Assignment: HW1 [40 points]

Assigned: 2006/09/20
Due: 2006/09/27

P1.1 [10 points]
Show that the Galilei transformations $g(\mathbf{R}(\psi, \hat{\mathbf{n}}), \mathbf{w}, \mathbf{a}, s)$,

$$\begin{pmatrix} \mathbf{r} \\ t \end{pmatrix} \xrightarrow{g} \begin{pmatrix} \mathbf{r}' = \mathbf{R}\mathbf{r} + \mathbf{w}t + \mathbf{a} \\ t' = \lambda t + s \end{pmatrix}, \tag{1}$$

with $\det \mathbf{R} = +1$, $\lambda = +1$, form a group.[1]

S1.1 Consider the composition of two successive transformations,

$$\begin{pmatrix} \mathbf{r}_0 \\ t_0 \end{pmatrix} \xrightarrow{g^{(0,1)}} \begin{pmatrix} \mathbf{r}_1 = \mathbf{R}^{(\mathbf{0},\mathbf{1})}\mathbf{r}_\mathbf{0} + \mathbf{w}^{(0,1)}t + \mathbf{a}^{(0,1)} \\ t_1 = t_0 + s^{(0,1)} \end{pmatrix}, \tag{2}$$

and

$$\begin{pmatrix} \mathbf{r}_1 \\ t_1 \end{pmatrix} \xrightarrow{g^{(1,2)}} \begin{pmatrix} \mathbf{r}_2 = \mathbf{R}^{(\mathbf{1},\mathbf{2})}\mathbf{r}_\mathbf{1} + \mathbf{w}^{(1,2)}t + \mathbf{a}^{(1,2)} \\ t_2 = t_1 + s^{(1,2)} \end{pmatrix}. \tag{3}$$

Writing the transformation from $\mathbf{r}_0$ to $\mathbf{r}_2$ in the same way,

$$\begin{pmatrix} \mathbf{r}_0 \\ t_0 \end{pmatrix} \xrightarrow{g^{(0,2)}} \begin{pmatrix} \mathbf{r}_2 = \mathbf{R}^{(\mathbf{0},\mathbf{2})}\mathbf{r}_\mathbf{0} + \mathbf{w}^{(0,2)}t + \mathbf{a}^{(0,2)} \\ t_2 = t_0 + s^{(0,2)} \end{pmatrix}, \tag{4}$$

we read off the following relations

$$\begin{aligned} \mathbf{R}^{(0,2)} &= \mathbf{R}^{(1,2)}\mathbf{R}^{(0,1)}, \\ \mathbf{w}^{(0,2)} &= \mathbf{R}^{(1,2)}\mathbf{w}^{(0,1)} + \mathbf{w}^{(1,2)}, \\ \mathbf{a}^{(0,2)} &= \mathbf{R}^{(1,2)}\mathbf{a}^{(0,1)} + s^{(0,1)}\mathbf{w}^{(1,2)} + \mathbf{a}^{(1,2)}, \\ s^{(0,2)} &= s^{(1,2)} + s^{(0,1)} \end{aligned} \tag{5}$$

We can now see how these transformations form a group by verifying that they satisfy the four group axioms:

1. There is an operation defining the composition of two Galilei transformations
$$g^{(0,1)}g^{(1,2)} = g^{(0,2)}, \tag{6}$$
as we have explicitly worked out in Eq. 5.
2. The composition is an associative operation: $g_3(g_2g_1) = (g_3g_2)g_1$. This is so because both addition and matrix multiplication have this property.

[1] This is the *proper, orthochronous Galilei group* $G_+^{\uparrow 4}$.

3. There exists a unit element $e = g(\mathbf{1}, \mathbf{0}, \mathbf{0}, 0)$, which satisfies the condititon $g_i e = e g_i = g_i$ for all $g_i \in G_+^{\uparrow 4}$.
4. For every $g \in G_+^{\uparrow 4}$, there is an inverse transformation g^{-1} such that $gg^{-1} = g^{-1}g = e$. If $g = g(\mathbf{R}, \mathbf{w}, \mathbf{a}, s)$, then it can be seen easily from Eq. 5 that $g^{-1} = g(\mathbf{R}^T, -\mathbf{R}^T\mathbf{w}, s\mathbf{R}^T\mathbf{w} - \mathbf{R}^T\mathbf{a}, -s)$.

