_f^{e^-} \approx -2p_i^{\mu^-} \cdot p_f^{\mu^-}, \\ u &= (p_f^{\mu^-} - p_i^{e^-})^2 = (p_f^{e^-} - p_i^{\mu^-})^2 \approx -2p_f^{\mu^-} \cdot p_i^{e^-} \approx -2p_f^{e^-} \cdot p_i^{\mu^-}. \end{aligned} \quad (5.21)$$

Thus, for scattering of unpolarized electrons and muons at $E \gg m_\mu (\approx 200m_e)$, we have a compact expression for the squared matrix element:

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 \approx 2e^4 \frac{(s^2 + u^2)}{t^2}. \quad (5.22)$$

Note that this diverges in the limit of $t \rightarrow 0$ (no momentum transfer).

What remains in the calculation of the cross section is the integration over the phase space. For final states consisting of any two given particles, the only spatial variable is the scattering angle (evaluated in the center of mass, unless otherwise specified.) The differential cross section is obtained by substituting the above $|\mathcal{M}_{fi}|^2$ in Eq. 4.21.

5.2 e^+e^- annihilation to $\mu^+\mu^-$

The Feynman diagram for the process $e^+e^- \rightarrow \mu^+\mu^-$ is shown in Fig. 5.2.

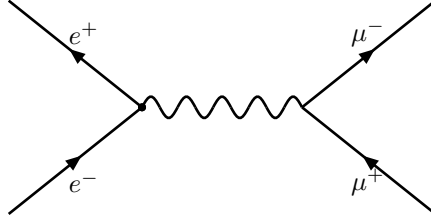


Figure 5.2: The Feynman diagram for $e^+e^- \rightarrow \mu^+\mu^-$.

The squared matrix element can be obtained simply by “crossing” the result for $e^-\mu^- \rightarrow e^-\mu^-$, which amounts to the interchange $s \leftrightarrow t$ in Eq. 5.22. Thus,

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 \approx 2e^4 \frac{(t^2 + u^2)}{s^2}, \quad (5.23)$$

where now $e^+e^- \rightarrow \mu^+\mu^-$ is the s -channel process.

We can calculate s , t , and u in terms of the center-of-mass energy E_{CM} and the scattering angle θ between the outgoing muons and the incoming electrons (*Exercise: derive these relations.*)

$$\begin{aligned} s &= E_{\text{CM}}^2, \\ t &= \frac{1}{2}E_{\text{CM}}^2(1 - \cos\theta), \\ u &= \frac{1}{2}E_{\text{CM}}^2(1 + \cos\theta). \end{aligned} \quad (5.24)$$

Therefore,

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 \approx e^4(1 + \cos^2\theta), \quad (5.25)$$

and the differential cross section is

$$\frac{d\sigma}{d\Omega_{\text{CM}}} = \frac{2e^4}{64\pi^2s} \left(\frac{1}{2}(1 + \cos^2\theta) \right) = \frac{\alpha^2}{4s}(1 + \cos^2\theta), \quad (5.26)$$

where we have made the substitution $\alpha = \frac{e^2}{4\pi}$. Note that, unlike $e^-\mu^- \rightarrow e^-\mu^-$, this cross section shows only a mild peaking in the forward direction.

The total interaction cross section can be obtained by integrating over θ :

$$\sigma(e^+e^- \rightarrow \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}. \quad (5.27)$$

It falls as the inverse-square of the CM energy.

This result has been verified by experimental data at E_{CM} 's up to several tens of GeV, until effects of weak interaction become significant. Up to that point, the error incurred by ignoring higher order terms in the perturbative calculations (corresponding Feynman diagrams have multiple internal lines) is much smaller than the experimental resolution.

Chapter 6

Gauge Theories in Particle Physics

In this chapter we will put to use the mathematical formalism of groups learnt in Chapter 2 to establish the $U(1)$, $SU(2)$, and $SU(3)$ symmetries of the Standard Model Lagrangian that give rise to the electromagnetic, weak, and strong interactions, respectively.

6.1 Gauge Invariance in Quantum Theory

We saw in Section 2.6 how a local phase transformation of the fermion wave function $\psi \rightarrow e^{iq\theta(x)}\psi$ along with a simultaneous redefinition of the electromagnetic field $A^\mu \rightarrow A^\mu + \partial_\mu\theta(x)$ leaves the Lagrangian invariant. The result can, in fact, be interpreted to say that the local gauge invariance (*phase invariance* would be more appropriate, but we'll honor the historical legacy) requires the presence of a field $A^\mu = (V, \vec{A})$. In *Example 1* of Sec. 2.6, we solved for the scalar field as a static function of space. However, we could have written it just as well as a plane wave normalized to a single quantum of a definite energy ω and momentum \vec{k} :

$$\phi = \frac{1}{\sqrt{2\omega}} \left(a e^{i(\vec{k}\cdot\vec{r}-\omega t)} + a^\dagger e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \right) = \frac{1}{\sqrt{2\omega}} \left(a e^{ik^\mu x_\mu} + a^\dagger e^{-ik^\mu x_\mu} \right), \quad (6.1)$$

where a^\dagger creates quanta associated with the field ϕ and a destroys them.

In a similar fashion, the solution for the vector field A^μ can be expanded in terms of particle creation and destruction operators. Consequently, there must be an associated particle, and since the field is described by a 4-vector, it must be associated with a vector, i.e. spin-1, particle. Since the same effect occurs with any charged particle (not just fermions), the interaction of the new particle, which we interpret as being the photon, is the same with any charged particle - it is a universal interaction. Thus, phase invariance of the theory for electrically charged particles requires that there must be a photon and an electromagnetic

interaction of precisely the observed kind. Note, however, that the numerical value of the charge is undetermined.

In a sense, the existence and form of the electromagnetic interaction has been derived. If a particle carries a charge and the theory is invariant under certain phase transformations, then associated fields, called *gauge fields*, and associated spin-1 particles, called *gauge bosons* must exist. These allow us to write the associated interaction Lagrangians. There are three known gauge transformations under which the theory is invariant, and three associated sets of gauge bosons. Why these three and not others, or whether there are others, is not known.¹

The Lagrangian (and wave equation) for a free charged particle can be modified to describe its interaction with a photon by replacing the ordinary 4-gradient with the *covariant derivative*:

$$D^\mu = \partial^\mu - iqA^\mu, \quad (6.2)$$

² This concept can be generalized to other (i.e. non-electromagnetic) charges as well. Suppose we want the theory to be invariant under a transformation where particle states change as

$$\psi' = U\psi \quad (6.3)$$

for some U . We want to define

$$D^\mu = \partial^\mu - igA^\mu, \quad (6.4)$$

where A^μ is the interacting field that has to be added to keep the theory invariant, but now we don't know how A^μ itself transforms. We also want

$$D^{\mu'}\psi' = U(D^\mu\psi), \quad (6.5)$$

i.e.

$$(\partial^\mu - igA^{\mu'})U\psi = U(\partial^\mu - igA^\mu)U\psi. \quad (6.6)$$

This can be solved for $A^{\mu'}$:

$$-igA^{\mu'}U\psi = -\partial^\mu(U\psi) + U\partial^\mu\psi - igUA^\mu\psi = -(\partial^\mu U)\psi - igUA^\mu\psi. \quad (6.7)$$

Since each term acts on an arbitrary state ψ , we can drop the ψ and multiply from the right by U^{-1} , so

$$A^{\mu'} = -\frac{i}{g}(\partial^\mu U)U^{-1} + UA^\mu U^{-1}. \quad (6.8)$$

This is how A^μ must transform for any U . (*Exercise: verify that this gives the expected answer for $g = q$ and $U = e^{-iq\theta}$*)

Equation 6.8 is very general and remains valid if U a matrix, rather than a scalar, in an internal space. The A^μ is also a matrix, so the order of multiplication is important.³

¹Another way to phrase the interpretation of what we have observed is that we cannot distinguish between the effects of a local change in phase convention and the effects of a new vector field.

²For charged fermions, this can be seen by comparing Eqs. 3.31 and 3.35.

³Of course, if U and A^μ are not matrices, then $UA^\mu U^{-1} = A^\mu$, as is the case in electromagnetism.

6.2 Strong Isospin: an example of $SU(2)$

Let us take for example the strong isospin symmetry of nucleons, pions, and other hadrons, that plays an important role in the understanding of nuclei and of hadrons. This will serve as an important aid in “visualizing” the more fundamental weak isospin symmetry, which is what we’re really after.

Consider the neutron (n) and the proton (p). Their masses,

$$m_n = 939.57 \text{ MeV}, \quad m_p = 938.27 \text{ MeV}, \quad (6.9)$$

differ by only $\sim 0.15\%$. No other particles have a similar mass. Both form nuclei and interact similarly. The only obvious difference is that the proton carries an electric charge and the neutron does not. However, their interactions in the nucleus are strong interactions. Strong interactions are not sensitive to electric charge and are ~ 100 times stronger than electromagnetic ones. So, the electric charge is not of much consequence.

This kind of reasoning led to the idea of picturing n and p as two states of the same entity, a nucleon, N . One could associate an *internal quantum space* called the *strong isospin space*, where the nucleon points in some direction: “up” if it is a p , “down” if n . If strong interactions do not distinguish between a n and a p , it follows that the theory that describes strong interactions is invariant under rotations in the strong isospin space.⁴

Since there are two nucleon states, it is like spin-up and spin-down. So, we can try to put the p and n as states of a spin-like, or $SU(2)$, doublet:

$$N = \begin{pmatrix} p \\ n \end{pmatrix}. \quad (6.10)$$

Another example of a hadron classified as states in $SU(2)$ multiplets is the pion, which has states π^\pm and π^0 , with masses $m_{\pi^\pm} = 139.57 \text{ MeV}$, $m_{\pi^0} = 134.96 \text{ MeV}$. It can be represented as an isospin-1 state, i.e. a triplet:

$$\pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}, \quad (6.11)$$

with charge states

$$\begin{aligned} \pi^\pm &= \frac{1}{\sqrt{2}}(\pi_1 \pm i\pi_2), \\ \pi^0 &= \pi_3. \end{aligned} \quad (6.12)$$

As for nucleons, the pion states have the same strong interactions, and differences in mass and interactions of sizes typical of electromagnetic effects. Later we will see how W bosons can be put in a similar classification under weak isospin.

⁴This can only be an approximate symmetry of nature since it is broken, however slightly, by electromagnetic interactions.

For the strong isospin to be a valid symmetry, it must also hold for interactions. Let us try to write an interaction Lagrangian to describe the most general pion-nucleon interaction at the lowest order. Let p^\dagger create a proton or destroy and antiproton, π^+ destroy a π^+ or create a π^- , n destroy a neutron or create an antineutron and so on. Then the most general 3-particle interaction Lagrangian that conserves nucleon number and electric charge is

$$\mathcal{L}_{\text{int}} = g_{pn}p^\dagger n\pi^+ + g_{np}n^\dagger p\pi^- + g_{pp}p^\dagger p\pi^0 + g_{nn}n^\dagger n\pi^0. \quad (6.13)$$

For this Lagrangian to be invariant under rotations in the isospin space, certain relations must hold among the g 's. For example, invariance under $p \rightarrow n$ rotation requires $g_{pp} = \pm g_{nn}$. Since π is a vector in the isospin space, we must make a vector from the nucleon, so the Lagrangian, made of the scalar product of the two, will be invariant. This is achieved by forming the vector $N^\dagger \vec{\sigma} N$, where σ_i are the Pauli spin matrices. Then

$$\mathcal{L}_{\text{int}} = g (N^\dagger \vec{\sigma} N) \cdot \vec{\pi} = g N^\dagger \vec{\sigma} \cdot \vec{\pi} N \quad (6.14)$$

is invariant under rotations in the isospin space since it is the scalar product of two vectors in that space. We can write the scalar product in an expanded form:

$$\begin{aligned} \vec{\sigma} \cdot \vec{\pi} &= \sigma_1 \pi_1 + \sigma_2 \pi_2 + \sigma_3 \pi_3 \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pi_1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \pi_2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pi_3 \\ &= \begin{pmatrix} \pi_3 & \pi_1 - i\pi_2 \\ \pi_1 + i\pi_2 & -\pi_3 \end{pmatrix} \\ &= \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ -\sqrt{2}\pi^- & \pi^0 \end{pmatrix}. \end{aligned} \quad (6.15)$$

Then

$$\begin{aligned} N^\dagger \vec{\sigma} \cdot \vec{\pi} N &= \begin{pmatrix} p^\dagger & n^\dagger \end{pmatrix} \begin{pmatrix} \pi^0 & -\sqrt{2}\pi^+ \\ -\sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix} \\ &= \begin{pmatrix} p^\dagger & n^\dagger \end{pmatrix} \begin{pmatrix} \pi^0 p - \sqrt{2}\pi^+ n \\ -\sqrt{2}\pi^- p - \pi^0 n \end{pmatrix} \\ &= p^\dagger p \pi^0 - \sqrt{2} p^\dagger n \pi^+ - \sqrt{2} n^\dagger p \pi^- - n^\dagger n \pi^0. \end{aligned} \quad (6.16)$$

Thus we see that for the interactions to be invariant under rotations in the strong isospin space, the couplings g_{pn} , g_{np} , g_{pp} , g_{nn} must be in ratios $1 : 1 : -\frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}}$.

This technique of writing interactions invariant under rotations in internal spaces to obtain the form of the interaction is used extensively. For weak isospin, it is the W bosons, rather than pions, that have isospin 1. That strong isospin is a nearly good symmetry turns out to be fortuitous, rather than a fundamental feature of nature.

6.3 Non-Abelian Gauge Theories

We can now put together the ideas of internal spaces and of phase invariance. For the moment, let us continue with the nucleon example. We can write a phase transformation where a rotation mixing the proton and neutron states is expressed as a unitary operator in the isospin space,

$$\begin{pmatrix} p' \\ n' \end{pmatrix} = e^{i\vec{\epsilon} \cdot \frac{\vec{\sigma}}{2}} \begin{pmatrix} p \\ n \end{pmatrix}. \quad (6.17)$$

The σ_i are the Pauli matrices and ϵ_i are three parameters that specify the rotation. We can expand the exponential as a power series. Since $\sigma_i^2 = 1$, all powers of any Pauli matrix are either itself or the unit matrix. Note, however, that the order of successive transformations matters, since the rotations do not commute. Formally, this is expressed by the commutator $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$. Whenever the order of transformations matters, they are called non-Abelian transformations.

