New Particle-In-Cell Code For Numerical Simulations of Coherent Synchrotron Radiation

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Outline of the Talk

• Motivation and Background of Coherent Synchrotron Radiation:
  • Physical problem
  • Computational challenges
  • Two approaches: point-to-point (P2P) and particle-in-cell (PIC)
    • Why develop a new PIC code from an existing P2P code designed by Rui Li?

• New Particle-In-Cell CSR Code
  • Outline of the new algorithm
  • First results: benchmarking against analytical results

• Still to do...

• Summary
Coherent Synchrotron Radiation: Motivation and Background

• When a charged particle beam travels along a curved trajectory (bending magnet), beam emits synchrotron radiation
  
• If the wavelength $\lambda$ of synchrotron radiation is longer than the bunch length $\sigma_s$, the resulting radiation is coherent (CSR)

incoherent (ISR)  
$\lambda < \sigma_s$  

coherent (CSR)  
$\lambda > \sigma_s$

ISR radiated power:  
$$ P = \frac{e^2 c}{6\pi \varepsilon_0 (m_0 c^2)^4} \frac{N E^4}{R^2} $$

CSR radiated power:  
$$ P = \frac{\Gamma (2/3)}{\pi} \left[ \frac{2^{4/3} 3^{1/6} \Gamma (2/3)}{R^{2/3} \sigma_s^{4/3}} \right] \frac{N^2 e^2}{R^{2/3} \sigma_s^{4/3}} $$

Synchrotron radiation mostly negligible in protons

largely cancels out

has systematic effects

\[ \frac{P_e}{P_p} = \left( \frac{m_p}{m_e} \right)^4 = 1.13 \times 10^{13} \]
Coherent Synchrotron Radiation: Motivation and Background

- CSR is the low frequency part of the synchrotron radiation power spectrum

- \( N \) particles in the bunch act in phase and enhance intensity by a factor \( N \) (typically \( N=10^9-10^{11} \))

- Therefore for shorter bunch (\( \sigma_s \) small), CSR is more pronounced

\[
P \propto \frac{N^2}{R^{2/3} \sigma_s^{4/3}}
\]
Coherent Synchrotron Radiation: Motivation and Background

• Short bunch lengths are desirable in many different contexts:
  • FEL, ERL, B-factories, linear colliders such as ILC...
  • The demand for short bunches is expected to increase in the future
• This presents a problem:
  Short beam bunch $\Rightarrow$ CSR is dominant
  $\Rightarrow$ Beam is a subject to adverse CSR effects
• Adverse CSR effects, which can seriously impair beam quality:
  – Energy spread
  – Longitudinal instability (microbunching)
  – Emittance degradation
• Having a trustworthy code to simulate CSR effects is of great importance
Coherent Synchrotron Radiation: Computational Challenges

- Numerical simulations of the CSR effects have proven to be extremely challenging because of:
  - Memory requirement associated with storing history of the beam bunch
  - Difficulty to accurately account for retardation
  - Large cancellation between $E$ and $B$ fields in the Lorentz force
  - Sensitivity to numerical noise
  - The manner in which self-interactions scale in numerical algorithms
- We present the new code which simultaneously deals with all of these issues
Coherent Synchrotron Radiation: Computational Challenges

- Dynamics of an electron bunch is governed by
  \[
  \frac{d}{dt}(\gamma m_e \vec{v}) = e (\vec{E} + \vec{\beta} \times \vec{B})
  \]
  \[
  \vec{\beta} = \vec{v} / c \\
  \vec{E} = \vec{E}^{\text{ext}} + \vec{E}^{\text{self}} \\
  \vec{B} = \vec{B}^{\text{ext}} + \vec{B}^{\text{self}}
  \]

