Assignment: HW8 [40 points]

Assigned: 2006/11/20
Due: 2006/11/27

P8.1 [$5 + 3 = 8$ points]

(a) Show that the combination of two successive Lorentz transformations with parallel velocities β_1 and β_2 is equivalent to a single Lorentz transformation with velocity $\beta = \dfrac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}$.

(b) Show that the set of all Lorentz transformations with this composition rule forms a group.

P8.2 [$8 + 2 + 2 = 12$ points]

(a) A rocket propels itself along a straight line in empty space by burning its fuel in such a way that the velocity of the exhaust gases ejected from the nozzle is v_0 in the instantaneous (inertial) rest frame of the rocket. If the rocket starts from rest with a total mass M_i (icluding fuel), find the relationship between its velocity and mass during its journey.

(b) Typical chemical fuels yield exhaust speeds of the order of 10^3 m/s. Let us imagine we had a fuel that gives $v_0 = 3 \times 10^5$ m/s. What initial mass of fuel would the rocket need in order to attain a final velocity of $0.1c$ for a final mass of 1 ton?

(c) Matter-antimatter fuel yields $v_0 = c \equiv 1$ (the exhaust consists of photons). What initial mass of fuel would be required in this case (in order to attain a final velocity of $0.1c$ for a final mass of 1 ton)?

P8.3 [5 points]
A particle moving with velocity $\vec{\beta}$ decays "in flight" into two particles. Determine the relation between the angle of emergence of either daughter particle and its energies in the laboratory frame and the rest frame of the parent particle.

P8.4 [7 points]
Determine the maximum energy which can be carried off by any one of the decay particles, when a particle of mass m_0 at rest decays into three particles with masses m_1, m_2, and m_3.

P8.5 [$6 + 2 = 8$ points]
The Lagrangian that yields the correct relativistic equations of motion of a free particle can be expressed in the non-relativistitc form as

$$\mathcal{L}_0 = -\frac{1}{\gamma} m = -mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \simeq -mc^2 + \frac{1}{2} m \mathbf{v}^2. \tag{1}$$

where $c = 1$.

(Note: This form of the Lagrangian refers to a fixed inertial frame. Thus, it is not a Lorentz invariant, which is why it is not used in practical applications. We choose to work with it in this problem only as an illustrative example. In this problem, $\dot{q} \equiv \frac{dq}{dt}$.)

Now consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} m \left(\psi \dot{\mathbf{q}}^2 - c_0^2 \frac{(\psi - 1)^2}{\psi} \right) \equiv \mathcal{L}(\dot{\mathbf{q}}, \psi), \tag{2}$$

which contains the additional, dimensionless degree of freedom ψ. The parameter c_0 has the physical dimension of a velocity.

(a) Show that the extremum of the action integral yields a theory obeying special relativity for which c_0 is the maximal velocity (in other words, one obtains the Lagrangian in Eq. 1 with the velocity of light c replaced by c_0).

(b) What happens when $c_0 \to \infty$?