We could equally well consider particles in a multiplet of any group, and demand invariance under the appropriate transformation. If particles a_1 , a_2 , and a_3 carry quantum numbers in an $SU(3)$ space, we could write

$$\begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} = e^{i\vec{\alpha} \cdot \frac{\vec{\lambda}}{2}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad (6.18)$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_8)$ are the eight rotation parameters, and λ_i , ($i = 1, 2, \dots, 8$) are the Gell-Mann matrices. Quarks possess such a degree of freedom. It is called *color* because some of its properties are analogous to those of colors, although it has no connection to anything in classical physics nor with anything we experience in the everyday world.

As of now, there is no theoretical principle that tells us what internal spaces to examine. Each internal space where particles carry non-trivial quantum numbers leads to an interaction between particles, mediated by a new set of gauge bosons. In the Standard Model, the complete set of spaces comprises of the $SU(3)$ color space, and the $SU(2)$ and $U(1)$ electroweak spaces. These have been discovered empirically, and do an amazingly good job of describing the all fundamental phenomena and features of the world we live in. Next, we will examine the implications of demanding invariance under transformations

in these spaces, but no one yet understands why it is these particular ones that apply and not others. Having gained some insight into the familiar case of the proton and the neutron, let us next examine the quarks and the leptons and their weak isospin.

6.4 Gauge Theories for Quarks and Leptons

Suppose that the quarks and leptons can be put in multiplets of a (*weak*) isospin space, and that the theory should be invariant under transformations of the form of Eq. 6.17. Proceeding as before, we demand invariance under local phase transformations. That is, technically, we make the parameters functions of space and time, $\epsilon_i(x^\mu)$ or $\alpha_i(x^\mu)$. That guarantees we can choose how we define the phase of the quark and lepton states at each point of space-time, rather than having a choice here fix how the phase must be defined somewhere else of sometime later. A theory with a local non-Abelian phase invariance is called a Yang-Mills gauge theory.

No free particle can have an invariance under a non-Abelian gauge transformation, since the derivatives in the wave equation will act on $\epsilon_i(x^\mu)$. We are led again to define a covariant derivative. All the logic of Sec. 6.1 carries over, but now instead of a function $q\theta(x)$ in the exponent of the wave function, we have a function $\epsilon_i\sigma_i$ that transforms non-trivially under a $SU(2)$ group or a function $\alpha_i\lambda_i$ that transforms non-trivially under a $SU(3)$ group. The states describe leptons or quarks.

To define the covariant derivative D^μ , it is necessary for the $SU(2)$ case to introduce a set of three fields, each of which behaves as a 4-vector under Lorentz transformations, in order that we can write a term that transforms as ∂^μ does. Before we needed one A^μ ; now we need a W_i^μ for each σ_i . We can define

$$D^\mu = \partial^\mu - ig_2 \frac{\vec{\sigma}}{2} \vec{W}^\mu. \quad (6.19)$$

This is the generalization of the Abelian case Eq. 6.4, to include the non-Abelian transformations. If both transformations were relevant, the appropriate terms would add in D^μ . The coupling g_2 is an arbitrary factor which will determine the interaction strengths. The W_i^μ must be introduced if the theory is to be invariant under weak isospin transformations. Since they correspond to a particle transforming under space rotations as a vector, they should be realized as spin-1 particles, like the photon. Since σ_i is there, Eq. 6.19 is a 2×2 matrix equation.

We want to find how W_i^μ changes under a gauge transformation, since we have no previous answer analogous to $A^\mu \rightarrow A^\mu + \partial_\mu\theta(x)$ in the Abelian case. We start with the basic physics requirement that

$$D^{\mu'}\psi' = e^{i\vec{\epsilon}(x)\cdot\frac{\vec{\sigma}}{2}} D^\mu\psi, \quad (6.20)$$

since ψ itself transforms that way. Assume that W_i^μ transforms so that

$$W_i^{\mu'} = W_i^\mu + \delta W_i^\mu, \quad (6.21)$$

and we want to solve for δW_i^μ . The derivation is quite similar to the steps followed in Eq. 6.5 through Eq. 6.8, and leads to

$$\sigma_i \delta W_i^\mu = \frac{1}{g_2} (\partial^\mu \epsilon_i) \sigma_i + \frac{i}{2} \epsilon_i W_j^\mu [\sigma_i \sigma_j - \sigma_j \sigma_i]. \quad (6.22)$$

Recognizing the commutator as $2i\varepsilon_{ijk}\sigma_k$, this becomes

$$\sigma_i \left(\delta W_i^\mu - \frac{1}{g_2} \partial^\mu \epsilon_i - \varepsilon_{ijk} \epsilon_j W_k^\mu \right) = 0, \quad (6.23)$$

so we can conclude

$$\delta W_i^\mu = \frac{1}{g_2} \partial^\mu \epsilon_i - \varepsilon_{ijk} \epsilon_j W_k^\mu = 0. \quad (6.24)$$

We will not use Eq. 6.24 in further derivations, though it would be useful in a more advanced treatment. For us, it demonstrates how a fully gauge invariant non-Abelian theory can be constructed.

In Eq. 6.19 the covariant derivative is written with the understanding that it will act on the souplet representation of $SU(2)$. That is appropriate for us as we will put left-handed fermions in such doublets. We have implicitly noted that by labeling the couplig g_2 . Two generalizations are necessary.

First, though still in the internal $SU(2)$ weak isospin space, D^μ could act on a state in a different representation. If ψ is a state of weak isospin t with $2t + 1$ components, let \vec{T} be the $(2t + 1) \times (2t + 1)$ matrix operator representation in that basis. Then

$$D^\mu = \partial^\mu - ig_2 \vec{T} \cdot \vec{W}^\mu. \quad (6.25)$$

For spin- $\frac{1}{2}$, $\vec{T} = \frac{\vec{\sigma}}{2}$. We will interchangeably write $\vec{T} \cdot \vec{W}^\mu$ or $T_i W_i^\mu$, where summation over $i = 1, 2, 3$ is implied in the latter.

Second, we could consider a different internal space, where the interactions are invariant under another set of transformations. For a $SU(n)$ invariance, with group generators \vec{F} in an $(n^2 - 1)$ -dimensional space, and $[F_i, F_j] = ic_{ijk} F_k$, the appropriate D^μ to act on the n -dimensional matter state ψ is

$$D^\mu = \partial^\mu - ig_n \vec{F} \cdot \vec{G}^\mu, \quad (6.26)$$

where the G_μ are the $(n^2 - 1)$ gauge bosons that must be introduced to have a gauge-invariant theory. We will interchangeably write $\vec{F} \cdot \vec{G}^\mu$ or $F_a G_a^\mu$, where summation over $a = 1, 2, \dots, 8$ is implied in the latter for $SU(3)$.

Apparently nature also knows about a $SU(3)$ internal space, which is called the ‘‘color’’ space as we have already mentioned, as well as the $SU(2)$ isospin space. The appropriate generators are the Gell-Mann matrices λ_i described in Section 2.3.

By adding several terms to ∂^μ we can guarantee that we obtain a covariant derivative D^μ that will allow us to write Lagrangians (and therefore equations) that are invariant under gauge transformations, simultaneously or separately, in

all the internal spaces. the full covariant derivative that we are presently aware of can be written as

$$D^\mu = \partial^\mu - ig_1 \frac{Y}{2} B^\mu - ig_2 \frac{\sigma_i}{2} W_i^\mu - ig_3 \frac{\lambda_a}{2} G_a^\mu. \quad (6.27)$$

The couplings are arbitrary real numbers. For the Abelian $U(1)$ symmetry we have written the field that must be introduced as B^μ rather than as the electromagnetic field A^μ , since we do not know ahead of time that nature's $U(1)$ invariance corresponds precisely to electromagnetism. We will use physics arguments to make this association later. The $U(1)$ term has been written with a *generator* Y in a form analogous to the other terms. For $U(1)$, Y is just a number, though it can depend on the states on which D^μ operates. Y is called the $U(1)$ *hypercharge* generator.

It is worth emphasizing that for the non-Abelian transformations, once the g_i are fixed for any representation, they are known for all representations. For example, measuring g_2 with muon decay fixes it for quark interactions. Once the coupling of W or g to one fermion is measured, their coupling to all fermions and gauge bosons is known.

The ∂^μ is a Lorentz 4-vector, as are all the terms in Eq. 6.27. The first two terms are singlets (i.e. they multiply the unit matrix) in the $SU(2)$ and $SU(3)$ spaces. The third term is a 2×2 matrix in $SU(2)$ and a singlet in the other spaces. The fourth term is a 3×3 matrix in $SU(3)$ and a singlet in the other spaces. There is no inconsistency in having different size matrices for different terms as they operate in different spaces.

Equation 6.27 is, in a sense, the main equation of the Standard Model. When used in a Lagrangian, it leads to the full theory of the SM. It is the culmination of several decades of creative thinking by a number of physicists, leading to the realization that the phase invariance of quantum theory must exist for transformations in new kinds of internal spaces, and that quarks and leptons apparently carry labels that distinguish among three internal spaces. The phase, or gauge, invariance is guaranteed by the form of D^μ , as we learned in Section 6.1. In each case, as in the discussion of gauge invariance for electromagnetism, additional spin-1 gauge boson fields B^μ , W_i^μ , and G_a^μ must exist (1, 3, and 8 respectively). All of these have been observed experimentally.

Chapter 7

The Electroweak theory and Quantum Chromodynamics

The standard model Lagrangian is invariant under a set of transformations, each forming a symmetry group, in three internal spaces of its particles. Each of these appears as a local gauge (phase) invariance of the fields, and the corresponding additional term in the covariant derivative requires a set of vector (i.e. spin-1) bosons to mediate the transformations.

First, all particles have a $U(1)$ symmetry that leads to an Abelian phase invariance. This is just like the phase invariance related to the electromagnetic interactions. Instead of identifying the two just yet, let us call the gauge boson required by this $U(1)$ invariance B^μ . The index μ indicates that B^μ must transform under Lorentz transformations the same way as an ordinary 4-vector, so it can be added to the 4-gradient ∂^μ that gives the kinetic term in the Lagrangian. The connection between B^μ and the photon field A^μ remains to be established.

The second internal symmetry forms an $SU(2)$ group of *weak isospin* that leads to a non-Abelian phase invariance. This is analogous to the strong isospin $SU(2)$ symmetry of n and p that we have studied in some detail. The associated gauge bosons, analogous to the pions of the strong isospin, are denoted by W_i^μ . There is one boson for each of the 3 generators of the $S(2)$. So, $i = 1, 2, 3$. Just as for the pions, the physical W particles have integral electromagnetic charges:¹

$$\begin{aligned}W^+ &= \frac{1}{\sqrt{2}}(-W^1 + iW^2), \\W^- &= \frac{1}{\sqrt{2}}(-W^1 - iW^2), \\W^0 &= W^3.\end{aligned}\tag{7.1}$$

¹Note that we now have a “charge” label for each symmetry, and must distinguish them.

As we shall see, the $U(1)$ and the $SU(2)$ combine in what is called the *Electroweak theory*.²

The third internal symmetry forms an $SU(3)$ group of *color* that leads to another, independent, non-Abelian phase invariance. The associated gauge bosons are labeled G_a^μ , $a = 1, 2, \dots, 8$, one for each of the 8 generators of the $SU(3)$ group. The bosons are called *gluons*, and the theory of particle interactions via gluon exchange is called *Quantum Chromodynamics* (QCD). The internal charge of the particles that gluons couple to is called “color”, and the associated force is called the *strong* force because the coupling constant has the largest value of all the gauge interactions.

This chapter is devoted to a pedagogical overview of how the Electroweak and QCD terms enter the standard model Lagrangian.

7.1 The Quark and Lepton states

By construction, the full Lagrangian should be derivable by replacing the ordinary (4-gradient) derivative in the free particle Lagrangian by the covariant derivative that is adjusted for the gauge symmetries. It will have a part, $\mathcal{L}_{\text{gauge}}$, for the kinetic energy of the gauge fields. The rest represents, in addition to the kinetic energies of the quark and leptons, their interactions with the gauge fields; we will call this part $\mathcal{L}_{\text{fermion}}$.³

Now that we have 3 internal charges in addition to the space-time properties of the fields, we need to establish some notations to indicate how the particles will transform in those internal spaces. These characterizations are based on observation - the standard model does not stipulate any physical principle behind the arrangement, but does a remarkable job of accommodating the experimental observations in a consistent manner.

All known quarks and leptons are observed to be either electroweak $SU(2)$ singlets or parts of electroweak doublets. The way a particle is assigned to $SU(2)$ states is subtle. Consider the electron spinor ψ_{e^-} . The left- and right-handed components can be separated by using the respective spin projection operators defined in Eq. 3.53 (see Sec 3.6):

$$\begin{aligned} e_R^- &= P_R \psi_{e^-}, \\ e_L^- &= P_L \psi_{e^-}. \end{aligned} \tag{7.2}$$

Remarkably, the left- and right-handed states transform differently under the electroweak $SU(2)$. Right-handed electrons are electroweak singlets, while the

²This should not be a total surprise since some of the weak bosons carry electromagnetic charge.

³Additional fields will contribute additional terms. As we shall see in the next chapter, a complex scalar (i.e. spin-0) field will be necessary to explain how the electroweak symmetry of the vacuum is broken, and mass is generated. For now, we focus on constructing a theory that describes all the gauge interactions, but where all particles are massless.

left-handed electrons are in electroweak doublets, with left-handed neutrinos as their partners. Thus, the electroweak $SU(2)$ is denoted as $SU(2)_L$, in which e_R^- is a singlet and

$$\ell_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad (7.3)$$

is a doublet. Rotations in $SU(2)_L$ space cause transitions $e_L^- \leftrightarrow \nu_{eL}$, but don't affect e_R^- (we are not talking about right-handed neutrinos since their existence is somewhat uncertain). We will denote $\ell_1 = \nu_{eL}$ and $\ell_2 = e_L^-$. The up and down quarks behave in an analogous way:

$$q_{L\alpha} = \begin{pmatrix} u_\alpha \\ d_\alpha \end{pmatrix}_L \quad (7.4)$$

are the $SU(2)_L$ doublets of left handed quarks, while the right-handed quark states, $d_{R\alpha}$ and $u_{R\alpha}$ are $SU(2)_L$ singlets. Note that we have introduced an additional index, α to label the state of a quark in the $SU(3)$ space of color. All quarks are color triplets, so α is equal to 1, 2, or 3. Sometimes the 3 color charges are referred to as r, g, b , and their “negative”s $\bar{r}, \bar{g}, \bar{b}$, so $r\bar{r} + g\bar{g} + b\bar{b}$ is a color singlet. A quark makes a transition from one color state to another by emitting or absorbing a gluon, of which there are eight (one corresponding to each Gell-Mann matrix, the generators of $SU(3)$ symmetry). The color index is not needed for leptons since they are colorless (i.e. singlets in color space).