- $\vec{E}^{\text{ext}}$, $\vec{B}^{\text{ext}}$: external EM fields
- $\vec{E}^{\text{self}}$, $\vec{B}^{\text{self}}$: self-interaction (CSR)

\[
\vec{E}^{\text{self}} = -\vec{\nabla} \phi - \frac{1}{c} \frac{\partial}{\partial t} \vec{A}
\]
\[
\vec{B}^{\text{self}} = \vec{\nabla} \times \vec{A}
\]

where
\[
\phi(\vec{r}, t) = \int \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|} \rho(\vec{r}', t')
\]
\[
\vec{A} = \frac{1}{c} \int \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|} \vec{J}(\vec{r}', t')
\]

charge density: $\rho(\vec{r}, t) = \int f(\vec{r}, \vec{v}, t) d\vec{v}$

current density: $\vec{J}(\vec{r}, t) = \int \vec{v} f(\vec{r}, \vec{v}, t) d\vec{v}$

beam distribution function (DF): $f(\vec{r}, \vec{v}, t)$

Need to track the entire history of the bunch
Coherent Synchrotron Radiation: Computational Challenges

- Dynamics of an electron bunch is governed by:
  \[
  \frac{d}{dt} (\gamma m_e \vec{v}) = e (\vec{E} + \vec{\beta} \times \vec{B})
  \]

  \[
  \vec{\beta} = \frac{\vec{v}}{c}
  \]

  \[
  \vec{E} = \vec{E}^{ext} + \vec{E}^{self}
  \]

  \[
  \vec{B} = \vec{B}^{ext} + \vec{B}^{self}
  \]

- \(\vec{E}^{ext}, \vec{B}^{ext}\): external EM fields
- \(\vec{E}^{self}, \vec{B}^{self}\): self-interaction (CSR)

\[
\vec{E}^{self} = -\nabla \phi - \frac{1}{c} \frac{\partial A}{\partial t}
\]

\[
\vec{B}^{self} = \nabla \times A
\]

where

\[
\phi (\vec{r}, t) = \int \frac{d \vec{r}'}{|\vec{r} - \vec{r}'|} \rho (\vec{r}', t')
\]

\[
A = \frac{1}{c} \int \frac{d \vec{r}'}{|\vec{r} - \vec{r}'|} \vec{J} (\vec{r}', t')
\]

charge density: \(\rho (\vec{r}, t) = \int f (\vec{r}, \vec{v}, t) d \vec{v}\)

current density: \(\vec{J} (\vec{r}, t) = \int \vec{v} f (\vec{r}, \vec{v}, t) d \vec{v}\)

beam distribution function (DF): \(f (\vec{r}, \vec{v}, t)\)

Need to track the entire history of the bunch

MEMORY REQUIREMENT

LARGE CANCELLATION

NUMERICAL NOISE DUE TO GRADIENTS
Coherent Synchrotron Radiation: Computational Challenges

• Storing and computing with a 4D (3 positions, 1 time) charge and current densities is prohibitively expensive
  ⇒ Need simplifications/approximations

• Possible simplifications to full dimensional CSR modeling:
  • 1D line approximation (IMPACT, ELEGANT): probably too simplistic
  • 2D approximation (vertically flat beam):
    • codes by Li 1998, Bassi et al. 2006

• Based on how the DF (and, consequently, charge and current densities) are represented, two approaches emerge:
  • Point-to-point methods:
    solve Maxwell's equations using retarded potentials with DF represented by macroparticles
  • Particle-In-Cell (PIC) (mean field, grid, mesh) methods:
    solve Maxwell's equations or retarded potentials on the grid, using finite difference, finite element, Green's function, retarded potentials...
Coherent Synchrotron Radiation: Point-to-Point Approach

- **Point-to-point approach (2D):** Li 1998

\[
\begin{align*}
  f (\vec{r}, \vec{v}, t) &= q \sum_{i=1}^{N} n_m (\vec{r} - \vec{r}_0^{(i)}(t)) \delta (\vec{v} - \vec{v}_0^{(i)}(t)) \\
  \rho (\vec{r}, t) &= q \sum_{i=1}^{N} n_m (\vec{r} - \vec{r}_0^{(i)}(t)) \\
  \vec{J} (\vec{r}, t) &= q \sum_{i=1}^{N} \vec{\beta}_0^{(i)}(t) n_m (\vec{r} - \vec{r}_0^{(i)}(t)) \\
  n_m (\vec{r} - \vec{r}_0^{(i)}(t)) &= \frac{1}{2 \pi \sigma_m^2} e^{-\frac{(x-x_0(t))^2+(y-y_0(t))^2}{2\sigma_m^2}}
\end{align*}
\]

