Assignment: HW2 [40 points]

Assigned: 2006/09/27
Due: 2006/10/04

Solutions

P2.1 [$1+3+3+3+2 = 12$ points]

Let the motion of a point mass be governed by the law

$$\ddot{\mathbf{r}} = \dot{\mathbf{r}} \times \mathbf{a}, \qquad \mathbf{a} = \text{ const.} \tag{1}$$

(a) Show that $\dot{\mathbf{r}} \cdot \mathbf{a}$ is constant in time.

(b) Reduce Eq. 1 to an inhomogeneous differential equation of the form $\ddot{\mathbf{r}} + \omega^2 \mathbf{r} = f(t)$.

(c) Solve the above equation by using a particular function of the form $\mathbf{r}_p = \mathbf{c}t + \mathbf{d}$.

(d) Express the constants of integration in terms of the initial values $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$.

(e) Describe the curve $\mathbf{r}(t) = \mathbf{r}_g(t) + \mathbf{r}_p(t)$, where $\mathbf{r}_g(t)$ is the solution of the homogeneous equation $\ddot{\mathbf{r}} + \omega^2 \mathbf{r} = 0$.

S2.1 **(a)**

$$\frac{d}{dt}(\dot{\mathbf{r}} \cdot \mathbf{a}) = \ddot{\mathbf{r}} \cdot \mathbf{a} = (\dot{\mathbf{r}} \times \mathbf{a}) \cdot \mathbf{a} = 0. \tag{2}$$

(b)

$$\frac{d}{dt}\ddot{\mathbf{r}} = \ddot{\mathbf{r}} \times \mathbf{a} = (\dot{\mathbf{r}} \times \mathbf{a}) \times \mathbf{a} = -(\mathbf{a} \cdot \mathbf{a})\dot{\mathbf{r}} + (\dot{\mathbf{r}} \cdot \mathbf{a})\mathbf{a}. \tag{3}$$

We have shown the second term to be constant. Thus, integrating this equation from 0 to t, we get

$$\ddot{\mathbf{r}}(t) - \mathbf{r}(0) = -\omega^2(\mathbf{r}(t) - \mathbf{r}(0)) + (\dot{\mathbf{r}}(0) \cdot \mathbf{a})\mathbf{a}t, \tag{4}$$

where $\omega^2 = \mathbf{a} \cdot \mathbf{a} = \mathbf{a}^2$. Since $\ddot{\mathbf{r}}(0) = \dot{\mathbf{r}}(0) \times \mathbf{a}$, we may write

$$\ddot{\mathbf{r}}(t) + \omega^2 \mathbf{r}(t) = (\dot{\mathbf{r}}(0) \cdot \mathbf{a})\mathbf{a}t + \dot{\mathbf{r}}(0) \times \mathbf{a} + \omega^2 \mathbf{r}(0). \tag{5}$$

(c) The general solution of the homogeneous differential equation is

$$\mathbf{r}_g(t) = \mathbf{c}_1 \sin(\omega t) + \mathbf{c}_2 \cos(\omega t). \tag{6}$$

So, the particular solution is simply the RHS of Eq. 5 divided by ω^2, which is of the form

$$\mathbf{r}_p(t) = \mathbf{c}t + \mathbf{d} \tag{7}$$

(d) We can get the constants in Eqs. 6 and 7 by comparing them with the corresponding terms in Eq. 5 and using the initial conditions

$$\begin{aligned}
\mathbf{c}_1 &= \frac{1}{\omega^3}(\mathbf{a}^2\dot{\mathbf{r}}(0) - (\dot{\mathbf{r}}(0)\cdot\mathbf{a})\mathbf{a}) = \frac{1}{\omega^3}\mathbf{a}\times(\dot{\mathbf{r}}(0)\times\mathbf{a}),\\
\mathbf{c}_2 &= \frac{1}{\omega^2}\dot{\mathbf{r}}(0)\times\mathbf{a},\\
\mathbf{c} &= \frac{1}{\omega^2}(\dot{\mathbf{r}}(0)\cdot\mathbf{a})\mathbf{a},\\
\mathbf{d} &= \frac{1}{\omega^2}\dot{\mathbf{r}}(0)\times\mathbf{a} + \mathbf{r}(0).
\end{aligned} \tag{8}$$

(e) It represents a helix winding around the vector $\mathbf{a}$.