We have only considered one generation (or family) of fermions: ν_e, e, d, u . The theory simply repeats itself for the two other known families: ν_μ, μ, s, c , and ν_τ, τ, b, t . All gauge interactions of a given type of quark or lepton (where a type is specified by its electric charge or weak isospin label) are identical across families, only their masses are different. The charged quarks and leptons of the 2nd and 3rd generations are unstable. They decay to the first generation particles via weak interactions.

7.2 The quark and lepton Lagrangian

To write the fermion Lagrangian in a compact form, we introduce one convention: whenever the terms in D^μ act on a fermion state of a different matrix form, they give zero, by definition. Thus, $\sigma_i W_i$, which is a 2×2 matrix in the $SU(2)_L$ space, gives zero upon acting on e_R, u_R, d_R . Similarly, $\lambda_a G_a$, which is a 3×3 matrix in the color $SU(3)$ space, gives zero upon acting on leptons. With this, the (massless) fermion Lagrangian can be written as

$$\mathcal{L}_{\text{fermion}} = \sum_f \bar{f} \gamma^\mu D_\mu f, \quad (7.5)$$

where the covariant derivative is given in Eq. 6.27, and $f = \ell_L, e_R, q_{L\alpha}, u_R, d_R$ plus similar terms for the 2nd and 3rd generations.

Let us look at this Lagrangian term-by-term, one type of gauge interactions at a time. For the $U(1)$ and $SU(2)$ terms, we will only consider the leptons.

Since the color labels of the quarks are not affected by transitions in the $U(1)$ or $SU(2)$ spaces, quarks will behave in the same way as leptons for $U(1)$ and $SU(2)$ interactions.

7.2.1 The $U(1)$ terms

For the first family of leptons, we have

$$\mathcal{L}_{\text{fermion}}(U(1), \ell) = \bar{\ell} i \gamma^\mu \left(i g_1 \frac{Y_L}{2} B_\mu \right) \ell + \bar{e}_R i \gamma^\mu \left(i g_1 \frac{Y_R}{2} B_\mu \right) e_R, \quad (7.6)$$

where we have (presciently) made the provision that the left- and right-handed fermions could carry different amounts of the charge Y . In the $SU(2)_L$ space, ℓ is a doublet, while $g_1 Y B^\mu$ is just a number, so

$$\bar{\ell} \gamma^\mu \ell = \bar{\nu}_L \gamma^\mu \nu_L + \bar{e}_L \gamma^\mu e_L. \quad (7.7)$$

Then

$$\mathcal{L}_{\text{fermion}}(U(1), \ell) = \frac{g_1}{2} (Y_L (\bar{\nu}_L \gamma^\mu \nu_L + \bar{e}_L \gamma^\mu e_L) + Y_R \bar{e}_R \gamma^\mu e_R) B_\mu. \quad (7.8)$$

Before we interpret this, we need the $SU(2)$ part as it will contain terms involving the same particles.

7.2.2 The $SU(2)$ terms

Since $\sigma_i W_i$ is a 2×2 matrix, the contribution from right-handed leptons is zero, and we have

$$\begin{aligned} \mathcal{L}_{\text{fermion}}(SU(2), \ell) &= \bar{\ell} i \gamma^\mu \left(i g_2 \frac{\sigma^i}{2} W_\mu^i \right) \ell \\ &= -\frac{g_2}{2} \begin{pmatrix} \bar{\nu}_L & \bar{e}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} W_\mu^3 & W_\mu^1 - i W_\mu^2 \\ W_\mu^1 + i W_\mu^2 & -W_\mu^3 \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\ &= -\frac{g_2}{2} \begin{pmatrix} \bar{\nu}_L & \bar{e}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} W_\mu^0 & -\sqrt{2} W_\mu^+ \\ -\sqrt{2} W_\mu^- & -W_\mu^0 \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\ &= -\frac{g_2}{2} \left(\bar{\nu}_L \gamma^\mu \nu_L W_\mu^0 - \sqrt{2} \bar{\nu}_L \gamma^\mu e_L W_\mu^+ - \sqrt{2} \bar{e}_L \gamma^\mu \nu_L W_\mu^- - \bar{e}_L \gamma^\mu e_L W_\mu^0 \right). \end{aligned} \quad (7.9)$$

The seven terms in eqs 7.8 and 7.9 describe all gauge interactions of leptons in the SM.

7.3 The Electroweak Neutral Current

We have seen that the electromagnetic interaction of a particle of charge Q is

$$\mathcal{L}_{\text{EM}} = QA_\mu (\bar{e}_L \gamma^\mu e_L + \bar{e}_R \gamma^\mu e_R). \quad (7.10)$$

There are terms of this form in Eqs. 7.8 and 7.9. So, we have to determine whether they combine properly. However, before doing so, we have to resolve the fact that there are similar terms involving $\bar{\nu}_L \nu_L$, and the neutrino does not partake in EM interaction. The neutrino-neutrino interaction term is

$$\left(-\frac{g_1}{2} Y_L B_\mu - \frac{g_2}{2} W_\mu^0\right) \bar{\nu}_L \gamma^\mu \nu_L. \quad (7.11)$$

So, in order to avoid putting $g_1 = g_2 = 0$, which would render the whole exercise meaningless, we contend that the electromagnetic field A_μ is a combination of B_μ and W_μ^0 , and that it is orthogonal to the combination in Eq. 7.11. Thus, we try defining

$$A_\mu \propto g_2 B_\mu - g_1 Y_L W_\mu^0. \quad (7.12)$$

If B_μ and W_μ^0 are orthogonal, normalized fields, then the coefficient of $\bar{\nu}_L \gamma^\mu \nu_L$, which we call Z_μ ,

$$Z_\mu \propto g_1 Y_L B_\mu + g_2 W_\mu^0, \quad (7.13)$$

is indeed orthogonal to A_μ , so the neutrino is free of EM interaction. We can normalize A_μ and Z_μ ,

$$A_\mu = \frac{g_2 B_\mu - g_1 Y_L W_\mu^0}{\sqrt{g_2^2 + g_1^2 Y_L^2}}, \quad (7.14)$$

$$Z_\mu = \frac{g_1 Y_L B_\mu + g_2 W_\mu^0}{\sqrt{g_2^2 + g_1^2 Y_L^2}}, \quad (7.15)$$

so if W_μ^i and B_μ are normalized to unity, so are A_μ and Z_μ .

Having combined the neutrino terms in a consistent way, we can return to electrons. From Eqs. 7.8 and 7.9, the term for electron-electron interaction is

$$\left(-\frac{g_1}{2} Y_L B_\mu + \frac{g_2}{2} W_\mu^0\right) \bar{e}_L \gamma^\mu e_L + \left(-\frac{g_1}{2} Y_R B_\mu\right) \bar{e}_R \gamma^\mu e_R. \quad (7.16)$$

Equations 7.14 and 7.15 give

$$B_\mu = \frac{g_2 A_\mu + g_1 Y_L Z_\mu}{\sqrt{g_2^2 + g_1^2 Y_L^2}}, \quad (7.17)$$

$$W_\mu^0 = \frac{-g_1 Y_L A_\mu + g_2 Z_\mu}{\sqrt{g_2^2 + g_1^2 Y_L^2}}. \quad (7.18)$$

Substituting these in Eq. 7.16, we can express the electron-electron interaction in terms of the electromagnetic field A_μ (and necessarily, therefore, the new field

Z_μ as well). Then Eq. 7.16 becomes

$$\begin{aligned}
& -A_\mu \left[\bar{e}_L \gamma^\mu e_L \left(\frac{g_1 g_2 Y_L}{\sqrt{g_2^2 + g_1^2 Y_L^2}} \right) + \bar{e}_R \gamma^\mu e_R \left(\frac{g_1 g_2 Y_R}{2\sqrt{g_2^2 + g_1^2 Y_L^2}} \right) \right] \\
& -Z_\mu \left[\bar{e}_L \gamma^\mu e_L \left(\frac{g_1^2 Y_L^2 - g_2^2}{2\sqrt{g_2^2 + g_1^2 Y_L^2}} \right) + \bar{e}_R \gamma^\mu e_R \left(\frac{g_1^2 Y_R Y_L}{2\sqrt{g_2^2 + g_1^2 Y_L^2}} \right) \right].
\end{aligned} \tag{7.19}$$

Comparing this with Eq. 7.10, we must have (e is positive, so $Q = -e$ for electrons),

$$-e = \frac{g_1 g_2 Y_L}{\sqrt{g_2^2 + g_1^2 Y_L^2}} \tag{7.20}$$

and

$$-e = \frac{g_1 g_2 Y_R}{2\sqrt{g_2^2 + g_1^2 Y_L^2}}. \tag{7.21}$$

These fix

$$\begin{aligned}
Y_R &= 2Y_L, \\
Y_L &= -e \frac{\sqrt{g_2^2 + g_1^2 Y_L^2}}{g_1 g_2}.
\end{aligned} \tag{7.22}$$

Since only the combination $g_1 Y_L$ appears, we can choose, for convenience, $Y_L = -1$ (we can always redefine g_1 to accommodate this choice), so

$$e = \frac{g_1 g_2}{\sqrt{g_2^2 + g_1^2}}. \tag{7.23}$$

This demonstrates that the theory we've been developing can indeed be interpreted to contain the usual electromagnetic interaction for electrons (Eq. 7.10), and neutrinos (none), plus an additional so-called *neutral current* interaction with Z_μ for both electrons and neutrinos. The form of Eq. 7.23 suggests the definitions

$$\begin{aligned}
\sin \theta_W &= \frac{g_1}{\sqrt{g_2^2 + g_1^2}}, \\
\cos \theta_W &= \frac{g_2}{\sqrt{g_2^2 + g_1^2}},
\end{aligned} \tag{7.24}$$

or,

$$\begin{aligned}
g_2 &= \frac{e}{\sin \theta_W}, \\
g_1 &= \frac{e}{\cos \theta_W}.
\end{aligned} \tag{7.25}$$

Note

$$\sqrt{g_2^2 + g_1^2} = \frac{e}{\cos \theta_W \sin \theta_W}. \quad (7.26)$$

So, now g_1 and g_2 have been written in terms of the known e ($\frac{e^2}{4\pi} \approx \frac{1}{137}$ in natural units), and an angle θ_W , called the *electroweak mixing angle*, which is not determined by the theory developed so far. θ_W has to be measured or calculated in some other way. Its value is given by $\sin^2 \theta_W = 0.22215 \pm 0.00076$.

Let us now examine the couplings of the new field Z_μ to both electrons and neutrinos. From Eqs. 7.8 and 7.9, using Eqs. 7.15, 7.22, the couplings, and Eq. 7.24, the term for neutrino-neutrino interaction becomes

$$-\frac{\sqrt{g_2^2 + g_1^2}}{2} Z_\mu \bar{\nu}_L \gamma^\mu \nu_L = \frac{g_2}{2 \cos \theta_W} Z_\mu \bar{\nu}_L \gamma^\mu \nu_L. \quad (7.27)$$

Thus, we can attach a strength factor of $\frac{g_2}{2 \cos \theta_W}$ to each ν_L - Z vertex.

For the interaction of electrons with Z , there's no reason to expect e_L and e_R to have the same coupling since they have been treated differently in the way the theory has been constructed. We are concerned now with the second half of Eq. 7.19. Using the identities derived above, we get

$$\begin{aligned} \frac{g_1^2 - g_2^2}{2\sqrt{g_2^2 + g_1^2} Y_L^2} &= \frac{e^2}{2\sqrt{g_2^2 + g_1^2} Y_L^2} \left(\frac{1}{\cos^2 \theta_W} - \frac{1}{\sin^2 \theta_W} \right) \\ &= \frac{e}{\cos \theta_W \sin \theta_W} \left(-\frac{1}{2} + \sin^2 \theta_W \right) \end{aligned} \quad (7.28)$$

for the first term, and

$$\begin{aligned} \frac{g_1^2}{\sqrt{g_2^2 + g_1^2} Y_L^2} &= \frac{e^2}{\cos^2 \theta_W} \frac{\cos \theta_W \sin \theta_W}{e} \\ &= \frac{e}{\cos \theta_W \sin \theta_W} (\sin^2 \theta_W). \end{aligned} \quad (7.29)$$

for the second. The last two results have been written in a convenient form to help one notice that they can both be written as

$$\frac{e}{\cos \theta_W \sin \theta_W} (T_{3f} - Q_f \sin^2 \theta_W). \quad (7.30)$$

In Eq. 7.30, T_{3f} is the eigenvalue of T_3 (the diagonal $SU(2)$ generator analogous to J_z for spin) for any fermion f . Formally, $T_i = \frac{\sigma_i}{2}$ for a left-handed doublet. If f is a $SU(2)_L$ singlet, (e_R, u_R, d_R etc), then $T_{3f} = 0$. If f is an upper member of a $SU(2)_L$ doublet, (ν_L, u_L etc), then $T_{3f} = +\frac{1}{2}$. If f is a lower member of

a $SU(2)_L$ doublet, (e_L, d_L) etc), then $T_{3f} = -\frac{1}{2}$. Q_f is the electric charge of the fermion: $Q_e = -1$, $Q_\nu = 0$, $Q_u = +\frac{2}{3}$, $Q_d = -\frac{1}{3}$, in units of e . Thus Eq. 7.30 gives the *electroweak charge* of any fermion, i.e. the strength of its coupling to Z .⁴

So, now we have constructed a theory of electroweak interactions so that it contains the ordinary EM interactions, plus an additional photon-like particle Z , which interacts with any fermion that has a non-zero electroweak charge (the Q_f term in Eq. 7.30) or weak isospin (the T_{3f} term in Eq. 7.30). The strength is not small – in fact, $\sin\theta_W \cos\theta_W < 1$ means that the Z interaction is stronger than the photon interaction! Why, then, was it not discovered long ago?

The new interactions are called *neutral current* interactions, since they are analogous to the charged current weak interactions. There must indeed be a new boson, Z , like the photon, if this theory is to be correct. The reason it was not discovered long ago (in low-energy experiments) is because, unlike the photon, it is not massless. In fact, weighing in at 91 GeV, it is much harder to produce than a photon. Since the mass appears in the denominator whenever a Z is exchanged in a neutral current interaction, the size of the neutral current effect decreases as the mass of the Z increases.