P1.2 $[2 + 4 + 4 = 10$ points$]$
A particle of mass m moves without friction along a symmetrical planar curve $s = s(\theta)$ whose axis of symmetry is parallel to a uniform gravitational field of acceleration $\mathbf{g}$. s is the displacement in arc length from the center, and θ in angle from the horizontal, as shown in Fig. 1.2.

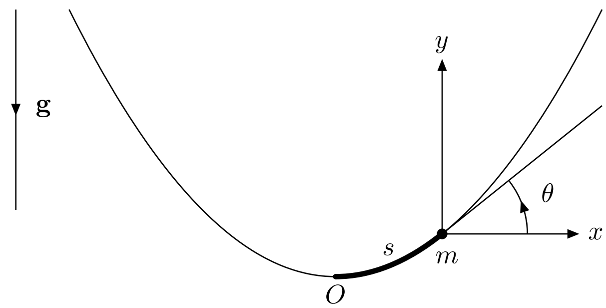

Figure 1.2

If the particle starts from rest at $s = s_0$ and executes simple harmonic oscillations with frequency ω,

(a) Derive the expression for $s(t)$.

(b) Relate $s(t)$ to $\theta(t)$ and comment on the resultant motion.

(c) From the explicit solution, calculate the force of constraint and the total force acting on the particle.

S1.2 **(a)** The equation for the simple harmonic motion is

$$\ddot{s} + \omega^2 s = 0. \tag{7}$$

The solution that satisfies the given initial condition $\dot{s}(0) = 0$ is

$$s(t) = s(0)\cos(\omega t) = s_0 \cos(\omega t) \tag{8}$$

(b) The Lagrangian function is

$$L(s, \dot{s}) = \frac{m}{2}\dot{s}^2 - U(s), \tag{9}$$

where the potential energy U is given by

$$U = mgy = mg\int_0^s \sin\theta ds. \tag{10}$$

Thus, the Euler-Lagrange equation is

$$\ddot{s} + g\sin\theta = 0. \tag{11}$$

Inserting the above expression for $s(t)$, we obtain

$$s_0\omega^2 \cos(\omega t) = g\sin\theta, \tag{12}$$

or,

$$\theta(t) = \sin^{-1}(\lambda\cos(\omega t)), \tag{13}$$

where

$$\lambda \equiv s_0\frac{\omega^2}{g} \leq 1. \tag{14}$$

The angular velocity is

$$\dot\theta(t) = \frac{-\lambda\omega\sin(\omega t)}{\sqrt{1-\lambda^2\cos^2(\omega t)}}. \tag{15}$$

In the limit $\lambda \to 1$, θ goes to zero and $\dot\theta$ goes to ω, except for $\omega t = n\pi$, where they are singular.

(c) The force of constraint is the one perpendicular to the trajectory. It is

$$\mathbf{F}_C(\theta) = mg\cos\theta \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{16}$$

The total force is then

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_g + \mathbf{F}_C(\theta) \\ &= mg\begin{pmatrix} 0 \\ -1 \end{pmatrix} + mg\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \\ &= -mg\sin\theta\begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix}. \end{aligned} \tag{17}$$

P1.3 [$5+5=10$ points]
Consider the equations (no sum over α)

$$\ddot x_\alpha + \omega_\alpha^2 x_\alpha = 0, \quad \alpha = 1,2,\ldots,n, \tag{18}$$

where $\omega_\alpha^2 = \dfrac{k_\alpha}{m}$, $k_\alpha \neq k_\beta$ for $\alpha \neq \beta$. In the absence of constraints this system can be thought of either as n uncoupled 1-D oscillators or as an anisotropic oscillator with n degrees of freedom, each with its own frequency ω_α. Take the second view.

(a) Find the constraint force $\mathbf{C}(x,\dot x)$ that will keep this oscillator on the sphere $\mathbf{S}^{n-1}$ of radius 1, in the Euclidean space $\mathbb{E}^n$, whose equation is $|\mathbf{x}|^2 \equiv \sum_{\alpha=1}^{n} x_\alpha^2 = 1$. Here $\mathbf{x}$ is the vector with components x_α in $\mathbb{E}^n$. Assume that $\mathbf{C}$ is normal to $\mathbf{S}^{n-1}$ (i.e., parallel to $\mathbf{x}$). Write down the equations of motion for the constrained oscillator.