- Charge density is sampled with \( N \) Gaussian-shaped 2D macroparticles (2D distribution without vertical spread)
- At each timestep, each macroparticle experiences fields generated by all other macroparticles throughout history
- **Expensive:** computation of retarded potentials and self fields \( \sim O(N^2) \) 
  \[ \Rightarrow \text{small number } N \Rightarrow \text{poor spatial resolution} \]
  \[ \Rightarrow \text{difficult to see small-scale structure} \]
- While useful in obtaining low-order moments of the beam, *point-to-point approach is not optimal for studying CSR*
Coherent Synchrotron Radiation: Particle-In-Cell Approach

- Particle-In-Cell approach with retarded potentials (2D):

  \[
  f(\vec{x}, \vec{v}, t) = q \sum_{i=1}^{N} \delta(\vec{x} - \vec{x}_0^{(i)}(t)) \delta(\vec{v} - \vec{v}_0^{(i)}(t))
  \]

  \[
  \rho(\vec{x}_k, t) = q \sum_{i=1}^{N} \int_{-h}^{h} \delta(\vec{x}_k - \vec{x}_0^{(i)}(t) + \vec{X}) p(\vec{X}) d\vec{X}
  \]

  \[
  \vec{J}(\vec{x}_k, t) = q \sum_{i=1}^{N} \vec{\beta}_0^{(i)}(t) \int_{-h}^{h} \delta(\vec{x}_k - \vec{x}_0^{(i)}(t) + \vec{X}) p(\vec{X}) d\vec{X}
  \]

- Charge and current densities are sampled with \( N \) point-charges (\( \delta \)-functions) and deposited on a finite grid \( \vec{x}_k \) using a deposition scheme \( p(\vec{X}) \)

- Two main deposition schemes:
  - Nearest Grid Point (NGP)
     (constant: deposits to \( 1^D \) points)
  - Cloud-In-Cell (CIC)
     (linear: deposits to \( 2^D \) points)

There exist higher-order schemes

- Particles do not directly interact with each other, but only through a mean-field of the gridded representation
Computational cost for P2P: \( \text{Total cost} \sim O(N^2) \)

- Integration over history (yields self-forces): \( O(N^2) \) operation

Computational cost for PIC: \( \text{Total cost} \sim O(N_{\text{grid}}^2) \)

- Particle deposition (yields charge & current densities on the grid): \( O(N) \) operation
- Integration over history (yields retarded potentials): \( O(N_{\text{grid}}^2) \) operation
- Finite difference (yields self-forces on the grid): \( O(N_{\text{grid}}) \) operation
- Interpolation (yields self-forces acting on \( N \) individual particles): \( O(N) \) operation
- Overall \( \sim O(N_{\text{grid}}^2) + O(N) \) operations (but in realistic simulations: \( N_{\text{grid}}^2 \gg N \))

- Favorable scaling allows for larger \( N \), and reasonable grid resolution \( \Rightarrow \) improved spatial resolution

- Fair comparison: P2P with \( N \) macroparticles and PIC with \( N_{\text{grid}} = N \)
Coherent Synchrotron Radiation: P2P Vs. PIC

- Difference in spatial resolution: An illustrative example
  - Analytical distribution sampled with
    - \( N = N_x N_y \) macroparticles (as in P2P)
    - On a \( N_x \times N_y \) grid (as in PIC)
  - 2D grid: \( N_x = N_y = 32 \)

\[
\text{Signal-to-Noise Ratio (SNR)} = \sqrt{\frac{\sum_i \bar{q}_i^2}{\sum_i (q_i - \bar{q}_i)^2}}
\]

\( \bar{q}_i = \text{exact} \)
\( q_i = \text{approx.} \)

- PIC approach provides superior spatial resolution to P2P approach
  \( \Rightarrow \) Modify Rui Li's P2P CSR code into a PIC
integrate over particle histories to compute retarded potentials and corresponding forces on each macroparticle

Outline of P2P Algorithm

N macroparticles at $t < t_k$

N macroparticles at $t = t_k$

system at $t = t_k + \Delta t$

advance particles by $\Delta t$

From P2P To PIC
Outline of PIC Algorithm

- **N point-particles at** $t=t_k$
- **Advance particles by** $\Delta t$
- **System at** $t=t_k + \Delta t$
- **Distributions on** $N_x \times N_y$ **grid for** $t=t_k$
- **Bin particles on** $N_x \times N_y$ **grid**
- **Integrate over grid histories to compute** retarded potentials and corresponding forces on the $N_x \times N_y$ **grid**
- **Interpolate to obtain forces on each particle**
Outline of PIC Algorithm