The role of mass in the theory will be discussed later. We will see that it is possible to give mass to the Z in a consistent way. Indeed, the mass of the Z can be predicted, and the Z was directly observed in 1983 at precisely the expected mass. The neutral current effects due to the interactions of neutrinos, electrons, quarks, and muons were found in the early 1970's, and provided the earliest confirmations of the approach we are following.

7.4 The Charged Current

The $U(1)$ part of the lepton Lagrangian only gave terms that are diagonal in the fermion type (i.e. $\nu_L \rightarrow \nu_L$, $e_L \rightarrow e_L$, $e_R \rightarrow e_R$), which we have called *neutral current transitions*. But, Eq. 7.9 also has an off-diagonal part leading to transitions between partners in the $SU(2)_L$ doublet structure (i.e. $\nu_L \leftrightarrow e_L$):

$$\mathcal{L}_{CC} = \frac{g_2}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu e_L W_\mu^+ + \bar{e}_L \gamma^\mu \nu_L W_\mu^-). \quad (7.32)$$

The two terms on the RHS are Hermitian conjugates of each other, so \mathcal{L} is Hermitian, as expected. The subscript “CC” reflects the fact that the transitions involve transfer of electric charge since $SU(2)$ doublet partners carry different amounts of it.

Note that only e_L are involved, e_R does not interact with charged W s at all. This is, of course, the parity violation of the weak interactions. In accordance

⁴Note that we can identify

$$Q_f = T_{3f} + \frac{Y_f}{2}. \quad (7.31)$$

with Eq. 3.56, we can remove the subscript “ L ” by explicitly applying the projection operator to the general lepton wave function that includes both the left- and right-handed components:

$$\bar{\nu}_L \gamma^\mu e_L = \frac{1}{2} \bar{\nu} \gamma^\mu (1 - \gamma_5) e. \quad (7.33)$$

Because of its vector (γ^μ) minus axial vector ($\gamma^\mu \gamma_5$) form, an interaction with a current of this type is called a $V - A$ charged current interaction. The neutron beta decay (to a proton, an electron and an antineutrino) is perhaps the most well-known example of a $V - A$ interaction. It is interpreted at the quark level as $n(udd) \rightarrow p(ud)W^- \rightarrow p(ud)e^- \bar{\nu}_e$.

From Eq. 7.32, one might expect the strength of the charged current interaction to be almost twice that of the electromagnetic interaction since

$$\frac{(g_2/\sqrt{2})^2}{4\pi} = \frac{(e^2/4\pi)}{2 \sin^2 \theta_W} \approx \frac{2}{137}.$$

What one finds instead, is that the *weak* interactions are much weaker (hence the name) at low energies. Just as for the Z^0 , this is explained if one assumes the W^\pm to be massive. Direct observation of W^\pm in 1983 confirmed this explanation.

We have now dealt with the basic leptonic part of the SM Lagrangian, except for the mass term. The mass terms of the form $m\bar{\psi}\psi$ for the fermions or $m^2 V^\mu V_\mu$ for the spin-1 gauge bosons haven't appeared yet, so all fermions and gauge bosons are massless, contrary to experience. We will address this problem after dealing with the quark content of the Lagrangian.

7.5 The Quarks and the QCD Lagrangian

Some comments on the electroweak terms involving quarks (in Eq. 7.5) are in order before we turn our attention to QCD. The quark wave function is a product of 5 factors: space, spin, and one each for the 3 gauge groups. The orthonormality of the wave function must hold separately for each factor. Since the space, spin, and $U(1)_Y \otimes SU(2)_L$ structures of quarks and leptons are the same, the treatment of leptons in the preceding sections apply without modifications to quarks (except for the fact that unlike ν_R , we cannot pretend that u_R does not exist, since the latter carries electric charge, and therefore, has photon interactions). Charged currents (W^\pm) couple to left-handed quarks only, causing $u_L \leftrightarrow d_L$ transitions, and neutral current transitions occur with a universal strength $\frac{e(T_{3f} - Q_f \sin^2 \theta_W)}{\sin \theta_W \cos \theta_W}$ for each left- or right-handed quark, with $Q = \frac{2}{3}$ for u_L, u_R ; $Q = -\frac{1}{3}$ for d_L, d_R ; $T_3 = \frac{1}{2}$ for u_L ; $T_3 = -\frac{1}{2}$ for d_L ; $T_3 = 0$ for d_L and d_R .

Since the leptons carry no color (i.e. they are singlets in the $SU(3)_C$ space), the $\lambda_a G_a$ term in the Lagrangian (or in the covariant derivative of Eq. 6.27), which is a sum of 3×3 matrices in the $SU(3)_C$ space, makes zero contribution

to the leptons. Its contribution to quarks is non-zero, since each quark is a member of a $SU(3)_C$ triplet. With color indices α and β equal to 1, 2, or 3, the terms have the form, for a particular quark q ,

$$\frac{g_3}{2} \bar{q}_\alpha \gamma^\mu \lambda_{\alpha\beta}^a G_\mu^a q_\beta. \quad (7.34)$$

Unlike some of the W_i , the eight gluons G^a are electrically neutral and have no electromagnetic interactions. Their interactions with quarks is somewhat like a photon to a charged fermion, but there's a major difference. Since not all the generators λ^a can be simultaneously diagonalized (only two can, as we have seen in Sec. 2.3), the interaction with a gluon can change the color of the quark. Color being a conserved quantum number, this implies that the gluons themselves carry color (unlike photons, which do not carry the charge they couple to), and therefore have (strong) self-interaction.

The color charge is forever confined inside bound states that carry no net color. This makes it impossible to observe directly (although its existence is firmly established by overwhelming circumstantial evidence), and we will not study the term in Eq. 7.34 as explicitly as we did the equivalent electroweak term. But, it can be seen from the term 7.34 that a quark or an antiquark makes transition from one color state to another by emitting or absorbing a gluon. The gluon, like other vector bosons, carry momentum and other additive quantum numbers. The $SU(3)_C$ term is strictly separated from the $U(1)_Y$ and $SU(2)_L$ terms and parity is strictly conserved in strong interactions. Since particles can be turned into antiparticles by reversing their direction (momentum and charge), one vertex with a strength $\frac{g_3}{2}$ describes all the transitions

$$qg \rightarrow q', \quad q \rightarrow q'g, \quad g \rightarrow q'\bar{q}, \quad \bar{q} \rightarrow \bar{q}'g, \quad \bar{q}g \rightarrow \bar{q}', \quad (7.35)$$

where q and q' differ only in their color labels.

7.6 The Fermion-Gauge Boson Lagrangian

So, now we have all the pieces to put together the Lagrangian for the interaction of quarks and leptons with photons, W^\pm , Z^0 , and gluons. The relevant part of the $SU(3) \times SU(2) \times U(1)$ Lagrangian is, for the first family,

$$\begin{aligned} \mathcal{L} = & \sum_{f=\nu_e, e, u, d} e Q_f (\bar{f} \gamma^\mu f) A^\mu \\ & + \frac{g_2}{\cos \theta_W} \sum_{f=\nu_e, e, u, d} [\bar{f}_L \gamma^\mu f_L (T_{3f} - Q_f \sin^2 \theta_W) + \bar{f}_R \gamma^\mu f_R (Q_f \sin^2 \theta_W)] Z_\mu \\ & + \frac{g_2}{\sqrt{2}} [(\bar{u}_L \gamma^\mu d_L + \bar{\nu}_L \gamma^\mu e_L) W_\mu^+ + \text{Hermitian Conjugate}] \\ & + \frac{g_3}{2} \sum_{q=u, d} q_\alpha \gamma^\mu \lambda_{\alpha\beta}^a q_\beta G_\mu^a. \end{aligned} \quad (7.36)$$

For the second and third families, the substitutions $(\nu_e, e, u, d) \rightarrow (\nu_\mu, \mu, c, s)$ or $(\nu_e, e, u, d) \rightarrow (\nu_\tau, \tau, t, b)$ give the appropriate results. Some numerical relations are

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8M_W^2}, \quad (7.37)$$

$$g_2 = \frac{e}{\sin \theta_W}, \quad (7.38)$$

$$g_1 = \frac{e}{\cos \theta_W}, \quad (7.39)$$

$$\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}, \quad (7.40)$$

$$\alpha_1 = \frac{g_1^2}{4\pi} \simeq \frac{1}{100}, \quad (7.41)$$

$$\alpha_2 = \frac{g_2^2}{4\pi} \simeq \frac{1}{30}, \quad (7.42)$$

$$\alpha_3 = \frac{g_3^2}{4\pi} \simeq 0.3-0.1. \quad (7.43)$$

The values of the couplings α , α_1 , α_2 , α_3 depend on the momentum transfer of the interaction. The values given above are for interactions up to a ~ 100 GeV.

Chapter 8

Masses and the Higgs Mechanism

The Z , like the W^\pm must be very heavy. This much can be inferred immediately because a massless Z boson would give rise to a peculiar parity-violating long-range force. However, when the $SU(2) \times U(1)$ gauge structure for the EM interactions was first written down by Glashow in 1960, it was not clear how to give mass to the W^\pm and the Z bosons without breaking the gauge symmetry explicitly. That adding mass terms for the W^\pm and the Z bosons by hand breaks the symmetry is not a problem as such. Nice as the idea of local symmetry is, we, as physicists, would gladly sacrifice it if that enables us to explain some facts. The main problem is that addition of such terms renders the theory unrenormalizable. It leads to dimension 5 operators that make infinite contributions to the interaction Hamiltonian when one attempts to take higher order quantum corrections into account. The longitudinal component of the Z field, Z_L , then appears only in the mass term, and not in the kinetic energy term, thus acting as an auxiliary field. Such a Z_L propagator does not fall off with momentum, and we get infinities that do not cancel and cannot be swept out of the observable realm. But all is not lost. If we somehow preserve the gauge-invariance structure and give mass to W^\pm and Z , we may be able to preserve renormalizability. This is what Weinberg and Salam accomplished by invoking the idea of *spontaneous symmetry breaking*. The result was not only a consistent theory, but the masses of the W^\pm and Z were calculated in terms of parameters that had been measured before we had the capability to produce those particles on mass shell. The theory was quickly put to test by building a machine powerful enough to produce those particles if their masses were not too different from their predicted values. When it emerged with flying colors, it marked one of the towering achievements in theoretical physics.

The mechanism of spontaneous symmetry breaking also provides a way for fermions to get their masses, but those masses cannot be predicted - they appear as free parameters in the SM.

Essentially, it is conjectured that the “vacuum”, defined as the ground state of nature, is not a complete void. Instead, it is permeated by a scalar (i.e. spin-0) field, called a Higgs field, that is a doublet in the $SU(2)$ space of weak isospin, and carries a non-zero $U(1)$ hypercharge, but is electrically neutral and a singlet in the $SU(3)$ space of color. The gauge bosons and fermions can interact with this field in a way so as to appear massive. This means that the $SU(2)$ and $U(1)$ quantum numbers of vacuum are non-zero, so those symmetries are effectively broken. The associated charges can appear from or disappear into the vacuum even though the corresponding currents are conserved. When the symmetry is broken in this way, i.e. the symmetry is valid for the Lagrangian but not for the ground state of the system, it is said to be a spontaneously broken symmetry.

8.1 Spontaneous Symmetry Breaking

As a simple example of spontaneous symmetry breaking, consider the theory with a single Hermitian scalar field and the Lagrangian

$$\mathcal{L}(\phi) = T - V = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \left(\frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right), \quad (8.1)$$

where μ and λ are parameters of the potential. With a single field, there can be no continuous internal symmetry, but the above Lagrangian is invariant under the “reflection” $\phi \rightarrow -\phi$. By general quantum mechanical principles, the potential must have minima if it is to describe a physical system. This requires $\lambda > 0$. Given a minimum (which is the classical ground state of the system), we can follow the normal perturbative procedure in quantum mechanics to expand the fields around their values at the minimum and determine the excitations. In quantum field theory, it is conventional to call this ground state the vacuum, and the excitations are particles. Their masses are determined by the form of the Lagrangian near the vacuum. The Lagrangian in Eq. 8.1 is not the most general, but it is more general than it may appear. It can be shown that higher powers of ϕ would lead to infinities in physical quantities and must therefore be excluded.

If $\mu^2 > 0$, then we have the vacuum at $\phi = 0$, and the Lagrangian describes a scalar field with mass μ that has a quartic self coupling of strength $\frac{\lambda}{4}$. But, what if $\mu^2 < 0$? There is no physical reason to exclude such a possibility. Now we have a local maximum, instead of a minimum, at $\phi = 0$. We would not want to perturb around such a point where the free theory contains a tachyon, a particle with imaginary mass! Rewriting the potential as

$$V(\phi) = \frac{\lambda}{4} \left(\phi^2 + \frac{\mu^2}{\lambda} \right)^2, \quad (8.2)$$

(the additive constant is of no consequence) we see that there are two degenerate minima at $\phi = \pm \sqrt{\frac{-\mu^2}{\lambda}}$. The potential in a symmetric neighborhood of $\phi = 0$ that contains both minima looks like a smooth “W”. Physically, the two minima

are equivalent. We can choose either one to perturb around, so let us pick $\phi = \sqrt{\frac{-\mu^2}{\lambda}}$. For the ϕ field,

$$v = \sqrt{\frac{-\mu^2}{\lambda}} \quad (8.3)$$

is a *vacuum expectation value* (VEV). To determine the particle spectrum, let us rewrite the theory in terms of a field with zero VEV:

$$\eta(x) = \phi(x) - v. \quad (8.4)$$

We could equally well have chosen $\eta(x) = \phi(x) + v$, but the physics conclusions would not be affected since the theory is symmetric under $\phi \rightarrow -\phi$. But having made a particular choice of η , the potential is not symmetric about its minimum. The Lagrangian is not invariant under $\eta \rightarrow -\eta$. The symmetry has been spontaneously broken by the choice of vacuum.