P2.2 $[2 + 5 = 7$ points$]$
Use variational calculus to find

(a) The shortest connection between two points (x_1, y_1) and (x_2, y_2) on the 2-dimensional Euclidean plane.

(b) The shape of a line of uniform mass density supported between (x_1, y_1) and (x_2, y_2) in a uniform gravitational field $\mathbf{g} = g\hat{\mathbf{e}}_y$.

S2.2 In the general case, where the action functional is the integral of a scalar function $f(y, y', x)$, with $y' = \frac{dy}{dx}$, the Euler-Lagrange equation reads

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'} \tag{9}$$

Multiplying by y' and adding the term $y''\frac{\partial f}{\partial y'}$ to both sides we get

$$y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} = \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) \tag{10}$$

If f does not depend explicitly on x, then $\frac{\partial f}{\partial x} = 0$. Adding $\frac{\partial f}{\partial x}$ to the LHS of the above equation gives $\frac{df(y,y')}{dx}$, while the RHS is left unchanged:

$$\frac{df(y,y')}{dx} = \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right), \tag{11}$$

or,

$$y'\frac{\partial f}{\partial y'} - f(y, y') = \text{const.} \tag{12}$$

(a) We have to minimize the arc length

$$L = \int ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\; dx. \tag{13}$$

So, $f(y, y') = \sqrt{1 + y'^2}$, and Eq. 12 gives

$$y'\frac{\sqrt{y'}}{\sqrt{1+y'^2}} - \sqrt{1 + y'^2} = \text{const.} \qquad \text{or,} \qquad y' = \text{const.} \tag{14}$$

Thus, $y = ax + b$. Applying the boundary conditions $y(x_1) = y_1$, $y(x_2) = y_2$ we get

$$y(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1. \tag{15}$$

(b) Equilibrium under gravity is reached when the y coordinate of the center of mass is at its lowest point. Thus, we must minimize the functional

$$L = \int y ds = \int_{x_1}^{x_2} y\sqrt{1 + y'^2}\, dx. \tag{16}$$

Once again, we use Eq. 12 with $f(y, y') = y\sqrt{1 + y'^2}$ and get

$$\frac{yy'^2}{\sqrt{1 + y'^2}} - y\sqrt{1 + y'^2} = -\frac{y}{\sqrt{1 + y'^2}} = C_1 = \text{const}, \tag{17}$$

or,

$$\frac{y^2}{C_1^2} - y'^2 = 1. \tag{18}$$

This is a separable differential equation whose general solution is

$$y(x) = C_1 \cosh\left(\frac{x}{C_1} + C_2\right) \tag{19}$$

The constants C_1 and C_2 are determined by the boundary conditions $y(x_1) = y_1$, $y(x_2) = y_2$.

P2.3 [$2 + 3 = 5$ points]
If a conservative force field in 3-dimensional space is axially symmetric, and we choose the z axis of a cylindrical coordinate system $\{r, \phi, z\}$ along the axis of symmetry, then show that

(a) The corresponding potential has the form $U = U(r, z)$,

(b) The force always lies in a plane containing the z axis.

S2.3 **(a)** A general conservative force in 3 dimensions can be expressed as

$$\nabla U(r, \phi, z) = \frac{\partial U}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\mathbf{e}}_\phi + \frac{\partial U}{\partial z}\hat{\mathbf{e}}_z \tag{20}$$

Since the force is axially symmetric, the coefficient of $\hat{\mathbf{e}}_\phi$ must be zero:

$$\frac{\partial U}{\partial \phi} = 0 \quad \Longrightarrow \quad U = U(r, z). \tag{21}$$

(b) The force lies in the plane spanned by the unit vectors $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_z$, which always contains the z axis.

P2.4 [$3 + 2 = 5$ points]
In Problem P1.4 (see Homework assignment 1), is

$$H \equiv \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L \tag{22}$$

conserved? Evaluate H and compare it to the energy $E \equiv T + V$.