- **N point-particles at** $t=t_k$
- **advance particles by** $\Delta t$
- **bin particles on** $N_x \times N_y$ grid
- **distributions on** $N \times N_y$ grid for $t=t_k$
- **interpolate to obtain forces** on each particle
- **integrate over grid histories to compute retarded potentials and corresponding forces** on the $N_x \times N_y$ grid

**From P2P To PIC**
To simplify calculations use 3 frames of reference:

- **Frenet frame** \((s, x)\)
  - \(s\) – along design orbit
  - \(x\) – deviation normal to direction of motion

- **Lab frame** \((X, Y)\)

- **Grid frame** \((\tilde{X}, \tilde{Y})\)
  - Scaled & rotated lab frame
  - particle deposition
  - grid interpolation
  - integration
  - always \([-0.5,0.5] \times [-0.5,0.5]\)
New PIC CSR Code: Outline

- **FRENET FRAME**
  - $(t, x, s)$
  - Beam bunch $(s_n, x_n, p_n^x, p_n^s)$

- **LAB FRAME**
  - $(t, X, Y)$
  - Compute $\alpha$ offset $X'_0, Y'_0$
  - Beam bunch $(X_n, Y_n, p_n^x, p_n^s)$
  - Vantage point: $(t_k, X'_k, Y'_k)$
  - Compute integration range

- **GRID FRAME**
  - $(t, \vec{X}, \vec{Y})$
  - Beam bunch $(\vec{X}_n, \vec{Y}_n, p_n^x, p_n^s)$
  - Bin particles $\rho, J$ on grid
  - Vantage point: $(t_k, \vec{X}'_k, \vec{Y}'_k)$
  - Integrate to get $A_x, A_y, \varphi$ on grid

- **FRENET FRAME**
  - $(t, s)$
  - Beam bunch $(s_n, x_n, p_n^x, p_n^s)$

- **LAB FRAME**
  - $(t, X, Y)$
  - Transform $\partial_{s_x} A_{s_x}$, $\partial_{s_y} A_{s_y}$
  - Get $E_x, E_y, B_z$ on grid

- **GRID FRAME**
  - $(t, \vec{X}, \vec{Y})$
  - Transform to get $\partial_{X_x} A_{X_x}, \partial_{X_y} A_{X_y}$
  - $\partial_{X_x} \varphi, \partial_{X_y} \varphi$ on grid

- **GRID FRAME**
  - $(t, \vec{X}, \vec{Y})$
  - Take partial derivatives: $\partial_{X_x} A_{X_x}, \partial_{X_y} A_{X_y}$
  - $\partial_{X_x} \varphi, \partial_{X_y} \varphi$ on grid

- **interp_density.f90**
  - $\rho, J$ on grid
  - $\rho, J$ at an arbitrary pt. (interp3D)
• Grid resolution is specified \textit{a priori} (fixed grid)
  
- $N_X : \# \text{ of gridpoints in } X$
- $N_Y : \# \text{ of gridpoints in } Y$
- $N_{\text{grid}} = N_X \times N_Y \text{ total gridpts}$
- Grid: $\vec{x}_k = [\tilde{X}_{ij}, \tilde{Y}_{ij}]$
  
  $i=1,..,N_X \quad j=1,..,N_Y$
- Inclination angle $\alpha$

• Grid is determined so as to tightly envelope all particles
  Minimizing number of empty cells $\Rightarrow$ optimizing spatial resolution
• Determine limits of integration in lab frame:

\[ t' = t - \frac{|\vec{r} - \vec{r}'|}{c} \]