Substituting Eq. 8.4 into Eq. 8.1 we get

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \eta \partial^\mu \eta) - \left(\frac{1}{2}\mu^2(v^2 + 2\eta v + \eta^2) \right. \\ &\quad \left. + \frac{1}{4}(v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4) \right) \\ &= \frac{1}{2}(\partial_\mu \eta \partial^\mu \eta) - \left(\frac{v^2}{2}(\mu^2 + \frac{1}{2}\lambda v^2) + \eta v(\mu^2 + \lambda v^2) \right. \\ &\quad \left. + \frac{\eta^2}{2}(\mu^2 + 3\lambda v^2) + \lambda v\eta^3 + \frac{1}{4}\lambda\eta^4 \right). \end{aligned} \quad (8.5)$$

The term linear in η vanishes (by Eq. 8.3), as it must near the minimum, and \mathcal{L} simplifies to

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta \partial^\mu \eta) - \left(\lambda v^2 \eta^2 + \lambda v \eta^3 + \frac{1}{4}\lambda \eta^4 \right) + \text{irrelevant constant}. \quad (8.6)$$

Now the term with η^2 has the correct sign so it can be interpreted as a mass term. This Lagrangian describes a scalar field η that appears as a particle of mass

$$m_\eta^2 = 2\lambda v^2 = -2\mu^2, \quad (8.7)$$

and with two interactions, a cubic one of strength λv and a quartic one of strength $\frac{\lambda}{4}$. Both of these depend on λ , which is a free parameter as far as we can tell, and are therefore interactions of undetermined strengths.

The two descriptions of the theory in terms of ϕ or η must be equivalent if the problem is exactly solvable. If we want a perturbative description, it is essential to perturb around the minimum to have a convergent description. The scalar particle described by the theory with $\mu^2 < 0$ is a real scalar, with a mass obtained by its self-interaction with other scalars, because at the minimum of the potential there is a non-vanishing VEV v .

Next we will repeat the analysis for increasingly complicated symmetries until we see what happens when the symmetry of \mathcal{L} is the SM $SU(2)_L \times U(1)_Y$ invariance and when we have the combined Lagrangian of gauge bosons, fermions, and Higgs fields. At each stage surprising new features will emerge.

8.2 Spontaneous breaking of a continuous symmetry: the Goldstone Theorem

Breakdown of continuous symmetries is slightly more subtle. Let us consider the fairly general situation described by the Lagrangian

$$\mathcal{L}(\phi) = T - V = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi), \quad (8.8)$$

where ϕ is some multiplet of spinless fields and $V(\phi)$ and thus $\mathcal{L}(\phi)$ is invariant under some symmetry group

$$\delta\phi = i\epsilon_a T^a \phi, \quad (8.9)$$

T^a being imaginary antisymmetric matrices (because ϕ are Hermitian).

As in the previous section, we want to perturb around a minimum of the potential $V(\phi)$. We expect the ϕ field to have a VEV, $\langle\phi\rangle$, which minimizes V . To simplify notations, we define

$$V_{j_1 \dots j_n} = \frac{\partial^n}{\partial\phi_{j_1} \dots \partial\phi_{j_n}} V(\phi). \quad (8.10)$$

Then we can write the condition that λ be an extremum of $V(\phi)$ as

$$V_j(\lambda) = 0. \quad (8.11)$$

For V to have a minimum at λ , we must also have

$$V_{jk}(\lambda) \geq 0. \quad (8.12)$$

The second derivative matrix $V_{jk}(\lambda)$ is the mass-squared matrix. We can see this by expanding $V(\phi)$ in a Taylor series in the shifted fields $\eta = \phi - \lambda$ and noting that the mass term is $\frac{1}{2} V_{jk}(\lambda) \eta_j \eta_k$. Thus, Eq. 8.12 assures us that there are no tachyons in the free field case about which we are perturbing.

Now comes the interesting part, the behavior of the VEV λ under the transformations in Eq. 8.9. There are two cases. If

$$T_a \lambda = 0 \quad (8.13)$$

for all a , the symmetry is not broken. This is certainly what happens if $\lambda = 0$. But Eq. 8.13 is the more general statement that the vacuum doesn't carry the charge T_a , so the charge cannot disappear into the vacuum. In the second case,

$$T_a \lambda \neq 0 \quad \text{for some } a. \quad (8.14)$$

Then the charge T_a can disappear into the vacuum even though the associated current is conserved. This is spontaneous breaking of a continuous symmetry.

Often there are some generators of the original symmetry that are spontaneously broken while others are not. The set of generators satisfying Eq. 8.13 is

closed under commutation (because $T_a\lambda = 0$ and $T_b\lambda = 0$ implies $[T_a, T_b]\lambda = 0$) and generates an unbroken subgroup of the original symmetry group.

Returning to the mass matrix, because V is invariant under the transformation in Eq. 8.9, we can write

$$V(\phi + \delta\phi) - V(\phi) = iV_k(\phi)\epsilon_a(T^a)_{kl}\phi_l = 0. \quad (8.15)$$

If we differentiate with respect to ϕ_j , we get (since ϵ^a are arbitrary),

$$V_{jk}(\phi)(T^a)_{kl}\phi_l + V_k(\phi)(T^a)_{kj} = 0. \quad (8.16)$$

Setting $\phi = \lambda$, we find that the second term drops out because of Eq. 8.11, and we obtain

$$V_{jk}(\phi)(T^a)_{kl}\phi_l = 0. \quad (8.17)$$

But $V_{jk}(\lambda)$ is the mass-squared matrix M_{jk}^2 for the spinless fields, so we can rewrite the last equation in the matrix form as

$$M^2 T^a \lambda = 0. \quad (8.18)$$

For T^a in the unbroken subgroup, this condition is trivially satisfied. But if $T^a\lambda \neq 0$, then it requires $T^a\lambda$ is an eigenvector of M^2 with zero eigenvalue. It corresponds to a massless boson field given by

$$\phi^T T^a \lambda. \quad (8.19)$$

This is called a *Goldstone boson* after J. Goldstone, who first established this connection between spontaneously broken continuous symmetries and massless particles. We will see a specific example in the next section.

8.3 Complex scalar field - a Global Symmetry

Suppose that ϕ is a complex scalar,

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad (8.20)$$

and

$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2. \quad (8.21)$$

This is invariant under a global gauge transformation,

$$\phi \rightarrow \phi' = e^{ix}\phi, \quad (8.22)$$

so the symmetry of \mathcal{L} is now a global $U(1)$ symmetry rather than a reflection as in Section 8.1. In terms of the real components, we have

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{\mu^2}{2}(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2. \quad (8.23)$$

In the ϕ_1, ϕ_2 plane, the potential clearly has a minimum at the origin if $\mu^2 > 0$, while for $\mu^2 < 0$, the minimum is along a circle of radius

$$\phi_1^2 + \phi_2^2 = \frac{\mu^2}{\lambda} = v^2. \quad (8.24)$$

The potential in a symmetric neighborhood of $\phi = 0$ that contains the minimum looks like a Mexican sombrero or the bottom of a wine bottle.

As before, to analyze the case with $\mu^2 < 0$ we have to expand around $\phi_1^2 + \phi_2^2 = v^2$. We could choose any point on the circle, but to proceed we have to choose some point, which will break the symmetry for the solutions. We pick, arbitrarily, the point $\phi_1 = v, \phi_2 = 0$, and write, with η and ρ real,

$$\phi = \frac{1}{\sqrt{2}}(v + \eta(x) + i\rho(x)). \quad (8.25)$$

Substituting this in Eq. 8.23, we again find that the Lagrangian can be written in a form that is readily interpreted in terms of particles and their interactions:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu \eta \partial^\mu \eta) + \frac{1}{2}(\partial_\mu \rho \partial^\mu \rho) + \mu^2 \eta^2 \\ & - \lambda v(\eta \rho^2 + \eta^3) - \frac{\lambda}{4}(\eta^4 + 2\eta^2 \rho^2 + \rho^4) \quad (8.26) \\ & + \text{irrelevant constant.} \end{aligned}$$

The first two terms represent the normal kinetic energy. The term $+\mu^2 \eta^2$ tells us that the η field corresponds to a physical particle of squared mass

$$m_\eta^2 = 2|\mu^2|. \quad (8.27)$$

Note that there's no equivalent term in ρ^2 , implying that the particle associated with the field ρ has zero mass. It is the Goldstone boson of the theory. As we expected, since we chose a particular direction in the ϕ_1, ϕ_2 plane to associate with the vacuum, the gauge invariance is no longer present in Eq. 8.26.

Physical interpretation of the massless boson is not difficult. Excitations in the radial direction requires moving up in the potential (away from the minimum), and a mass term arises from the resistance against that effort. Along the circle, the potential does not vary, so there is no resistance to motion along the circle. Thus, excitation along the circle amounts to creation of a massless boson. The Goldstone phenomenon is widespread in physics. We have encountered a simple example. The $U(1)$ symmetry is broken because we had to choose a particular point on the circle to perturb around. The presence and the particular form of the remaining (interaction) terms in Eq. 8.26 carry the link to the original unbroken symmetry, but not in an obvious way.

8.4 The Abelian Higgs Mechanism

Having worked out the breaking of a global gauge symmetry, let us now try a local one, i.e. let us consider a Lagrangian that is invariant under local gauge

transformations. We know from our earlier discussions that such an invariance requires the introduction of a massless vector field, say A_μ , and we know that we should write the Lagrangian in terms of the covariant derivative,

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - igA_\mu. \quad (8.28)$$

The gauge field transforms as

$$A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{g}\partial_\mu\chi(x) \quad (8.29)$$

and ϕ as

$$\phi(x) \rightarrow \phi'(x) = e^{i\chi(x)}\phi(x). \quad (8.30)$$

The Lagrangian is then

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (8.31)$$

For $\mu^2 > 0$ this describes the interaction of a charged scalar particle (with $g = e$) of mass μ with the electromagnetic field A_μ , for example. Note that there is no mass term for A_μ . The kinetic energy terms for the vector field are contained in $F_{\mu\nu}F^{\mu\nu}$ and we shall carry them along, but they do not play a role in the spontaneous breakdown of the symmetry. As in the previous sections, we are primarily interested in the scenario where $\mu^2 < 0$. Note that this Lagrangian contains four independent degrees of freedom: the two components ϕ_1 and ϕ_2 of the scalar field, and the two transverse polarization states of the massless vector boson (as expected, if A_μ represents a photon). We could proceed as before. The algebra gets increasingly complicated, however, so in order to simplify the analysis, let us use what we have already learned.

In general, ϕ can be written in the form $\phi(x) = \eta(x)e^{i\rho(x)}$, where η, ρ are real, so we can rewrite ϕ as

$$\phi(x) = \frac{1}{\sqrt{2}}(v + h(x)), \quad (8.32)$$

with h real, having used a transformation as in Eq. 8.30, knowing that if necessary, we could find a χ to accomplish that. We could not have done this in the previous section, since the Lagrangian there was only invariant under a global symmetry, not a local one. Now we substitute this in \mathcal{L} . Since the original choice of the field A_μ was not fixed by physics, we do not bother to carry its transformation (Eq. 8.29) along. So

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}((\partial_\mu - igA_\mu)(v + h))((\partial^\mu + igA^\mu)(v + h)) \\ &\quad - \frac{\mu^2}{2}(v + h)^2 - \frac{\lambda}{4}(v + h)^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{2}g^2v^2A_\mu A^\mu - \lambda v^2h^2 - \lambda v h^3 - \frac{\lambda}{4}h^4 \\ &\quad - g^2v h A_\mu A^\mu + \frac{1}{2}g^2h^2A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \end{aligned} \quad (8.33)$$

The surprising result is the second term on the RHS: we now have a mass term for the gauge boson! But since we started with a gauge invariant theory and only made algebraic transformations, we expect the resulting theory to be gauge invariant as well. The gauge boson mass is the square root of the coefficient of $\frac{1}{2}A_\mu A^\mu$,

$$M_A = gv, \quad (8.34)$$

which is non-zero only when the gauge symmetry is spontaneously broken as a result of the Higgs field acquiring a non-zero VEV, v . So, the theory is only gauge invariant in a restricted sense. As before, the Lagrangian is gauge invariant, but the vacuum is not, because we had to choose a particular direction in the ϕ_1, ϕ_2 plane and the minimum of the potential along that direction to label “the vacuum”.

The spectrum now consists of a single real Higgs boson h , that has a mass $\sqrt{2\lambda}v$, various self interactions and cubic and quartic interactions with the gauge field A_μ . Since the massive boson has three spin states (corresponding to $J_3 = 1, 0, \text{ or } -1$ in its rest frame), the number of independent fields is still four.

What has happened here is that the Goldstone boson of the previous section has become the longitudinal polarization state of the gauge boson. This can be seen a little more explicitly if the calculation of this section is carried out without the simplifying step of Eq. 8.32, but using Eq. 8.25 instead. Then the mass appears for the gauge vector boson, and a term $A_\mu \partial^\mu \rho$, which apparently allows A_μ to turn into ρ as it propagates. When such cross terms appear, one can go to eigenstates by a diagonalization, which can be accomplished here by a gauge transformation, and which eliminates ρ from the Lagrangian. This phenomenon is sometimes referred to as the gauge boson having “eaten” the Goldstone boson.

The mechanism we have just studied is called the *Higgs mechanism*. Technically it is well understood, but at a physical level its meaning is not yet fully grasped in particle physics. In some sense, the longitudinal polarization state of the gauge boson (which must exist if it is to be massive in a Lorentz invariant theory where it is possible to go to its rest frame) is the Goldstone boson that would have appeared as a physical particle if the theory were not a local gauge theory. There is also a neutral spin-0 boson left over that apparently should exist as a physical particle; it is called the Higgs boson. Note that the gauge boson mass is fixed if g^2 and v are known, but the mass of the Higgs boson h depends on the unknown parameter λ . In the next section we will add the last bit of complexity needed to fully incorporate the Higgs mechanism into the SM.

8.5 The Higgs Mechanism in the Standard Model

Now we are ready to work out the Higgs mechanism in the SM by adding one further degree of complexity. In the last section, we saw how the Higgs field can break a local symmetry, but it was an Abelian symmetry and the Higgs field

carried no charge: all of its quantum numbers, other than 4-momentum, were zero. In the SM, the Higgs field is assigned to a $SU(2)$ doublet. We can choose

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad (8.35)$$

where the superscripts denote the electric charge. ϕ^+ and ϕ^0 are each complex fields:

$$\begin{aligned} \phi^+ &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \\ \phi^0 &= \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4). \end{aligned} \quad (8.36)$$

In the $SU(2)$ space ϕ^+ and ϕ^0 are related by a rotation, like a spin-up and a spin-down state, or the left-handed ν_e to the left-handed electron. The scalar part of the Lagrangian has the same form as in last section

$$\mathcal{L}_\phi = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2, \quad (8.37)$$

but now ϕ is a column vector and ϕ^\dagger a row, each with 2 complex components, so

$$\phi^\dagger \phi = \begin{pmatrix} \phi^{+*} & \phi^{0*} \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \phi^{+*} \phi^+ + \phi^{0*} \phi^0 = \frac{\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2}{2}. \quad (8.38)$$

As before, we study the potential

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2. \quad (8.39)$$

$V(\phi)$ is invariant under the local gauge transformation

$$\phi(x) \rightarrow \phi'(x) = e^{i\vec{\alpha}(x) \cdot \frac{\vec{\sigma}}{2}} \phi(x), \quad (8.40)$$

where σ_i are the Pauli matrices and α_i are parameters. Proceeding as before, $V(\phi)$ has a minimum for $\mu^2 < 0$ at

$$\phi^\dagger \phi = \frac{-\mu^2}{2\lambda} = \frac{v^2}{2}. \quad (8.41)$$

From Eq. 8.38, we see that there are many ways to satisfy Eq. 8.41.