S2.4 Recall the Lagrangian

$$L = \frac{m}{2}(R^2\dot{\theta}^2 + R^2\Omega^2 \sin^2\theta) + mgR\cos\theta \tag{23}$$

and note that it is time-independent. Hence, H is conserved. Direct calculation yields

$$H(\theta, \dot{\theta}) = \frac{m}{2}(R^2\dot{\theta}^2 - R^2\Omega^2 \sin^2\theta) - mgR\cos\theta. \tag{24}$$

This is not the energy. The energy E is

$$E \equiv T + V = \frac{m}{2}(R^2\dot{\theta}^2 + R^2\Omega^2 \sin^2\theta) - mgR\cos\theta = H + mR^2\Omega^2\sin^2\theta. \tag{25}$$

Since H is constant in time, but θ may vary, E is not constant in time. Energy must enter and leave the system to keep the hoop rotating at a constant speed, and the term $mR^2\Omega^2\sin^2\theta$ represents this varying amount of energy.

P2.5 [$3 + 6 + 2 = 11$ points]
The Lagrangian for a particle of mass m and electric charge e moving in the xy plane is given by

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + eEy - \frac{eB}{c}y\dot{x}.$$

This describes the motion of the particle in a uniform electric field E in the y direction and a uniform magnetic field B in the z direction. c is the speed of light.

(a) Write down the Euler-Lagrange equations.

(b) Find the Hamiltonian. Simplify the expression (eliminating $\dot{x}$ by making use of a first integral of motion). What can you say about the general allowed motions in $y(t)$?

(c) At $t = 0$, $x = y = 0$. What critical value v_c must $\dot{x}(0)$ take in order for the particle to go in a uniform motion? What is the corresponding value of $\dot{y}$? (You can either make use of the results in part (a) or that in part (b).)

S2.5 **(a)** The Euler-Lagrange equations are

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(m\dot{x} - \frac{eB}{c}y\right) = 0 \quad \Rightarrow \quad \ddot{x} = \frac{eB}{mc}\dot{y} \tag{26}$$

and

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0 \quad \Rightarrow \quad \frac{d}{dt}(m\dot{y}) = eE - \frac{eB}{c}\dot{x} \quad \Rightarrow \quad \ddot{y} = -\frac{eB}{mc}\dot{x} + \frac{eE}{m}. \tag{27}$$

(b)

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{eB}{c}y \quad \Rightarrow \quad \dot{x} = \frac{p_x}{m} + \frac{eB}{mc}y. \tag{28}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \quad \Rightarrow \quad \dot{y} = \frac{p_y}{m}. \tag{29}$$

Hence, the Hamiltonian is

$$\begin{aligned} H &= \dot{x}p_x + \dot{y}p_y - L \\ &= \frac{p_x^2}{m} + \frac{eB}{mc}p_x y + \frac{p_y^2}{m} - \frac{m}{2}\left[\left(\frac{p_x}{m} + \frac{eB}{mc}y\right)^2 + \left(\frac{p_y}{m}\right)^2\right] \\ &\quad - eEy + \frac{eB}{c}y\left(\frac{p_x}{m} + \frac{eB}{mc}y\right) \\ &= \frac{1}{2m}(p_x^2 + p_y^2) + \left(\frac{eB}{mc}p_x - eE\right)y + \frac{e^2B^2}{2mc^2}y^2. \end{aligned} \tag{30}$$

Since x does not explicitly appear in the Hamiltonian, the corresponding momentum p_x must be conserved:

$$\dot{p_x} = -\frac{\partial H}{\partial x} = 0 \quad \Rightarrow \quad p_x = \text{const.} \tag{31}$$

Using Eqs. 28 and 29 in another Hamilton's equation,

$$\ddot{y} = \frac{\dot{p_y}}{m} = -\frac{1}{m}\frac{\partial H}{\partial y} = -\frac{eB}{m^2c}p_x + \frac{eE}{m} - \frac{e^2B^2}{m^2c^2}y \tag{32}$$

or,

$$\ddot{y} + \frac{e^2B^2}{m^2c^2}y = \frac{e}{m}\left(E - \frac{p_x}{mc}B\right) = \text{const.} \tag{33}$$

Equation 33 represents a simple harmonic oscillator along y, with a characteristic frequency $\omega_0 = \dfrac{eB}{mc}$, acted on by a constant driving force. Note that the same equation could be obtained by integrating both sides of Eq. 26 once with respect to t to get $\dot{x}$ and substituting it in Eq. 27.

(c) Uniform motion $\Rightarrow \quad \ddot{x} = 0$; $\ddot{y} = 0$. By Eqs. 27 and 26, this means

$$v_c = \dot{x} = \frac{cE}{B} \quad \text{and} \quad \dot{y} = 0. \tag{34}$$