(b) Show that the n functions

$$F_\alpha = x_\alpha^2 + \sum_{\beta\neq\alpha}\frac{(x_\alpha\dot x_\beta - \dot x_\alpha x_\beta)^2}{\omega_\alpha^2 - \omega_\beta^2} \tag{19}$$

are constants of the motion.

S1.3 **(a)** Since $\mathbf{C}$ is parallel to $\mathbf{x}$, we may write $\mathbf{C}(x, \dot{x}) = mf(x, \dot{x})\mathbf{x}$, where f is a scalar function. Then the equations of motion are

$$\ddot{x}_\alpha + \omega_\alpha^2 x_\alpha = f(x, \dot{x}) x_\alpha \qquad \text{(no sum over } \alpha\text{)}. \tag{20}$$

Multiplying both sides by x_α and summing over α,

$$\sum_\alpha (x_\alpha \ddot{x}_\alpha + \omega_\alpha^2 x_\alpha^2) = f(x, \dot{x}) \sum_\alpha x_\alpha^2 = f(x, \dot{x}). \tag{21}$$

But

$$\frac{d^2}{dt^2} \sum_\alpha x_\alpha^2 = 2 \sum_\alpha x_\alpha \ddot{x}_\alpha + 2 \sum_\alpha \dot{x}_\alpha^2 = \frac{d^2}{dt^2}(1) = 0, \tag{22}$$

or,

$$\sum_\alpha x_\alpha \ddot{x}_\alpha = -\sum_\alpha \dot{x}_\alpha^2. \tag{23}$$

So, the scalar function is

$$f(x, \dot{x}) = \sum_\alpha (\omega_\alpha^2 x_\alpha^2 - \dot{x}_\alpha^2) \tag{24}$$

and the constraint force itself is

$$\mathbf{C}(x, \dot{x}) = m\mathbf{x} \sum_\alpha (\omega_\alpha^2 x_\alpha^2 - \dot{x}_\alpha^2). \tag{25}$$

The equations of motion become

$$\ddot{x}_\alpha + \omega_\alpha^2 x_\alpha = x_\alpha \sum_\beta (\omega_\beta^2 x_\beta^2 - \dot{x}_\beta^2) \qquad \text{(no sum over } \alpha\text{)}. \tag{26}$$

The equations of motion are nonlinear, which makes the problem nontrivial.

(b)

$$\begin{aligned}
\frac{dF_\alpha}{dt} &= 2x_\alpha \dot{x}_\alpha + \sum_{\beta \neq \alpha} \frac{2(x_\beta \dot{x}_\alpha - x_\alpha \dot{x}_\beta)(\dot{x}_\beta \dot{x}_\alpha + x_\beta \ddot{x}_\alpha - \dot{x}_\alpha \dot{x}_\beta - \ddot{x}_\beta x_\alpha)}{\omega_\alpha^2 - \omega_\beta^2} \\
&= 2x_\alpha \dot{x}_\alpha + 2 \sum_{\beta \neq \alpha} \frac{x_\beta (f - \omega_\alpha^2) x_\alpha - x_\alpha (f - \omega_\beta^2) x_\beta}{\omega_\alpha^2 - \omega_\beta^2} (x_\beta \dot{x}_\alpha - x_\alpha \dot{x}_\beta) \\
&= 2x_\alpha \dot{x}_\alpha - 2 \sum_{\beta \neq \alpha} x_\alpha x_\beta (x_\beta \dot{x}_\alpha - x_\alpha \dot{x}_\beta) \\
&= 2x_\alpha \dot{x}_\alpha + 2x_\alpha^2 \sum_{\beta \neq \alpha} x_\beta \dot{x}_\beta - 2x_\alpha \dot{x}_\alpha \sum_{\beta \neq \alpha} x_\beta^2 \\
&= 2x_\alpha \dot{x}_\alpha + 2x_\alpha^2 (-x_\alpha \dot{x}_\alpha) - 2x_\alpha \dot{x}_\alpha (1 - x_\alpha^2) \\
&= 0.
\end{aligned} \tag{27}$$

To get the second line, use the equations of motion. To get the last line, use the constraint equation.