• Carry out integration:

\[
\begin{bmatrix}
\phi(\vec{r}, t) \\
A(\vec{r}, t)
\end{bmatrix} = \int \begin{bmatrix}
\rho \left( \vec{r}', t - \frac{R'}{c} \right) \\
J \left( \vec{r}', t - \frac{R'}{c} \right)
\end{bmatrix} \frac{d\vec{r}'}{|\vec{r}' - \vec{r}|} = \sum_{i=1}^{M_{\text{int}}} \int_{0}^{R_{\text{max}}} \int_{\theta_{\text{min}}^{i}}^{\theta_{\text{max}}^{i}} \begin{bmatrix}
\rho \left( \vec{r}', t - \frac{R'}{c} \right) \\
J \left( \vec{r}', t - \frac{R'}{c} \right)
\end{bmatrix} dR' d\theta'.
\]
All this yields scalar and vector potentials on the grid:

\[ \Phi \]

\[ A_x \]

\[ A_y \]

\[ N = 512000 \]

\[ N_x = N_y = 64 \]
New PIC CSR Code: Computing Self-Forces

- From retarded potentials on the grid to forces on individual particles:
  - Compute derivatives on the grid: $\partial_{\tilde{x}, \tilde{y}} A_{X,Y}, \partial_{\tilde{t}} A_{X,Y}, \partial_{\tilde{x}, \tilde{y}} \phi$
  - Transform to lab frame: $\partial_{X,Y} A_{X,Y}, \partial_{\tilde{t}} A_{X,Y}, \partial_{X,Y} \phi$
  - Transform to Frenet frame: $\partial_{s,x} A_{s,x}, \partial_{\tilde{t}} A_{s,x}, \partial_{s,x} \phi$
  - Interpolate from grid to individual particle positions to obtain forces acting on them:
    $$\vec{E}^{\text{self}} = -\hat{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial \tilde{t}}$$
    $$\vec{B}^{\text{self}} = \hat{\nabla} \times \vec{A}$$
    $$\frac{d}{dt} (\gamma m_e \vec{v}) = e (\vec{E} + \vec{\beta} \times \vec{B})$$
- Finally able to advance particles in time
New PIC CSR Code: Large Cancellation At Work

- Traditionally difficult to track large quantities which mostly cancel out:

- High accuracy of the implementation able to accurately track these cancellations over 5 orders of magnitude

\[ N = 128000 \quad N_x = N_y = 32 \]
New PIC CSR Code: Benchmarking Against Analytic Results

- Analytic steady state solution available for a rigid line Gaussian bunch (Derbenev & Shiltsev 1996)

- Excellent agreement between analytic solution and the computed provides a proof of concept for the new code

\[ N = 512000 \]
\[ N_x = \text{64} \]
\[ N_y = \text{64} \]
New PIC CSR Code: Still To Do...

- Polish:
  - Optimize and parallelize

- Additional functionalities:
  - Wavelet denoising and compression:
    - Less numerical noise ⇒ cleaner results
    - Grid representation compactly stored (compression ~ 100 times)
      decrease memory requirement
      optimize computations

- Further benchmarking and testing

- Applications:
  - Simulating real machines: JLab FEL, JLAMP, LCSL, ...

- Long term plan:
  - 3D
  - Image charges
Motivated the need for accurate CSR codes

Demonstrated that the PIC approach is better because of:
  – Better spatial resolution (a “must” for small-scale instabilities)
  – Better scaling with the number of particles \( N \)

Presented the new PIC code:
  – Some details of the inner workings of the code
  – How the new code resolves traditional computational difficulties
  – Proof of concept: excellent agreement with analytical results

Outlined the path ahead toward simulating real machines

Closing in on our immediate goal: having an accurate, efficient and trustworthy code which faithfully simulates CSR effects

Near-term goal: being able to quantitatively simulate CSR in real machines, as the first step toward controlling its adverse effects
Backup Slides
Numerical Noise in the PIC Simulations

- There are the two major sources of numerical noise in MF simulations:
  - *graininess of the distribution function*: \( N_{\text{simulation}} \ll N_{\text{physical}} \)
  - *discreteness of the computational domain*: quantities defined on a finite grid
- One must first understand the profile of the numerical noise associated with the discreteness of the computational in order to be able to remove it
- Systematic removal of numerical noise from the PIC simulations leads to physically more reliable results, equivalent to simulations with many more particles
Numerical Noise in the Mean Field Simulations

- If many random realizations of a given particle distribution have are deposited onto a grid, the number of particles in each gridpoint is Poisson-distributed (variance = mean) $\Rightarrow$ noise is signal-dependent

