Assignment: HW5 [40 points]

Assigned: 2006/11/02
Due: 2006/11/08

P5.1 [10 points]

A simple plane pendulum has a mass m hanging at the end of a massless string of length ℓ in a field of constant gravitational acceleration $\vec{g}$. While the pendulum is in motion, the length of the string is changed at a constant rate $\dot{\ell} = v_0$. Find the Lagrangian and the Hamiltonian, determine whether or not $T + V$ and H are conserved, and comment on the physical interpretation of your results. This is a rather famous problem discussed by Einstein, Lorentz, and others at the 1911 Solvay Conference.
Hint: Since ℓ is not an independent generalized coordinate, but is constrained to be a simple linear function of time, the problem lends itself to be treated as one with just one degree of freedom.

P5.2 [$2 + 4 + 4 = 10$ points]

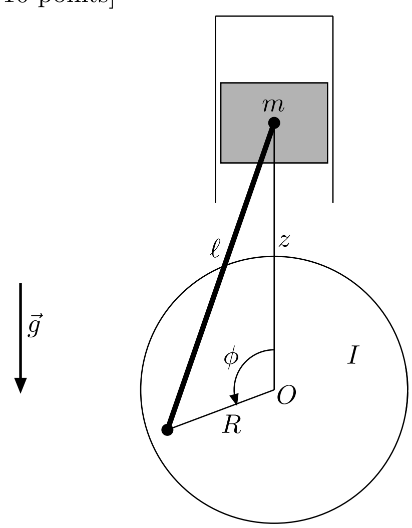

Figure 5.1

A lawn-mower engine contains a piston of mass m that moves along $\hat{z}$ in a field of constant gravitational acceleration $\vec{g} = g\hat{\mathbf{z}}$. The center of mass of the piston is connected to a flywheel of moment of intertia I at a distance R from its center by a rigid and massless rod of length ℓ, as shown in Fig. 5.1. The system has only one degree of freedom but two natural coordinates, ϕ and z.

(a) Express the Lagrangian in terms of $q_1 = z$, $q_2 = \phi$ and write the constraint equation that connects the two coordinates.

(b) From the above results, write down the two coupled equations of motion using the method of “undetermined multiplier”s. Then eliminate the undetermined multiplier to obtain a single equation of motion (it can still involve both coordinates).

(c) Find $p_\phi(z, \phi, \dot{\phi})$.

P5.3 $[4+4+2=10 \text{ points}]$

Liouville's Theorem gives information about the statistical properties of systems containing a very large number of particles. The theorem can be expressed as $\dot{D}=0$, where D is the phase space density of possible systems in that region of phase space. There is no equivalent theorem that can be expressed in terms of quantities in configuration space. Thus, problems in statistical mechanics are important examples where the Hamiltonian approach offers a solution while the Lagrangian approach does not.

Now consider the example of a beam of identical charged particles with momentum P, produced by an accelerator. Suppose that in the plane perpendicular to the incident direction, the beam initially has a uniformly populated circular cross section of radius r_1 in configuration space, and a uniformly populated circular cross section of radius p_1 in momentum space. A pair of quadrupole magnets with appropriate relative orientation can focus a beam of charged particles, i.e., can reduce the transverse radius from r_1 to r_2, where $r_2 < r_1$.

(a) What does Liouville's theorem tell us about the consequences of this focusing operation?

(b) Suppose that the beam pipe has an internal radius R. At what maximum distance downstream from the focus must another focusing element be located in order to avoid some of the beam particles scraping the pipe?

(c) A collimator (an absorber with a hole in it) could also be used to produce a beam with radius r_2. Contrast the consequences of using a focusing element vis-a-vis a collimator.

P5.4 $[3+3+4=10 \text{ points}]$

For a three-particle system with masses m_i, coordinates $\vec{r}_i$, and canonical momenta $\vec{p}_i$ $(i=1,2,3)$, we introdce the following (Jacobian) coordinates:

$$
\begin{aligned}
\vec{\rho}_1 &\equiv \vec{r}_2-\vec{r}_1 \quad \text{(the coordinate of particle 2 relative to 1)}, \\
\vec{\rho}_2 &\equiv \vec{r}_3-\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2} \quad \text{(the coordinate of particle 3 relative to the c.m. of 1 and 2)}, \\
\vec{\rho}_3 &\equiv \frac{m_1\vec{r}_1+m_2\vec{r}_2+m_3\vec{r}_3}{m_1+m_2+m_3} \quad \text{(the c.m. of the three particles)}, \\
\vec{\pi}_1 &\equiv \frac{m_1\vec{p}_2-m_2\vec{p}_1}{m_1+m_2}, \\
\vec{\pi}_2 &\equiv \frac{(m_1+m_2)\vec{p}_3-m_3(\vec{p}_1+\vec{p}_2)}{m_1+m_2+m_3}, \\
\vec{\pi}_3 &\equiv \vec{p}_1+\vec{p}_2+\vec{p}_3.
\end{aligned}
\tag{1}
$$

Assume that the canonical momenta $\vec{p}_i$ are the same as the kinetic momenta (i.e., $\vec{p}_i = m_i\dot{\vec{r}}_i$).

(a) What are the physical interpretatiton of the momenta $\vec{\pi}_i$?

(b) How could we define such (Jacobian) coordinates and momenta for any arbitrary number of particles?

(c) Show that the transformation

$$\{\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{p}_1, \vec{p}_2, \vec{p}_3\} \to \{\vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_3, \vec{\pi}_1, \vec{\pi}_2, \vec{\pi}_3\} \tag{2}$$

is canonical.

Hint: Use reduced masses.