So, once again, we must choose a direction, this time in $SU(2)$ space, and expand around the minimum. We make the following choice and call it the vacuum (in the sense that it is the ground state of free space), ϕ_0 :

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad (8.42)$$

i.e. $\phi_3 = v$, $\phi_1 = \phi_2 = \phi_4 = 0$.

To study the spectrum by perturbing around the vacuum, we write

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \quad (8.43)$$

and we will look for the equations satisfied by H . We can make this simple choice because for an arbitrary $\phi(x)$ we could apply a gauge transformation, $\phi \rightarrow \phi' = \exp\left(\frac{i\vec{\sigma} \cdot \vec{\theta}(x)}{v}\right)\phi$, and rotate $\phi(x)$ into the form of Eq. 8.43. This amounts to “gauging away” three fields, which is consistent with what the Goldstone theorem. The original symmetry, from Eq. 8.41,

$$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = \text{invariant} \quad (8.44)$$

is an $O(4)$ symmetry. By choosing a direction, we have broken three continuous symmetries, so three massless bosons, and three fields, are gauged away. We shall soon see that these three are just what are needed to endow the W^\pm and Z bosons with longitudinal polarization, and, therefore, mass.

Before we write the covariant derivative and complete the calculation, let us examine a bit further what is happening. The electric charge Q , the weak isospin eigenvalue T_3 , and the $U(1)$ hypercharge Y (for the Higgs field) are related by

$$Q = T_3 + \frac{Y}{2} \quad (8.45)$$

The electric charge assignment of Eq. 8.35 corresponds to $Y_H = 1$. The choice that only the neutral component ϕ^0 gets a VEV is very important, since whatever quantum numbers ϕ carries can vanish into the vacuum. If ϕ^+ had a non-zero VEV, then electric charge would not be conserved, contrary to observation.

If the vacuum ϕ_0 is invariant under some subgroups of the original $SU(2) \times U(1)$, any gauge bosons associated with that subgroup will still be massless. Since the Higgs field $\phi(x)$ is a doublet, but only one component gets a VEV, clearly the $SU(2)$ symmetry is broken. Since $Y_H \neq 0$, the $U(1)$ symmetry is broken. However, if we operate with the electric charge operator on ϕ_0 , we get

$$Q\phi_0 = \left(T_3 + \frac{Y}{2}\right)\phi_0 = 0, \quad (8.46)$$

so ϕ_0 (i.e. the vacuum) is invariant under a transformation

$$\phi_0 \rightarrow \phi_0' = e^{i\alpha(x)Q}\phi_0 = \phi_0. \quad (8.47)$$

This is also a $U(1)$ transformation, so the vacuum is invariant under a certain $U(1)_Q$ whose generator is a particular linear combination of the generators of the original $SU(2)_L$ and $U(1)_Y$. Of course, this is the $U(1)$ of electromagnetism, and the gauge boson that remains massless is the photon (because it does not acquire a longitudinal polarization by “eating” a Goldstone boson). The presence of a massless gauge boson was a necessary consequence of electric charge conservation, which forces us to choose a vacuum that is electrically neutral.

Finally, let us carry out the algebra to see the consequences of the Higgs mechanism. For the full Lagrangian to be invariant under the transformation

in Eq. 8.40, we have to replace ∂_μ by the covariant derivative D_μ where

$$D^\mu = \partial^\mu - ig_1 \frac{Y}{2} B^\mu - ig_2 \frac{\sigma_i}{2} W_i^\mu \quad (8.48)$$

(this is the same as Eq. 6.27 except that we have dropped the color $SU(3)$ term since that symmetry is not broken) and the gauge fields B_μ and \vec{W}_μ transform as in Chapter 6. Then when ϕ gets a VEV, proceeding as in the earlier sections, the Lagrangian contains extra terms

$$\phi^\dagger \left(ig_1 \frac{Y}{2} B^\mu + ig_2 \frac{\sigma_i}{2} W_i^\mu \right)^\dagger \left(ig_1 \frac{Y}{2} B^\mu + ig_2 \frac{\sigma_i}{2} W_i^\mu \right) \phi. \quad (8.49)$$

Putting $Y = 1$, writing the 2×2 matrices explicitly as in Chapter 7, and using Eq. 8.43, we get the contribution to \mathcal{L} ,

$$\begin{aligned} & \frac{1}{8} \left| \begin{pmatrix} g_1 B_\mu + g_2 W_\mu^3 & g_2 (W_\mu^1 - iW_\mu^2) \\ g_2 (W_\mu^1 + iW_\mu^2) & g_1 B_\mu - g_2 W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 \\ &= \frac{1}{8} v^2 g_2^2 ((W_\mu^1)^2 + (W_\mu^2)^2) + \frac{1}{8} v^2 (g_1 B_\mu - g_2 W_\mu^3)^2. \end{aligned} \quad (8.50)$$

The first term can be rewritten as

$$\left(\frac{1}{2} v g_2 \right)^2 W_\mu^+ W^{-\mu} \quad (8.51)$$

carefully keeping track of the $\sqrt{2}$ factors as in Eq. 7.9. For a charged boson, the expected mass term in a Lagrangian would be $m^2 W^+ W^-$, so we can conclude that the W^\pm has indeed acquired a mass

$$M_W = \frac{1}{2} v g. \quad (8.52)$$

The second term in Eq. 8.50 is not diagonal. So, we have to define new eigenvalues to find the particles with definite mass (B and W^3 are the neutral states with diagonal weak hypercharge and weak isospin interactions). In fact, we already have the answer in hand, because the linear combination of B and W^3 appearing in Eq. 8.50 is just the combination we have called Z_μ (see Sec. 7.3 and note the choice of $Y_L = -1$). We expect mass terms for Z_μ and the photon A_μ . For a neutral field there is a factor of $\frac{1}{2}$ relative to the charged ones, so mass terms $\frac{1}{2}(M_Z^2 Z_\mu Z^\mu + M_\gamma^2 A_\mu A^\mu)$ should appear. From Eq. 8.50 and the normalization of Z in Eq. 7.15, we can conclude that

$$M_Z = \frac{1}{2} v \sqrt{g_1^2 + g_2^2} \quad (8.53)$$

and

$$M_\gamma = 0. \quad (8.54)$$

It's good to see that the photon remains massless since no $A_\mu A^\mu$ term appears. Using the identities of Chapter 7, we can also write

$$\frac{M_W}{M_Z} = \cos \theta_W. \quad (8.55)$$

Since B and W^3 mix, the neutral state is not degenerate in mass with the charged ones, unless $\theta_W = 0$. Once θ_W is measured, SM fixes either of the *Intermediate Vector Boson* masses in terms of the other, and the result has been verified by experiments.

It's useful to define

$$\rho \equiv \frac{M_W}{M_Z \cos \theta_W}. \quad (8.56)$$

The SM predicts $\rho = 1$. In fact, it can be shown that $\rho = 1$ is guaranteed even if additional Higgs doublets are present. Any deviation from $\rho = 1$ would be an important signal of new physics.

8.6 Fermion Masses

Now that we have available the Higgs field in a $SU(2)$ doublet, it is possible to write a $SU(2)$ -invariant interaction of fermions with the Higgs field. For example, we can add the following interaction term to the Lagrangian for the first-generation leptons:

$$\mathcal{L}_{\text{int}} = g_e \left(\bar{L} \phi e_R^- + \phi^\dagger e_R^- L \right). \quad (8.57)$$

Since $L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$ and $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$, $\bar{L} \phi = \bar{\nu}_{eL} \phi^+ + \bar{e}_L^- \phi^0$ is a $SU(2)$ invariant.

Multiplying by the singlet e_R^- does not change the $SU(2)$ invariance. The second term is the Hermitian conjugate of the first. The coupling g_e is arbitrary; neither the presence of such terms nor g_e follows from a gauge principle.

Following the previous sections, we can calculate the observable effects of adding this term by replacing

$$\phi \rightarrow \begin{pmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{pmatrix}, \quad (8.58)$$

where v is the Higgs VEV, and H is the neutral, physical Higgs particle. Substituting it into Eq. 8.57 gives

$$\mathcal{L}_{\text{int}} = \frac{g_e v}{\sqrt{2}} \left(\bar{L} \phi e_R^- + \phi^\dagger e_R^- L \right) + \frac{g_e}{\sqrt{2}} \left(\bar{L} \phi e_R^- + \phi^\dagger e_R^- L \right) H. \quad (8.59)$$

The first term in Eq. 8.59 has exactly the same form as expected for a fermion of mass

$$m_e = \frac{g_e v}{\sqrt{2}}. \quad (8.60)$$

Thus the theory can now accommodate a non-zero electron mass. It should be noted here that it is only through such mass terms that a fermion can make a transition from a right-handed helicity state to a left-handed one (or vice versa). The amplitude is proportional to the fermion mass. This explains why $\frac{B(\pi^- \rightarrow e\bar{\nu}_e)}{B(\pi^- \rightarrow \mu\bar{\nu}_\mu)} \approx 1.23 \times 10^{-4}$.

Since g_e is arbitrary, the value of the electron mass has not been calculated. Rather, we can invert Eq. 8.60, so

$$g_e = \frac{\sqrt{2}m_e}{v}. \quad (8.61)$$

The second term in Eq. 8.59 says that the theory has an electron-Higgs vertex of strength $\frac{g_e}{\sqrt{2}} = \frac{m_e}{v}$, which determines the probability for an electron or positron to radiate a Higgs boson, or for a Higgs boson to decay into e^+e^- . We can eliminate g_e from the Lagrangian and rewrite it as

$$\mathcal{L}_{\text{int}} = m_e \bar{e}e + \frac{m_e}{v} \bar{e}eH. \quad (8.62)$$

Note that no mass term appears for the neutrino since we have assumed that the theory contains no right-handed neutrino state ν_R , so a term analogous to Eq. 8.57 cannot be written that will subsequently lead to a mass term $\bar{\nu}_R\nu_L$. This implies that the neutrinos do not interact with H . If there were a ν_R , it would be hard to observe; since it would have $T_3 = 0$ and $Q = 0$, it would not couple to W^\pm , Z^0 , or γ . However, a neutrino of non-zero mass, and therefore a ν_R , can be accommodated in the SM.

For quarks, this leads to another subtlety (one that would also have occurred for leptons had ν_R existed). If $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$ is a $SU(2)$ doublet, then so is

$$\psi_c = -i\sigma_2\psi^* = \begin{pmatrix} -b^* \\ a^* \end{pmatrix}. \quad (8.63)$$

Then we can also write terms of \mathcal{L}_{int} using

$$\phi_c = \begin{pmatrix} -\phi^{0*} \\ \phi^- \end{pmatrix}, \quad (8.64)$$

which becomes, after invoking the Higgs mechanism,

$$\phi_c \rightarrow \begin{pmatrix} -\frac{v+H}{2} \\ 0 \end{pmatrix}. \quad (8.65)$$

ϕ has hypercharge $Y = +1$, ϕ_c has $Y = -1$, and still satisfies for each state, $Q = T_3 + \frac{Y}{2}$.

Then, for quarks, we have

$$\mathcal{L}_{\text{int}} = g_d \bar{Q}_L \phi d_R + g_u \bar{Q}_L \phi_c u_R + \text{Hermitian conjugate}. \quad (8.66)$$

Substituting from Eqs. 8.61 and 8.63, and for Q_L , gives

$$\mathcal{L}_{\text{int}} = m_d \bar{d}d + m_u \bar{u}u + \frac{m_d}{v} \bar{d}dH + \frac{m_u}{v} \bar{u}uH, \quad (8.67)$$

where g_d and g_u have been eliminated in favor of their masses, following the same steps that led to Eq. 8.61. Again, the quark masses can be included in the description, but since g_d and g_u are arbitrary parameters, not related to each other or to g_e , the masses have to be measured. The last two terms of Eq. 8.67 describe the interaction of d and u quarks with H^0 .

The entire procedure of this section can be repeated for the second and third generations, giving further pieces of \mathcal{L}_{int} which come from taking Eq. 8.62 with $e \rightarrow \mu, \tau$ and Eq. 8.67 with $u \rightarrow c, t$ and $d \rightarrow s, b$. Since the Higgs-fermion coupling strength is proportional to the latter's mass, it couples most strongly to the heaviest fermions.

8.7 Comment on Vacuum Energy

The Higgs mechanism contributes to an important problem when cosmological considerations are introduced. We found the Higgs VEV $v = \sqrt{-\mu^2/\lambda}$. Putting $\phi = v$ in Eq. 8.39, we can calculate the Higgs potential at its minimum:

$$V(\phi = v) = V_0 = \frac{\lambda v^4}{2}. \quad (8.68)$$

$v = 2M_W/g_2 \approx 246$ GeV $\Rightarrow V_0 \approx 2 \times 10^9 \lambda$ GeV⁴, and 1 GeV³ $\approx 1.3 \times 10^{41}$ cm⁻³. So, $V_0 \approx 2.6 \times 10^{50} \lambda$ GeV/cm³. This is apparently the contribution of spontaneous symmetry breaking to the vacuum energy density of the universe.

But, from astrophysics, it is known that the energy density of luminous matter in the universe is about one proton per cubic meter on the average, and that the total density of matter is no more than ~ 100 times this number. Thus, empirically, the total energy density is less than about 10^{-4} GeV/cm³.

To compare this number with V_0 , we need the value of λ , the arbitrary Higgs self-coupling in the Higgs potential. While λ is not known, if it were eventually to be determined by some fundamental argument, such as a gauge principle, presumably $\lambda > \sim 1/10$. Combining these, we find $V_0 \approx 2 \times 10^{49}$ GeV/cm³, larger than the experimentally observed value by a factor of 10^{54} !

Technically, this is not a contradiction, because we can always add a constant to the potential in theories without gravity and cancel V_0 , but to do so involves tuning the constant to a part in 10^{54} , which is hardly satisfactory. This is (essentially) what is referred to as the problem of the cosmological constant. If gravity is included, terms of the kind we are considering will contribute to the energy-momentum tensor and, through Einstein's equations, dramatically affect the geometry of space-time. This is another clue that in spite of the remarkable descriptive power of the SM, it is a theory that is incomplete at the fundamental level.