- Wavelet denoising is at its most powerful (and mathematically strongest ground) when the noise is Gaussian-distributed (signal-independent, white)

- Signal contaminated with Poissonian noise can be transformed to signal with Gaussian noise by a variance-stabilizing Anscombe transform (1948):

\[
Y_G = 2\sqrt{Y_P + \frac{3}{8}}
\]

- After the transformation the noise in each gridpoint is (nearly) Gaussian-distributed with variance $\sigma=1$

- Essentially, we have pre-processed the signal before denoising it

- This error/noise estimate $\sigma$ is crucial for optimal wavelet noise removal

[For more details see Terzić, Pogorelov & Bohn 2007, PR STAB, 10, 034201]
Removing Numerical Noise from PIC Simulations

• It is desirable to remove noise from the PIC simulations
  less numerical noise ⇔ running simulations with more particles
  ⇒ increased sensitivity to physical small-scale structure

• Noise removal from the PIC simulations can be done in several ways:
  • Particle deposition schemes:
    • Higher order deposition schemes serve as smoothing filters
  • Filtering:
    • Savitzky-Golay smoothing filter (local polynomial regression)
  • In Fourier space:
    • Truncating the highest Fourier frequencies
  • In wavelet space:
    • Wavelet coefficient thresholding

• Wavelets provide a natural setting for judicious noise removal
  (other methods indiscriminantly smooth over/truncate small scale structures)
How Do Wavelets Work?

Wavelet analysis (wavelet transform):

- Approximation – apply low-pass filter to Signal and down-sample
- Detail – apply high-pass filter to Signal and down-sample
- Wavelet synthesis (inverse wavelet transform): up-sampling & filtering
- Complexity: \(4MN\), \(M\) the size of the wavelet, \(N\) number of cells
  - Recall: FFT complexity \(4N \log_2 N\)

S - signal
A - approximation
D - detail
Advantages of wavelet formulation:

- Wavelet basis functions have compact support ⇒ signal localized in space
- Wavelet basis functions have increasing resolution levels ⇒ signal localized in frequency
  ⇒ simultaneous localization in space and frequency (FFT only frequency)
- Wavelet basis functions correlate well with various signal types (including signals with singularities, cusps and other irregularities) ⇒ compact and accurate representation of data (compression)
- Wavelet transform preserves hierarchy of scales
- In wavelet space, discretized operators (Laplacian) are also sparse and have an efficient preconditioner ⇒ solving some PDEs is faster and more accurate
- Wavelets provide a natural setting for noise removal ⇒ wavelet denoising
Brief Overview of Wavelets

- **Wavelets**: orthogonal basis composed of scaled and translated versions of the same localized wavelet \( \psi(x) \):
  \[
  \psi^k_i(x) = 2^{k/2} \psi(2^k x - i) \quad k, i \in \mathbb{Z}
  \]
  \[
  f(x) \approx \sum_k \sum_i d^k_i \psi^k_i(x)
  \]
- Each new resolution level \( k \) is orthogonal to the previous levels
- Wavelets are derived from the *scaling function* \( \phi(x) \) which satisfies
  \[
  \phi(x) = \sqrt{2} \sum_j h_j \phi(2x - j)
  \]
  \[
  \psi(x) = \sqrt{2} \sum_j g_j \phi(2x - j)
  \]
  (only finite number of filter coefficients \( h_j \) and \( g_j \) are non-zero: compact support)
- In order to attain orthogonality of different scales, their shapes are strange
  - Makes them suitable to represent irregularly shaped functions
- For discrete signals (gridded quantities), fast Discrete Wavelet Transform (DFT) is an \( O(MN) \) operation, \( M \) size of the wavelet filter, \( N \) signal size

\[
\begin{align*}
  \psi^k_i(x) &= 2^{k/2} \psi(2^k x - i) \quad k, i \in \mathbb{Z} \\
  f(x) &\approx \sum_k \sum_i d^k_i \psi^k_i(x) \\
  \phi(x) &= \sqrt{2} \sum_j h_j \phi(2x - j) \\
  \psi(x) &= \sqrt{2} \sum_j g_j \phi(2x - j)
\end{align*}
\]
Brief Overview of Wavelets