P1.4 [$3 + 4 + 3 = 10$ points]
A bead of mass m slides without friction in a uniform gravitational field of acceleration $\mathbf{g}$ on a vertical circular hoop of radius R. The hoop is constrained to rotate at a fixed angular velocity Ω about its vertical diameter. Take the center of the hoop as the pole (origin) of a spherical polar coordinate system in which $\mathbf{r} = \{r, \theta, \phi\}$ represents the radius vector of the bead, with $\theta = 0$ along the direction of gravity.

(a) Write down the Lagrangian $L(\theta, \dot{\theta})$.

(b) Find how the equilibrium values of θ depend on Ω. Which are stable and which are unstable?

(c) Find the frequencies of small vibrations about the stable equilibrium positions (hint: use the first term in a Taylor series expansion). What happens when $\Omega = \sqrt{\frac{g}{R}}$?

S1.4 **(a)** The constraint equations in spherical polar coordinates are

$$r = R \tag{28}$$

and

$$\phi = \Omega t, \tag{29}$$

where ϕ is the azimuth angle, but we do not use them explicitly. The Lagrangian is

$$L = \frac{m}{2}(R^2\dot{\theta}^2 + R^2\Omega^2 \sin^2\theta) + mgR\cos\theta. \tag{30}$$

The first term is the kinetic energy T and the second is the negative potential $-V$ relative to $\theta = \frac{\pi}{2}$. In T the first term comes from the motion along the hoop, and the second from the rotation of the hoop. The constraints are built into T, for the r is constrained to be R, $\dot{r}$ is constrained to be zero (it does not appear), and $\dot{\phi}$ is constrained to be Ω. This is a system with one degree of freedom.

(b) Lagrange's equation (after dividing both sides by mR^2) is

$$\ddot{\theta} = \Omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta \equiv F(\theta). \tag{31}$$

Equilibrium occurs where $\ddot{\theta} = 0$, i.e., at $\theta = 0$, π, and θ_0 where

$$\theta_0 = \cos^{-1}\left(\frac{g}{R\Omega^2}\right) \tag{32}$$

is defined only if $\Omega \geq \sqrt{\frac{g}{R}}$. As Ω approaches zero, it merges with the equilibrium point at $\theta = 0$ when $\Omega = \sqrt{\frac{g}{R}}$. As Ω increases, θ_0 approaches $\frac{\pi}{2}$. To understand the stability, think of $F(\theta)$ as the magnitude of a force, the negative derivative of a potential. To see whether that potential is at a minimum or a maximum at equilibrium, take the second derivative of the potential, namely, $\frac{dF}{d\theta}$. It is found that θ_0 is stable for $0 < \theta_0 < \frac{\pi}{2}$, i.e., all allowed values of θ_0. The equilibrium point at $\theta = 0$ is stable for $\Omega < \sqrt{\frac{g}{R}}$ and unstable otherwise. The equilibrium point at $\theta = \frac{\pi}{2}$ is always unstable. Thus, $\theta = 0$ is the only stable equilibrium at $\Omega = 0$. It remains so as Ω increases to $\sqrt{\frac{g}{R}}$ at which point $\theta = 0$ becomes unstable, but two new stable equilibria appear at $\theta_0 = \pm \cos^{-1} \sqrt{\frac{g}{R\Omega^2}}$. As Ω increases, these two points approach $\cos\theta_0 = 0$, i.e., the horizontal plane, from opposite directions.

(c) For $\Omega > \sqrt{\frac{g}{R}}$, the first term in a Taylor series expansion about θ_0 yields

$$\begin{aligned} \Delta\ddot{\theta} &= \Omega^2(\sin^2\theta_0 - \cos^2\theta_0) - \frac{g}{R}\sin^2\theta_0 \\ &= -\Delta\theta\Omega^2\cos^2\theta_0 \\ &\equiv \Omega^2\left(1 - \frac{g^2}{R^2\Omega^2}\right)\Delta\theta, \end{aligned} \tag{33}$$

so the (circular) frequency of small vibrations is $\Omega\sin\theta_0$. For $\Omega < \sqrt{\frac{g}{R}}$, a Taylor series expansion about $\theta = 0$ yields

$$\ddot{\theta} = -\left(\frac{g}{R} - \Omega^2\right)\theta, \tag{34}$$

so the (circular) frequency is $\sqrt{\frac{g}{R} - \Omega^2}$.

For $\Omega = \sqrt{\frac{g}{R}}$, the first (or the linear) term in a Taylor series expansion about the stable equilibrium vanishes, so in this case the vibration is not harmonic.