Assignment: HW7 [40 points]

Assigned: 2006/11/15
Due: 2006/11/22

P7.1 [$6 + 8 = 14$ points]

A particle of mass m can move in one dimension under the influence of two springs connected to two walls that are a distance a apart, as shown in Fig. 7.1. The springs obey Hooke's law, have zero unstretched lengths and spring constants k_1 and k_2 respectively.

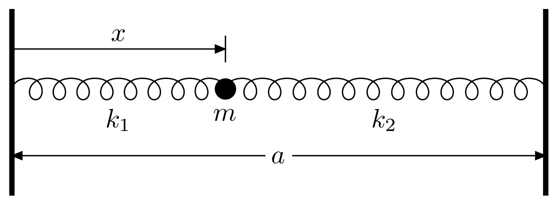

Figure 7.1

(a) Let x be the length of the first spring, and b its equilibrium value. Using the displacement from the equilibrium, $q = x - b$, as the generalized coordinate, find the Hamiltonian and the total energy of the system, and examine if these quantities are conserved.

(b) Now consider the coordinate transformation

$$Q = q - b\sin(\omega t), \tag{1}$$

where ω is some constant (not necessarily the natural frequency of the system). Find the Hamiltonian and the total energy of the system in terms of Q and its conjugate momentum P, and examine if these quantities are conserved.

P7.2 [$4 + 2 = 6$ points]

Let (q, p) be the phase-space coordinates of a system with one degree of freedom.

(a) Under what conditions is the transformation

$$Q = \alpha\frac{p}{q}; \qquad P = \beta q^2 \tag{2}$$

canonical (α and β are constants)?

(b) Find a suitable generating function of type 1, $F_1(q, Q)$, for the above transformation.

P7.3 [6 points]

Consider the (continuous and regular) one-parameter group of canonical transformations ψ_θ defined as the solution to the differential equation

$$\frac{\partial\psi^\mu(\omega_0;\theta)}{\partial\theta} = \epsilon^{\mu\nu}\left[\frac{\partial\phi(\omega)}{\partial\omega^\nu}\right]_{\omega^\mu=\psi^\mu(\omega_0;\theta)} = \epsilon^{\mu\nu}\frac{\partial\phi(\omega_0;\theta)}{\partial\psi^\nu}, \tag{3}$$

where ϕ^μ are some functions of ω, independent of θ. Indeed, ϕ^μ is the inverse of the ratio of the infinitesimal change in the parameter θ and the corresponding change in the phase space coordinate ω^μ:

$$\omega'^\mu = \omega^\mu + \delta\theta\phi^\mu(\omega). \tag{4}$$

Show that a function $A(\omega) = A(\psi(\omega_0;\theta)) = A_\theta(\omega_0)$ will obey the differential equation

$$\frac{\partial A_\theta(\omega_0)}{\partial\theta} = \left\{A_\theta(\omega_0), \phi(\omega_0)\right\}_{\omega_0}, \tag{5}$$

which then leads to the power series solution

$$A(\omega) = A_\theta(\omega_0) = A(\omega_0)+\theta\left\{A_\theta(\omega_0), \phi(\omega_0)\right\}+\frac{\theta^2}{2!}\left\{\left\{A_\theta(\omega_0), \phi(\omega_0)\right\}, \phi(\omega_0)\right\}+\cdots. \tag{6}$$

P7.4 [$10 + 4 = 14$ points]
A particle of mass m moves in one dimension under a potential $V(x) = kx^{-2}$, where x is the Cartesian coordinate and k a constant.

(a) Find $x(t)$ using the Poisson bracket form of the equation of motion for the quantity $y = x^2$ given the initial conditions $x(0) = x_0$, $p(0) = 0$.

(b) Show that the quantity $D = xp - 2Ht$ is a constant of the motion.