- Wavelet transform separates scales

![Wavelet Transform Diagram]
Wavelet Denoising

- In wavelet space:
  - signal $\Rightarrow$ few large wavelet coefficients $c_{ij}$
  - noise $\Rightarrow$ many small wavelet coefficients $c_{ij}$

- Denoising by wavelet thresholding:
  - if $|c_{ij}| < T$, set to $c_{ij} = 0$

- A great deal of study has been devoted to estimating optimal $T$

$$T = \sqrt{2 \log N_{\text{grid}} \sigma}$$

($\sigma=1$ after Anscombe transform)

Denoising factor ($DF$):

$$DF = \frac{\text{Error}_{\text{original}}}{\text{Error}_{\text{denoised}}}$$

[Terzić, Pogorelov & Bohn 2007, PR STAB, 10, 034201]
Wavelet Denoising and Compression

- When the signal is known, one can compute Signal-to-Noise Ratio (SNR):

  \[ \text{SNR} = \sqrt{\frac{\sum_i \bar{q}_i^2}{\sum_i (q_i - \bar{q}_i)^2}} \]
  \[ \bar{q}_i = \text{exact} \]
  \[ q_i = \text{approx.} \]

- \( \text{SNR} \sim \sqrt{N_{\text{ppc}}} \)  
  \( N_{\text{ppc}} \): avg. # of particles per cell  
  \( N_{\text{ppc}} = \frac{N}{N_{\text{cells}}} \)
Wavelet Denoising and Compression

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  \( \bar{q}_i = \text{exact} \quad q_i = \text{approx.} \)

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2D superimposed Gaussians on 256×256 grid
Wavelet Denoising and Compression

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  \[
  \text{SNR} = \sqrt{\frac{\sum_i \bar{q}_i^2}{\sum_i (q_i - \bar{q}_i)^2}}
  \]
  \(\bar{q}_i = \text{exact}\)
  \(q_i = \text{approx.}\)

- \(\text{SNR} \sim \sqrt{N_{\text{ppc}}}\)  \(N_{\text{ppc}}\): avg. # of particles per cell  \(N_{\text{ppc}} = N / N_{\text{cells}}\)

2D superimposed Gaussians on 256x256 grid

ANALYTICAL

\(N_{\text{ppc}} = 3\) \(\text{SNR} = 2.02\)
When the signal is known, one can compute \textit{Signal-to-Noise Ratio} (SNR):

\[ \text{SNR} = \sqrt{\frac{\sum q_i^2}{\sum (q_i - \bar{q}_i)^2}} \]

\( N_{\text{ppc}} \): avg. # of particles per cell \hspace{1cm} N_{\text{ppc}} = \frac{N}{N_{\text{cells}}}

2D superimposed Gaussians on 256\( \times \)256 grid

\[ \bar{q}_i = \text{exact} \hspace{1cm} q_i = \text{approx.} \]

\begin{align*}
\text{ANALYTICAL} & \quad N_{\text{ppc}} = 3 \quad \text{SNR} = 2.02 \\
& \quad N_{\text{ppc}} = 205 \quad \text{SNR} = 16.89
\end{align*}
Wavelet Denoising and Compression

- When the signal is known, one can compute Signal-to-Noise Ratio (SNR):

\[
\text{SNR} = \sqrt{N_{\text{ppc}}} \quad N_{\text{ppc}} : \text{avg. \# of particles per cell} \quad N_{\text{ppc}} = \frac{N}{N_{\text{cells}}}
\]

2D superimposed Gaussians on 256×256 grid

- denoising by wavelet thresholding: if \(|c_{ij}| < T\), set to 0

ANALYTICAL

\[
\begin{array}{c|c|c}
N_{\text{ppc}} & \text{SNR} & N_{\text{ppc}} \\
3 & 2.02 & 205 \quad \text{SNR}=16.89
\end{array}
\]
When the signal is known, one can compute Signal-to-Noise Ratio (SNR):

\[
SNR = \sqrt{N_{\text{ppc}}} \\
N_{\text{ppc}}: \text{avg. # of particles per cell} \\
N_{\text{ppc}} = N / N_{\text{cells}}
\]

2D superimposed Gaussians on 256×256 grid

Wavelet denoising yields a representation which is:

- Appreciably more accurate than non-denoised representation
- Sparse (if clever, we can translate this sparsity in computational efficiency)