Assignment: HW3 [40 points]

Assigned: 2006/10/04
Due: 2006/10/11

Solutions

P3.1 [7 points]
A smooth rod of length ℓ rotates in a plane with a constant angular velocity ω about an axis fixed at one end of the rod and perpendicular to the plane of motion. A bead of mass m, free to move along the rod, is initially positioned at the fixed end of the rod and given a slight push such that its initial speed directed towards the other end of the rod is $\omega\ell$. Using Lagrange's method, find the time it takes the bead to reach the other end of the rod.

S3.1 Let $\{r, \theta\}$ denote the polar coordinates of the bead in the frame whose origin is at the fixed point on the rod ($\theta = 0$ is arbitrary). Then, its kinetic energy is

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{m}{2}(\dot{r}^2 + r^2\omega^2) \tag{1}$$

and potential energy

$$U = 0. \tag{2}$$

Thus, the Lagrangian is

$$L = T - U = \frac{m}{2}(\dot{r}^2 + r^2\omega^2). \tag{3}$$

And the Euler-Lagrange equation gives

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \quad \Rightarrow \quad \ddot{r} - \omega^2 r = 0 \tag{4}$$

The general solution to this equation of motion is

$$r(t) = Ae^{\omega t} + Be^{-\omega t} \tag{5}$$

to which we apply the initial conditions:

$$r(0) = 0 \quad \Rightarrow \quad A + B = 0 \tag{6}$$

and

$$\dot{r}(0) = \ell\omega \quad \Rightarrow \quad A - B = \ell. \tag{7}$$

Solving Eqs. 6 and 7, we get

$$A = \frac{\ell}{2}; \qquad B = -\frac{\ell}{2}. \tag{8}$$

So,

$$r(t) = \frac{\ell}{2}\left(e^{\omega t} - e^{-\omega t}\right). \tag{9}$$

If τ is the time the bead takes to reach the other end, then

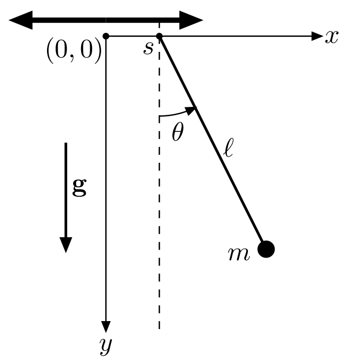
$$r(\tau) = \ell \quad \Rightarrow \quad \frac{1}{2}\left(e^{\omega\tau} - e^{-\omega\tau}\right) = 1, \tag{10}$$

or,

$$\tau = \frac{1}{\omega} \ln \left(\sqrt{2} + 1 \right). \tag{11}$$

P3.2 [7 points]
Using Lagrange's method, find the two-dimensional equation of motion of a pendulum of mass m suspended at the end of a massless rod of length ℓ in a gravitational field of uniform acceleration $\mathbf{g}$, whose point of support is executing a simple harmonic motion in the direction perpendicular to gravity, as shown in the figure below, i.e., the coordinates of the point of support are given as functions of time by

$$x_s(t) = x_0 \cos(\omega t); \qquad y_s(t) = 0.$$



Use θ, the angle between the pendulum and the direction of gravity, as the generalized coordinate, and express your answer in terms of θ (and its time derivatives). Assume θ to be small and use the corresponding approximations to simplify your answer. Compare your result to the equation of motion of a forced harmonic oscillator.

S3.2 The Cartesian coordinates and velocities of the bob are

$$\begin{aligned} & x = x_s + \ell \sin\theta = x_0 \cos(\omega t) + \ell \sin\theta, && y = \ell\cos\theta \\ \Rightarrow \quad & \dot{x} = -x_0\omega \sin(\omega t) + \ell\dot{\theta}\cos\theta, && \dot{y} = -\ell\dot{\theta}\sin\theta. \end{aligned} \tag{12}$$

So, the kinetic energy is

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}[x_0^2\omega^2 \sin^2(\omega t) - 2x_0\omega\ell\dot{\theta}\sin(\omega t)\cos\theta + \ell^2\dot{\theta}^2], \tag{13}$$

and the potential energy

$$U = -mgy = -mg\ell\cos\theta, \tag{14}$$

and the Lagrangian (we omit m which would eventually drop out anyway when we write the equation of motion)

$$L = T - V = \frac{1}{2}[x_0^2\omega^2 \sin^2(\omega t) - 2x_0\omega\ell\dot{\theta}\sin(\omega t)\cos\theta + \ell^2\dot{\theta}^2] + g\ell\cos\theta. \tag{15}$$

$$\frac{\partial L}{\partial \theta} = x_0\omega\ell\dot{\theta}\sin(\omega t)\sin\theta - g\ell\sin\theta, \tag{16}$$

$$\frac{\partial L}{\partial \dot{\theta}} = -x_0\omega\ell\sin(\omega t)\cos\theta + \ell^2\dot{\theta}, \tag{17}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -x_0\omega^2\ell\cos(\omega t)\cos\theta + x_0\omega\ell\dot{\theta}\sin(\omega t)\sin\theta + \ell^2\ddot{\theta}, \tag{18}$$