Chapter 9

Quark (and Lepton) Mixing

It may be noted that in all of the Standard Model described so far, we have made no provision for any transition between different generations (or families) of fermions. The charged current (W^\pm) only causes transitions between the two states of a $SU(2)$ doublet in the weak isospin space, while the neutral currents (Z^0, γ, g, H) do nothing to alter the flavor of a fermion.¹ This would imply that the lighter partner of a weak isospin doublet should be stable. While this seems to be acceptable for leptons (ignoring for the moment the possibility of neutrino mass, oscillations etc.), it is certainly not so for quarks. The b and s quarks are not stable. In fact, there is ample evidence that the strange quark connects to the up quark via charged current interactions in a manner very similar to that of the down quark. Like the π^+ (the scalar $u\bar{d}$ bound state), K^+ , the lightest charged strange meson (scalar $u\bar{s}$) decays predominantly to $\mu^+\nu$. The next major decay mode is $\pi^+\pi^0$.² What's more, the mean life of K^+ is less than half that of π^+ .³ The shorter lifetime of the kaon is explained by its larger mass, which translates to a larger phase space, but it also means that the underlying charged current interaction across quark generations is not very strongly suppressed.

9.1 The Cabibbo Mixing of d and s Quarks

The apparent contradiction is easily resolved if we abandon the implicit assumption that the weak isospin eigenstates of the quarks are also their mass eigenstates. In fact they are not. So far we have have put only the left-handed d quark in a doublet with a left handed u quark:

$$\psi_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \quad (9.1)$$

¹The emission or absorption of a gluon will change the color of a quark, but the internal space of color is orthogonal to flavor.

² $B(\pi^+ \rightarrow \mu^+\nu) \approx 1$, $B(K^+ \rightarrow \mu^+\nu) \approx 0.63$, $B(K^+ \rightarrow \pi^+\pi^0) \approx 0.21$.

³ $\tau_{K^\pm} \approx 1.24 \times 10^{-8}$ s, $\tau_{\pi^\pm} \approx 2.60 \times 10^{-8}$ s, $m_{K^\pm} \approx 493.7$ MeV, $m_{\pi^\pm} \approx 139.6$ MeV.

This is clearly inadequate to explain strange quark weak interactions, such as the kaon decays described above. But since d and s quarks are identical to each other in every respect other than mass, we can generalize Eq. 9.1 so that the field that appears in the weak isospin doublet with the u quark is a linear combination of the d and s mass eigenstates:

$$\psi_L = \begin{pmatrix} u \\ \cos\theta_c d + \sin\theta_c s \end{pmatrix}_L, \quad (9.2)$$

where the parameter θ_c is known as *the Cabibbo angle*. For a doublet of the form Eq. 9.2, the coupling strength of strangeness-conserving processes is proportional to $\cos\theta_c$ while that for the parallel strangeness-altering processes (i.e. those in which the d quark is replaced by the s quark) is proportional to $\sin\theta_c$. Experimentally, $\sin\theta_c \approx 0.223$, so that a strangeness-altering process is inherently about 20 times ($\sim \frac{1}{\sin^2\theta_c}$) weaker than the corresponding strangeness-conserving process.

9.2 The c quark and the GIM Mechanism

So far we have discussed only the W^\pm exchange, but if Eq. 9.2 were the whole story, there would be decay processes mediated by Z^0 exchange as well. For example, the Z^0 coupling $\sin\theta_c \cos\theta_c Z^0 \bar{d}_L s_L$ can give rise to the decay $K_L^0 \rightarrow \mu^+ \mu^-$ at a rate comparable to that of $K^+ \rightarrow \mu^+ \nu_\mu$. In fact, the former is suppressed by a factor of 10^{-8} ! This is one of the symptoms of a very general problem – the absence of *flavor-changing neutral current* effects (FCNC). It prompted Glashow, Iliopoulos, and Maiani (GIM) to propose a radical solution, the existence of a fourth quark, the charm quark, in a $SU(2)$ doublet with the orthogonal combination of s and d , so the doublet is

$$\psi'_L = \begin{pmatrix} c \\ \cos\theta_c s - \sin\theta_c d \end{pmatrix}_L, \quad (9.3)$$

As we will soon see, now the Z^0 coupling to this doublet precisely cancels the strangeness-altering coupling from the other doublet. This eliminates the decay $K_L^0 \rightarrow \mu^+ \mu^-$ in tree approximation. However, the problem reappears in one-loop level if m_c is too large. To fit the data, the c quark had to appear with a mass no more than a few GeV. And it did! Other charmed mesons and baryons were discovered just with the right properties. The c quark should decay primarily to s because of 9.3, and it does.

This was one of the great triumphs of theoretical physics. It made a tidy little world with a correspondence between quarks and leptons – one light family with u, d, ν_e, e^- and one heavy family with c, s, ν_μ, μ^- . Nature, however, chose not to stop here. Discovery of the τ^- lepton ruined the correspondence, and the subsequent discovery of the b quark did not come as much of a surprise. Direct observation of t and ν_τ in due course completed the third family. We do not know if there are any more families, but before considering the full spectrum

consisting of all 3 families, let us initiate the mathematical approach by first considering what may happen if there are only two families of quarks.⁴ Then we could write the charged current of Chapter 7 associated with the W^\pm boson as

$$J_{CC}^\mu = (\bar{u} \quad \bar{c}) \gamma^\mu P_L \begin{pmatrix} d \\ s \end{pmatrix} = \bar{u} \gamma^\mu P_L d + \bar{c} \gamma^\mu P_L s, \quad (9.4)$$

where we have used row and column vectors in a flavor space. By u, c, d, s we mean the mass eigenstates, the energy levels of the system. But then we should call the weak eigenstates something else – to allow them to be different. So we replace $\begin{pmatrix} d \\ s \end{pmatrix}$ by $\begin{pmatrix} d' \\ s' \end{pmatrix}$, where the q' states are defined to be the weak interaction eigenstates. One set of eigenstates can be expanded in terms of the other, so we write

$$\begin{pmatrix} d' \\ s' \end{pmatrix}_L = V \begin{pmatrix} d \\ s \end{pmatrix}_L, \quad (9.5)$$

where V must be a unitary 2×2 matrix. The most general unitary 2×2 matrix can be written with three angles $\theta, \alpha,$ and $\gamma,$

$$V = \begin{pmatrix} \cos \theta e^{i\alpha} & \sin \theta e^{i\gamma} \\ -\sin \theta e^{-i\gamma} & \cos \theta e^{-i\alpha} \end{pmatrix}, \quad (9.6)$$

so

$$\begin{aligned} d' &= \cos \theta e^{i\alpha} d + \sin \theta e^{i\gamma} s \\ &= e^{i\alpha} (d \cos \theta + s \sin \theta e^{i(\gamma-\alpha)}), \\ s' &= -\sin \theta e^{-i\gamma} d + \cos \theta e^{-i\alpha} s \\ &= e^{-i\gamma} (-d \sin \theta + s \cos \theta e^{i(\gamma-\alpha)}). \end{aligned} \quad (9.7)$$

We can redefine the relative phases of the quark states without changing any observables. So we multiply d' by $e^{-i\alpha}$, s' by $e^{i\gamma}$, and s by $e^{-i(\gamma-\alpha)}$. If mass terms are present, these phases can be absorbed by similar transformations of s_R and d_R . With these replacements,

$$\begin{aligned} d' &= d \cos \theta + s \sin \theta, \\ s' &= -d \sin \theta + s \cos \theta, \end{aligned} \quad (9.8)$$

and

$$V = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (9.9)$$

⁴In fact, this is pretty close to the historical development. The quarks were not discovered all at once. The quark picture of hadrons and mesons developed in the same period as the $SU(2) \times U(1)$ model. The first theory of quark mixing was formulated when only two families of quarks were known.

Then the form of Eq. 9.4 that should have been used from the beginning is

$$\begin{aligned}
J_{\text{CC}}^\mu &= \begin{pmatrix} \bar{u} & \bar{c} \end{pmatrix} \gamma^\mu P_L \begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \bar{u} & \bar{c} \end{pmatrix} \gamma^\mu P_L V \begin{pmatrix} d \\ s \end{pmatrix} \\
&= \bar{u} \gamma^\mu P_L d \cos \theta + \bar{u} \gamma^\mu P_L s \sin \theta - \bar{c} \gamma^\mu P_L d \sin \theta + \bar{c} \gamma^\mu P_L s \cos \theta.
\end{aligned} \tag{9.10}$$

There are two new terms, both containing $\sin \theta$, and the old terms are reduced by $\cos \theta$. The angle θ is the same as the Cabibbo angle θ_c encountered before. Had θ been zero, then Eq. 9.10 would reduce to Eq. 9.4 and the s quark would have been stable. Now it can decay via the coupling to u in the second term above.

Note that we chose, as is convention, to rotate the down-type quarks, but one could rotate the up-type quarks, or, in general, both types simultaneously. If we had rotated both $\begin{pmatrix} \bar{u} & \bar{c} \end{pmatrix}$ and $\begin{pmatrix} d \\ s \end{pmatrix}$, we would have had a current of the form $\begin{pmatrix} \bar{u} & \bar{c} \end{pmatrix} \gamma^\mu P_L V_{\text{up}}^\dagger V_{\text{down}} \begin{pmatrix} d \\ s \end{pmatrix}$, but the product of two rotations is a rotation so we can replace $V_{\text{up}}^\dagger V_{\text{down}}$ by a single rotation matrix V .

Next we need to check the effect of this rotation on the neutral current. Still working with two families, we have

$$J_{\text{NC}}^\mu = \sum_{f=u,c,d,s} (\bar{f}_L \gamma^\mu (T_{3L} - Q \sin^2 \theta_W) f_L + \bar{f}_R \gamma^\mu (-Q \sin^2 \theta_W) f_R). \tag{9.11}$$

Now we replace d by d' and s by s' . Then the terms that would change are

$$\begin{aligned}
&(\bar{d}_L \cos \theta + \bar{s}_L \sin \theta) \gamma^\mu (T_{3L} - Q \sin^2 \theta_W) (d_L \cos \theta + s_L \sin \theta) \\
&+ (-\bar{d}_L \sin \theta + \bar{s}_L \cos \theta) \gamma^\mu (T_{3L} - Q \sin^2 \theta_W) (-d_L \sin \theta + s_L \cos \theta) \\
&+ (L \rightarrow R) \\
= &\bar{d}_L \gamma^\mu (T_{3L} - Q \sin^2 \theta_W) d_L (\cos^2 \theta + \sin^2 \theta) \\
&+ \bar{s}_L \gamma^\mu (T_{3L} - Q \sin^2 \theta_W) s_L (\cos^2 \theta + \sin^2 \theta) \\
&+ \bar{d}_L \gamma^\mu (T_{3L} - Q \sin^2 \theta_W) s_L (\cos \theta \sin \theta - \cos \theta \sin \theta) \\
&+ \bar{s}_L \gamma^\mu (T_{3L} - Q \sin^2 \theta_W) d_L (\cos \theta \sin \theta - \cos \theta \sin \theta) \\
&+ (L \rightarrow R) \\
= &\sum_{f=d,s} (\bar{f}_L \gamma^\mu (T_{3L} - Q \sin^2 \theta_W) f_L + (L \rightarrow R)).
\end{aligned} \tag{9.12}$$

We are back to the original result! So, the neutral current is diagonal in mass eigenstates or weak eigenstates. This is called the GIM mechanism. It has very profound consequences for decays, since the SM has no FCNC at the tree level. Searches for FCNC are therefore interesting as possible probes for new interactions.

9.3 Kobayashi-Maskawa Mixing in the Six-Quark Model

We can extend the results of the previous sections to the three families of quarks that we know of. In analogy with Eq. 9.10, we can write the full charged current for quarks as

$$J_{CC}^\mu = (\bar{u} \quad \bar{c} \quad \bar{t}) \gamma^\mu P_L V \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad (9.13)$$

where V is now a 3×3 unitary matrix. A general $n \times n$ unitary matrix has n^2 independent real parameters, which means 9 for us. We can redefine the phases of five quark states; the sixth would amount to an overall phase for all the states, so it does not help. That leaves four parameters to describe the matrix. Also, an orthogonal $n \times n$ matrix has $n(n-1)/2$ real parameters, 3 here. Thus, one of the parameters in V must enter as a relative phase.⁵ Then the terms in the Hamiltonian $\sim W_\mu J_{CC}^\mu$ can be complex, which implies that the theory will not be invariant under transformations involving time reversal, or equivalently, CP .

The matrix V that connects the weak interaction eigenstates of the down-type quarks to their mass eigenstates is called the Kobayashi-Maskawa, or Cabibbo-Kobayashi-Maskawa (CKM) quark-mixing matrix:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \quad (9.14)$$

The elements of V are presently measured to have magnitudes

$$\begin{pmatrix} 0.9739 \text{ to } 0.9751 & 0.221 \text{ to } 0.227 & 0.0029 \text{ to } 0.0045 \\ 0.221 \text{ to } 0.227 & 0.9730 \text{ to } 0.9744 & 0.039 \text{ to } 0.044 \\ 0.0048 \text{ to } 0.014 & 0.037 \text{ to } 0.043 & 0.9990 \text{ to } 0.9992 \end{pmatrix} \quad (9.15)$$

Ordering the families by mass, we see that transitions by one unit are small, and transitions by two units are much smaller. These values assume that no additional quarks exist that are coupled to the known ones (so that V is a 3×3 unitary matrix).

