Assignment: HW1 [40 points]

Assigned: 2006/09/20
Due: 2006/09/27

P1.1 [10 points]

Show that the Galilei transformations $g(\mathbf{R}(\psi, \hat{\mathbf{n}}), \mathbf{w}, \mathbf{a}, s)$,

$$\begin{pmatrix} \mathbf{r} \\ t \end{pmatrix} \xrightarrow{g} \begin{pmatrix} \mathbf{r}' = \mathbf{R}\mathbf{r} + \mathbf{w}t + \mathbf{a} \\ t' = \lambda t + s \end{pmatrix}, \tag{1}$$

with $\det \mathbf{R} = +1$, $\lambda = +1$, form a group.[1]

P1.2 [$2 + 4 + 4 = 10$ points]

A particle of mass m moves without friction along a symmetrical planar curve $s = s(\theta)$ whose axis of symmetry is parallel to a uniform gravitational field of acceleration $\mathbf{g}$. s is the displacement in arc length from the center, and θ in angle from the horizontal, as shown in Fig. 1.2.

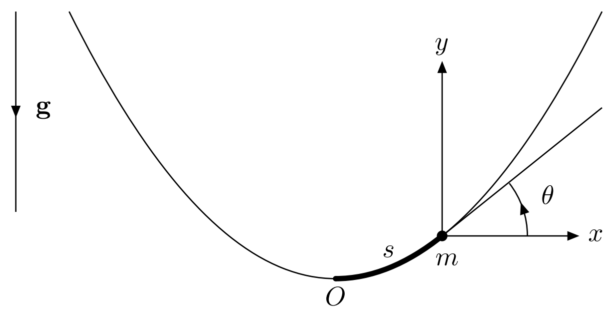

Figure 1.2

If the particle starts from rest at $s = s_0$ and executes simple harmonic oscillations with frequency ω,

(a) Derive the expression for $s(t)$.

(b) Relate $s(t)$ to $\theta(t)$ and comment on the resultant motion.

(c) From the explicit solution, calculate the force of constraint and the total force acting on the particle.

P1.3 [$5 + 5 = 10$ points]

Consider the equations (no sum over α)

$$\ddot{x}_\alpha + \omega_\alpha^2 x_\alpha = 0, \quad \alpha = 1, 2, \ldots, n, \tag{2}$$

where $\omega_\alpha^2 = \dfrac{k_\alpha}{m}$, $k_\alpha \neq k_\beta$ for $\alpha \neq \beta$. In the absence of constraints this system can be thought of either as n uncoupled 1-D oscillators or as an anisotropic oscillator with n degrees of freedom, each with its own frequency ω_α. Take the second view.

[1] This is the *proper, orthochronous Galilei group* $G_+^{\uparrow 4}$.

(a) Find the constraint force $\mathbf{C}(x, \dot{x})$ that will keep this oscillator on the sphere $\mathbf{S}^{n-1}$ of radius 1, in the Euclidean space $\mathbb{E}^n$, whose equation is $|\mathbf{x}|^2 \equiv \sum_{\alpha=1}^{n} x_\alpha^2 = 1$. Here $\mathbf{x}$ is the vector with components x_α in $\mathbb{E}^n$. Assume that $\mathbf{C}$ is normal to $\mathbf{S}^{n-1}$ (i.e., parallel to $\mathbf{x}$). Write down the equations of motion for the constrained oscillator.

(b) Show that the n functions

$$F_\alpha = x_\alpha^2 + \sum_{\beta \neq \alpha} \frac{(x_\alpha \dot{x}_\beta - \dot{x}_\alpha x_\beta)^2}{\omega_\alpha^2 - \omega_\beta^2} \tag{3}$$

are constants of the motion.

P1.4 $[3 + 4 + 3 = 10$ points$]$

A bead of mass m slides without friction in a uniform gravitational field of acceleration $\mathbf{g}$ on a vertical circular hoop of radius R. The hoop is constrained to rotate at a fixed angular velocity Ω about its vertical diameter. Take the center of the hoop as the pole (origin) of a spherical polar coordinate system in which $\mathbf{r} = \{r, \theta, \phi\}$ represents the radius vector of the bead, with $\theta = 0$ along the direction of gravity.

(a) Write down the Lagrangian $L(\theta, \dot{\theta})$.

(b) Find how the equilibrium values of θ depend on Ω. Which are stable and which are unstable?

(c) Find the frequencies of small vibrations about the stable equilibrium positions (hint: use the first term in a Taylor series expansion). What happens when $\Omega = \sqrt{\dfrac{g}{R}}$?