Hence, the Euler-Lagrange equation of motion gives

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \quad \Rightarrow \quad -g\ell\sin\theta + x_0\omega^2\ell\cos(\omega t)\cos\theta - \ell^2\ddot{\theta} = 0 \tag{19}$$

If θ is small, so all terms containing θ^2 or higher powers of θ can be neglected, then $\sin\theta \approx \theta$, $\cos\theta \approx 1$, and Eq. 19 reduces to

$$\ddot{\theta} + \omega_0^2\theta = \frac{x_0}{\ell}\omega^2\cos(\omega t) \tag{20}$$

where $\omega_0 = \sqrt{\frac{g}{l}}$. Equation 20 represents a simple harmonic oscillator driven by a sinusoidal force (no damping).

P3.3 [6+3 = 9 points]

(a) Obtain the Hamiltonian and the canonical equations for a particle in a central force field (in 3 dimensions).

(b) Take two of the initial conditions to be $p_\phi(0) = 0$ and $\phi(0) = 0$ (this is essentially the choice of a particular spherical coordinate system). Discuss the resulting simplification of the canonical equations.

S3.3 **(a)** The Lagrangian is

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + (r^2\sin^2\theta)\dot{\phi}^2\right) - U(r), \tag{21}$$

where the first of the two terms on the right hand side is the kinetic energy T, and $U(r)$ is the potential energy. The conjugate momenta are

$$\begin{aligned} p_r &\equiv \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \\ p_\theta &\equiv \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \\ p_\phi &\equiv \frac{\partial L}{\partial \dot{\phi}} = m(r^2\sin^2\theta)\dot{\phi}, \end{aligned} \tag{22}$$

which give us the generalized velocities in terms of the generalized momenta:

$$\begin{aligned} \dot{r} &= \frac{p_r}{m}, \\ \dot{\theta} &= \frac{p_\theta}{mr^2}, \\ \dot{\phi} &= \frac{p_\phi}{mr^2\sin^2\theta}. \end{aligned} \tag{23}$$

The Hamiltonian is

$$H = p_i \dot{q}_i - L = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\theta} + U(r) \equiv T + U. \quad (24)$$

Hamilton's canonical equations in these variables are

$$\begin{aligned}
\dot{r} &\equiv \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, & \dot{p}_r &\equiv -\frac{\partial H}{\partial r} = \frac{1}{mr^3}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - \frac{\partial U}{\partial r}; \\
\dot{\theta} &\equiv \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, & \dot{p}_\theta &\equiv -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{mr^2 \sin^3\theta}; \\
\dot{\phi} &\equiv \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2 \sin^2\theta}, & \dot{p}_\phi &\equiv -\frac{\partial H}{\partial \phi} = 0.
\end{aligned} \quad (25)$$

Notice that the three equations on the left follow immediately from the definitions of the generalized momenta. This is always the case with Hamilton's canonical equations: by studying their derivation it becomes clear that half of them are simply inversions of those definitions.

(b) If $p_\phi(0) = 0$, the last of the canonical equations implies that $p_\phi(t) = 0$ for all t. Then the canonical equations become

$$\begin{aligned}
\dot{r} &= \frac{p_r}{m}, & \dot{p}_r &= \frac{p_\theta^2}{mr^3} - \frac{\partial U}{\partial r}; \\
\dot{\theta} &= \frac{p_\theta}{mr^2}, & \dot{p}_\theta &= 0; \\
\dot{\phi} &= 0, & \dot{p}_\phi &= 0.
\end{aligned} \quad (26)$$

From $\phi(0) = 0$ and these equations it follows that $\phi(t) = 0$ for all t, i.e., the motion is confined to the $\phi = 0$ plane. The result is a system with two degrees of freedom described by the first four canonical equations, the last of which implies that p_θ, i.e., the angular momentum in the $\phi = 0$ plane, is a constant of the motion.

P3.4 [$5 + 5 = 10$ points]
The 3-dimensional motion of a particle of mass m is described by the Lagrangian function

$$L = \frac{m}{2}\dot{x}_i^2 + \omega l_3, \quad (27)$$

where l_3 represents the third (z) component of the angular momentum, and ω is the corresponding constant angular velocity.

(a) Find the equations of motion, write them in terms of the complex variable $u \equiv x_1 + ix_2$, and of x_3, and solve them.

(b) Find the *kinetic* and *canonical* momenta, and construct the Hamiltonian. Show that the particle has only kinetic energy, and that the canonical momenta are conserved.