Note that the b quark can decay as well. One of the terms in Eq. 9.13 is $\bar{c} \gamma^\mu P_L V b$, with coefficient $|V_{cb}| \approx 0.042$. The corresponding decay $b \rightarrow \bar{c} W^-$ will be followed by the W^- decaying into $e^- \nu_e$, $\mu^- \nu_\mu$, $\tau^- \nu_\tau$, $\bar{u} d$, or $\bar{c} s$, each of the last two carrying 3 times the weight of each of the first three due to the color degree of freedom available to the quarks. At the $b \bar{c} W$ vertex there is a

⁵A general $n \times n$ unitary rotation matrix contains $n(n-1)/2$ angles and $(n-1)(n-2)/2$ phases.

factor of V_{cb} moderating the usual coupling $g_2/\sqrt{2}$. Thus the width of the b is approximately

$$\Gamma_b \simeq \frac{9|V_{cb}|^2 G_F^2 m_b^5}{192\pi^3}, \quad (9.16)$$

where the 9 comes from the 9 open channels (neglecting phase space corrections due to the daughter quarks and lepton masses). All of the W decay modes except $\tau^- \nu_\tau$ and $\bar{c}s$ are also open in decays of the τ^- . However, being the heavier partner in its $SU(2)$ doublet, τ decay is not suppressed by an off-diagonal element of a (lepton) mixing matrix. This would imply

$$\frac{\Gamma_b}{\Gamma_\tau} \simeq \frac{9}{5}|V_{cb}|^2 \left(\frac{m_b}{m_\tau}\right)^5 \approx 0.23. \quad (9.17)$$

So, we'd expect the b lifetime to be ~ 4.5 times longer than that of the τ because the b can only decay via the rotation of the mass eigenstates to the weak eigenstates. In reality, the factor is closer to 5.8. Our naive estimate is not off by much considering that we ignored several (small) effects. A similar comparison can be carried out with the c quark as well. Indeed, despite being 3 times lighter than b and having fewer decay options, c has a shorter lifetime because it has a large diagonal element namely V_{cs} , available.

At present the elements of V are parameters that have to be measured, just like the fermion masses. Eventually, it is hoped that the relation between the weak eigenstates and the mass eigenstates will be calculable, so the elements of V can be expressed in terms of ratios of masses.

We could have carried out a similar procedure for leptons. If, however, any pair of the same-type quarks are degenerate in mass, then we cannot tell the mass eigenstates apart, so we could perform a rotation and make the corresponding angle or element of V zero. Thus, if neutrinos were massless, there is no need to have mixing angles; the weak eigenstates and mass eigenstates can be the same. Over the past few years it has been emerging that the neutrinos are not massless, after all. Oscillations among different flavors of neutrinos have been observed by independent experiments, using different techniques. The mass hierarchy in the lepton sector being entirely different from that in the quark sector (the heaviest neutrino is at least 3 orders of magnitude lighter than the lightest charged lepton), mixing of weak eigenstates and mass eigenstates has different consequences between the two. Understanding neutrino mixing is one of the highest priorities in particle physics today.

Chapter 10

Interaction of Particles with Matter

A scattering process at an experimental particle physics facility is called an *event*. Stable particles emerging from an event are identified and their momenta measured by their interactions in the material media of a suitably constructed detector.¹ Unstable particles are identified by adding the 4-momenta of their daughters, if they can be isolated with sufficient confidence. Particle identification can be either a unique assignment, or a broader classification. In this chapter we shall discuss some fundamental characteristics of how different kinds of particles interact with different kinds of material. In the next chapter we will see how this knowledge is utilized to design detectors.

Electromagnetic interactions are most heavily relied upon for particle detection. A charged particle loses energy as it tries to make its way through a material medium. Several phenomena contribute to this process, and their relative importance depends on the properties of the particle and of the medium. For energies of interest to us, the most important phenomenon, for any charged particle other than electrons, is ionization. In addition to ionization, electrons also lose a significant fraction of their energies by photon emission (a.k.a. *bremsstrahlung*). Being electrically neutral, photons do not cause ionization, but at high energies, they transfer their energy to a medium by such electromagnetic interactions as the photoelectric effect, Compton scattering, and production of electron-positron pairs. Another important process is the Coulomb scattering of a charged particle with atomic nuclei, which is responsible for *multiple scattering*. We will discuss these first.

There are other electromagnetic processes that are used in particle identification, but less generally. These include scintillation, Cerenkov radiation, and transition radiation. These will be discussed separately along with some special

¹Here “stable” is defined by the time it takes for a high-energy particle (K.E. $> \sim 1$ GeV) to traverse distances comparable to the dimensions of the detector. Therefore, a high-energy muon is a stable particle for our purpose.

applications of the general effects mentioned above. Strong and weak nuclear interactions of particles in matter will be discussed in subsequent sections.

10.1 Electromagnetic Interaction of Particles with Matter

A rigorous treatment of electromagnetic interactions based on QED has been done to a good extent. For subtle effects, empirical parametrizations of extensive data give reasonably good basis for interpolation. Details of these procedures are beyond our scope. We will only summarize the key results.

10.1.1 Energy Loss by Ionization

The form of the rate of energy loss by ionization can be seen from a semi-classical argument. The mean rate of energy loss, or *stopping power*, of moderately relativistic charged particles other than electron by ionization and atomic excitation of a material medium is given by the Bethe-Bloch equation, which follows from a quantum treatment of energy loss based on a first-order Born approximation, with some reasonable simplifying assumptions:

$$-\frac{dE}{dx} = Kz^2 \frac{Z}{A} \frac{1}{\beta^2} \left(\frac{1}{2} \ln \frac{2m_e c^2 \beta^2 \gamma^2 T_{\max}}{I^2} - \beta^2 - \frac{\delta}{2} \right). \quad (10.1)$$

(see “Passage of particle through matter” in <http://pdg.lbl.gov> for definitions of various terms). The first two terms in the parentheses depend on the particle velocity β , while the last one reflects a modest *density effect*.

The Bethe-Bloch formula is a good approximation (accurate to $\sim 1\%$) for particles with $\beta\gamma = \frac{p}{Mc}$ in the range of about 0.05 to 500. At lower energies various corrections need to be taken into account, while at higher energies radiative losses dominate. The effect of the sign of the particle’s charge, known as the “Barkas effect” begins to enter the picture only near the lower boundary of the Bethe-Bloch region.² Except in hydrogen, particles of the the same velocity have similar rates of energy loss in different materials, decreasing at a slow rate with increasing Z . The stopping power functions are characterized by broad minima whose position drops from $\beta\gamma = 3.5$ to 3.0 as Z goes from 7 to 100. In practical cases, most relativistic particles (e.g. cosmic-ray muons) have mean energy loss rates close to the minimum, and are said to be minimum ionizing particles, or MIP’s.

Equation 10.1 can be integrated to find the total (or partial) “continuous slowing-down approximation” (CSDA) range R for a particle which loses energy only through ionization and atomic excitation. Since for a given medium, $\frac{dE}{dx}$ depends only on β , R/M is a function of E/M or pc/M in the Bethe-Bloch region

²In principle, one might expect some particle-antiparticle asymmetry, since the detector is made entirely of matter, with no trace of antimatter.

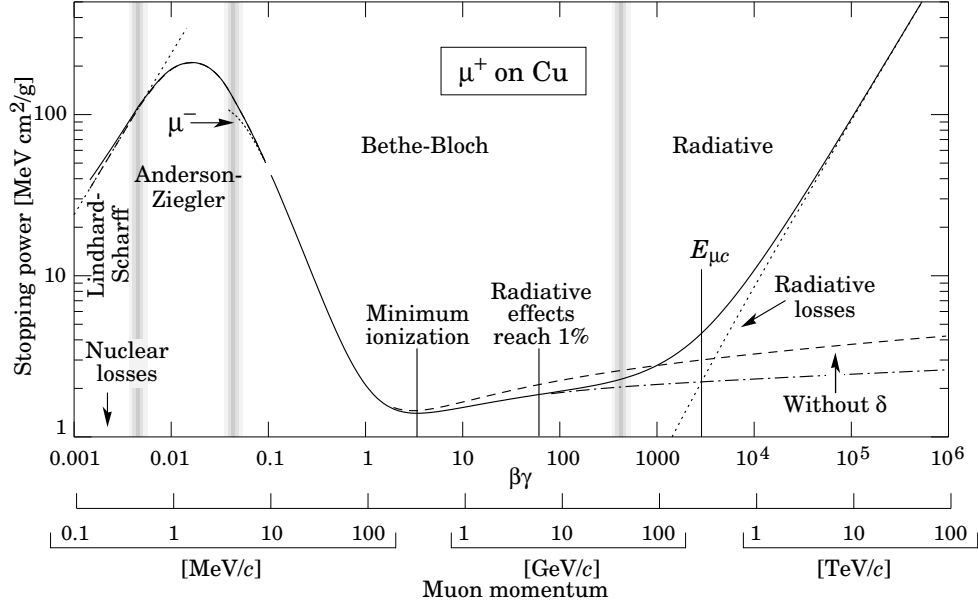


Figure 10.1: Stopping power ($= \langle -dE/dx \rangle$) for μ^+ 's in Cu as a function of $\beta\gamma = p/Mc$ over 9 orders of magnitude in momentum (12 orders in KE). Solid curves indicate the total stopping power. Vertical bands indicate boundaries between different approximations.

In practice, range is a useful concept only for low-energy hadrons, for which it is typically less than the *interaction length* (defined as the length through which the probability of the hadron not participating in a strong nuclear interaction drops by a factor of e), and for muons below a few hundred GeV (above which radiative effects dominate).

For a particle with mass M and momentum $M\beta\gamma c$, T_{\max} is given by

$$T_{\max} = \frac{2m_e c^2 \beta^2 \gamma^2}{1 + 2\gamma m_e/M + (m_e/M)^2}. \quad (10.2)$$

The determination of the mean excitation energy is the principal non-trivial task in the evaluation of the Bethe-Bloch formula. Estimates based on fits to experimental measurements for various charged particles are used.

The simplifying assumptions used in the derivation of Eq. 10.1 begin to lead us astray in regions of low energy. Atomic shell corrections are necessary when the velocity of the incident particle becomes comparable to the velocity of the bound electrons. Above the upper boundary of the Bethe-Bloch region, it is necessary to account for radiation, kinematics, and the structure of the incident particle.

The energy transferred to electrons increases with the incident particle energy. Secondary “knock-on” electrons with $T \gg I$ are known as δ rays. The

number of δ rays produced with energy greater than T_0 in a thickness x is

$$N(T \geq T_0) = \int_{T_0}^{T_{\max}} \xi \frac{dT}{T^2} = \xi \left(\frac{1}{T_0} - \frac{1}{T_{\max}} \right), \quad (10.3)$$

where

$$\xi = \frac{2\pi n_e Z^2 e^4}{M\beta^2} x. \quad (10.4)$$

So, for $T_0 \ll T_{\max}$, the number of energetic δ rays falls off inversely with the energy and that the parameter ξ is the energy above which there will be, on average, one δ ray produced. As such, it represents a “typical” value of energy loss in the material.

Usually, δ rays of appreciable energy are rare, but occasionally they can carry energies of $O(1 \text{ GeV})$, sufficient to start a process that requires independent treatment. A δ ray with kinetic energy T_e and corresponding momentum p_e is produced at an angle θ given by

$$\cos \theta = \frac{T_e p_{\max}}{p_e T_{\max}}, \quad (10.5)$$

where p_{\max} is the momentum of an electron with the maximum possible energy transfer T_{\max} .

Several other processes, such as Cerenkov radiation, transition radiation, bremsstrahlung, and pair-production also become important at high energies.

10.1.2 Fluctuations in Ionization Energy Loss

Equation 10.1 only gives the mean energy lost by a charged particle per unit thickness of matter (*absorber*). The actual amount of energy lost by a charged particle that has traversed a given thickness of absorber will vary due to the stochastic nature of the process. For moderately relativistic incident charged particles, collisions with small energy transfers are much more likely than those with large transfers. As a result, the single-collision spectrum is highly skewed and the $\frac{dE}{dx}$ distribution has a long tail on the high energy side. The probability density function $f(\Delta; \beta\gamma, x)$ describing the distribution of energy loss Δ in absorber thickness x is called the “Landau distribution”. If $\chi(W, x)dW$ is the probability that a particle loses an energy between W and $W + dW$ after crossing a thickness x of the absorber, then

$$\chi(W, x)dW = \frac{1}{\xi} f_L(\lambda), \quad (10.6)$$

where

$$\lambda = \frac{1}{\xi} \left(W - \xi \left(\ln \frac{\xi}{\epsilon'} + 1 - C_E \right) \right),$$

$$\ln \epsilon' = \ln \frac{(1 - \beta^2) I^2}{2mv^2} + \beta^2, \quad (10.7)$$

$$C_E = 0.577 \quad (\text{Euler's constant}).$$

The quantity ϵ' is the low energy cutoff of possible energy losses, chosen by Landau so that the mean energy loss agreed with the Bethe-Bloch theory. The function $f_L(\lambda)$ can be expressed as

$$f_L(\lambda) = \frac{1}{\pi} \int_0^\infty \exp(-u(\ln u + \lambda)) \sin(\pi u) du. \quad (10.8)$$

The most probable value of the energy loss is given by

$$W_{\text{MP}} = \xi \left(\ln \frac{\xi}{\epsilon'} + 0.198 + \delta \right). \quad (10.9)$$

The full width at half maximum (FWHM) of the distribution is 4.02ξ .

The Landau formula an approximation based on the assumptions that successive collisions are statistically independent, that the absorber medium is homogeneous, and that the total energy loss is small compared to the incident particle's energy. Experimental energy loss distributions in gases are broader than predicted by the Landau formula. Still, pulse height spectra of high-energy charged particles in gaseous proportional chambers follow the general form of the distribution. More elaborate "straggling" functions that provide a better fit are available, but the Landau distribution often serves well enough. When ξ/E_{max} is $O(0.01)$ or less, the number of δ rays with energies near E_{max} is small, and single large energy loss events give an asymmetric high-energy tail to the energy loss distribution. The distribution approaches a Gaussian for ξ/E_{max} of $O(1)$ or more, when the number of δ rays with energies near E_{max} is large.

Physicists often relate total energy loss to the number of ion pairs produced near the particle's track. This relation becomes complicated for extremely relativistic particles due to the wandering of energetic δ rays whose ranges exceed the dimensions of the fiducial volume. The mean local energy dissipation per ion-pair produced, W , is essentially constant for moderately relativistic particles, but increases at slower particle speeds. For gases, W can be highly sensitive to trace amounts of contaminants or dopants. Also, ionization yields in practical cases may be influenced by such factors as subsequent recombination.

Because of fluctuations in energy loss, a beam of particles of fixed energy will have a distribution of ranges in a thick absorber. This is another manifestation of the straggling phenomenon. The two fluctuations are related by

$$\langle (E - \bar{E})^2 \rangle = \left(\frac{dE}{dx} \right)^2 \langle (R - \bar{R})^2 \rangle. \quad (10.10)$$

The range distributions of moderately relativistic hadrons in metals are nearly Gaussian. For a pure, monoenergetic beam of particles, the fractional straggling σ_R/R increases with Z of the absorber. The fractional straggling in a given absorber decreases with increasing kinetic energy and approaches a value $\sigma_R/R \approx \frac{1}{2} \sqrt{m_e/M}$ at high energy, where M is the mass of the incident particle.