S3.4 **(a)**

$$\dot{u}\dot{u}^* = \dot{x}_1^2 + \dot{x}_2^2. \tag{28}$$

$$l_3 = m(x_1\dot{x}_2 - x_2\dot{x}_1) = \frac{m}{2i}(\dot{u}u^* - u\dot{u}^*). \tag{29}$$

So,

$$L(u, x_3, \dot{u}, \dot{x}_3) = \frac{m}{2}\left((\dot{u}\dot{u}^* + \dot{x}_3^2) - i\omega(\dot{u}u^* - u\dot{u}^*)\right). \tag{30}$$

The equations of motion are

$$\ddot{u}^* - i\omega\dot{u}^* = 0; \quad \text{and} \quad \ddot{x}_3 = 0. \tag{31}$$

So,

$$\dot{u}^* = e^{i\omega t}, \quad \text{or,} \quad u = \frac{i}{\omega}e^{-i\omega t} + C, \tag{32}$$

where C is a complex constant. The solutions for x_1 and x_2 follow:

$$x_1 = \frac{1}{\omega}\sin(\omega t) + C_1, \quad x_2 = \frac{1}{\omega}\cos(\omega t) + C_2, \tag{33}$$

where $C_1 = \mathrm{Re}(C)$, $C_2 = \mathrm{Im}(C)$. Also,

$$x_3 = C_3 t + C_4, \tag{34}$$

where C_3 and C_4 are real constants.

(b) The canonically conjugate momenta are obtained as $\dfrac{\partial L}{\partial \dot{q}_i}$:

$$p_1 = m(\dot{x}_1 - \omega x_2), \quad p_2 = m(\dot{x}_2 - \omega x_1), \quad p_3 = m\dot{x}_3, \tag{35}$$

while the kinetic momenta are given by $\pi_i = m\dot{x}_i$. In order to construct the Hamiltonian, we first express the velocities in terms of the canonical momenta

$$\dot{x}_1 = \frac{p_1}{m} + \omega x_2, \quad \dot{x}_2 = \frac{p_2}{m} - \omega x_1, \quad \dot{x}_3 = \frac{p_3}{m}. \tag{36}$$

When we substitute these into L and in the Hamiltonian

$$H = p_i\dot{x}_i - L, \tag{37}$$

all position-dependent terms cancel, leaving

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2). \tag{38}$$

Thus, the particle has only kinetic energy, and the canonical momenta p_i are conserved (since $\dot{p}_i = \dfrac{\partial H}{\partial x_i} = 0$).

P3.5 [$3 + 4 = 7$ points]
Invariance under time translations and Noether's theorem. The theorem of E. Noether can be applied to the case of translations in *time* by means of the following procedure. Make t a coordinate-like variable by parametrizing both q and t as functions of a common independent variable τ:

$$q_i = q_i(\tau) \qquad (i = 1, \ldots, n); \qquad t = t(\tau), \tag{39}$$

and by defining a new Lagrangian function in terms of the old one:

$$\tilde{L}\left(q_i, t, \frac{dq_i}{d\tau}, \frac{dt}{d\tau}\right) \equiv L\left(q_i, \frac{1}{\frac{dt}{d\tau}}\frac{dq_i}{d\tau}, t\right)\frac{dt}{d\tau} \tag{40}$$

(a) Show that Hamilton's variational principle applied to $\tilde{L}$ yields the same equations of motion as it does for L.

(b) Assume L to be invariant under time translations:

$$h^s(q_i, t) = (q_i, t + s). \tag{41}$$

Apply Noether's theorem to $\tilde{L}$ and find the constant of motion corresponding to the invariance.

S3.5 **(a)** Hamilton's variational principle, when applied to $\tilde{L}$, requires

$$\tilde{S} \equiv \int_{\tau_1}^{\tau_2} \tilde{L} d\tau \tag{42}$$

to be an extremum. Now, since

$$\int_{\tau_1}^{\tau_2} \tilde{L} d\tau = \int_{t_1}^{t_2} L dt, \quad \text{with} \quad t_1 = t_1(\tau_1),\ t_2 = t_2(\tau_2), \tag{43}$$

The action integral $\tilde{S}$ is extremal if and only if the Lagrangian equations that follow from L are fulfilled.

(b) Treating t as another generalized co-ordinate, with the correspondence between all coordinates marked by a common independent parameter τ, we have $t = q_{n+1}(\tau)$. The corresponding generalized momentum is given by

$$p_{n+1} = \frac{\partial \tilde{L}}{\partial (dt/d\tau)} = L + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \left(-\frac{1}{(dt/d\tau)^2} \right) \frac{\partial L}{\partial \dot{q}_i} \frac{dq_i}{dt}. \tag{44}$$

Since $\tilde{L}$ does not depend explicitly on t (according to the stated invariance under time translation), Noether's theorem dictates that p_{n+1} be conserved. Note that except for a sign, the RHS is just the expression for the total energy.