Simulations of Coherent Synchrotron Radiation and Wavelet Methodology

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Outline of the Talk

• Coherent Synchrotron Radiation (CSR):
  • Physical problem
  • Mathematical problem
  • Computational problem
    • Two approaches: point-to-point (P2P) and mean field (MF)
    • We present reasons why we choose to develop a MF code from an existing P2P code designed by Rui Li
    • Demand for increased sensitivity necessitates numerical noise removal
• Wavelet Methodology
  • Brief outline of wavelets
  • Wavelet denoising: examples and applications
  • Harnessing the power of wavelets: past, present and the future
• Summary
Coherent Synchrotron Radiation: A Physical Problem

- When a charged particle beam travels along a curved trajectory (bending magnet), beam emits synchrotron radiation.

- If the wavelength $\lambda$ of synchrotron radiation is longer than the bunch length $\sigma_s$, the resulting radiation is coherent synchrotron radiation (CSR).

- Incoherent synchrotron radiation: largely cancels out.

- Coherent synchrotron radiation: has systematic effects.
Coherent Synchrotron Radiation: A Physical Problem

- CSR is the low frequency (long wavelength) part of the power spectrum

- $N$ particles in the bunch act in phase and enhance intensity by a factor $N$ (typically $N=10^9\text{-}10^{11}$)

- Therefore for shorter bunch ($\sigma_s$ small), CSR is more pronounced
Coherent Synchrotron Radiation: A Physical Problem

• Short bunch lengths are desirable in many different contexts:
  • FEL require high peak current for a given bunch charge
  • ERL often require a short duration of radiation
  • B-factories and linear colliders require short bunch to achieve higher luminosities
• The demand for short bunches is expected to increase in the future
• This presents a problem:
  short beam bunch ⇒ CSR is dominant ⇒
  ⇒ beam is a subject to adverse CSR effects
• Adverse CSR effects, which can seriously impair beam quality:
  Energy change ⇒ energy spread ⇒ longitudinal instability (microbunching)
  ⇒ emittance degradation
• Having a trustworthy code to simulate CSR is of great importance
Coherent Synchrotron Radiation:  
A Mathematical Problem

- Dynamics of an electron bunch is governed by

\[
\frac{d}{dt} \left( \gamma m_e \vec{v} \right) = e \left( \vec{E} + \vec{\beta} \times \vec{B} \right) \\
\vec{\beta} = \vec{v} / c \\
\vec{E} = \vec{E}^{\text{ext}} + \vec{E}^{\text{self}} \\
\vec{B} = \vec{B}^{\text{ext}} + \vec{B}^{\text{self}}
\]

- \( \vec{E}^{\text{ext}}, \vec{B}^{\text{ext}} \): external EM fields
- \( \vec{E}^{\text{self}}, \vec{B}^{\text{self}} \): self-interaction (CSR)

\[
\vec{E}^{\text{self}} = - \vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \\
\vec{B}^{\text{self}} = \vec{\nabla} \times \vec{A}
\]

where

\[
\phi(\vec{r}, t) = \int \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|} \rho(\vec{r}', t') \\
\vec{A} = \frac{1}{c} \int \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|} \vec{J}(\vec{r}', t')
\]

\[
\text{charge density: } \rho(\vec{r}, t) = \int f(\vec{r}, \vec{v}, t) d\vec{v} \\
\text{current density: } \vec{J}(\vec{r}, t) = \int \vec{v} f(\vec{r}, \vec{v}, t) d\vec{v}
\]

\[
\text{Need to know the history of the bunch}
\]

beam distribution function (DF): \( f(\vec{r}, \vec{v}, t) \)
Coherent Synchrotron Radiation: A Computational Problem

• Storing and computing with a 4D (3 positions, 1 time) charge and current densities is prohibitively expensive
  ⇒ Need simplifications/approximations

• Possible simplifications to full dimensional CSR modeling:
  • 1D line approximation (IMPACT, ELEGANT): probably too simplistic
  • 2D approximation (vertically flat beam):
    • codes by Li 1998, Bassi et al. 2006

• Based on how the DF (and, consequently, charge and current densities) are represented, two approaches emerge:
  • *Point-to-point (tracking) methods*: solving microscopic Maxwell's equation using retarded potentials
  • *Mean field (PIC, grid, mesh) methods*: solving Maxwell equation using finite difference, finite element, Green's function, retarded potentials...
Coherent Synchrotron Radiation: A Computational Problem

- **Point-to-point approach (2D):** Li 1998

\[ f(\vec{r}, \vec{v}, t) = q \sum_{i=1}^{N} n_m(\vec{r} - \vec{r}_0^{(i)}(t)) \delta(\vec{v} - \frac{\vec{v}_0^{(i)}(t)}{c}) \]  
  DF

\[ \rho(\vec{r}, t) = q \sum_{i=1}^{N} n_m(\vec{r} - \vec{r}_0^{(i)}(t)) \]  
  charge density

\[ \vec{J}(\vec{r}, t) = q \sum_{i=1}^{N} \vec{\beta}_0^{(i)}(t) n_m(\vec{r} - \vec{r}_0^{(i)}(t)) \]  
  current density

\[ n_m(\vec{r} - \vec{r}_0^{(i)}(t)) = \frac{1}{2\pi \sigma_m^2} e^{-\frac{(x-x_0(t))^2+(y-y_0(t))^2}{2\sigma_m^2}} \]  
  Gaussian macroparticle

- Charge density is sampled with \( N \) Gaussian-shaped 2D macroparticles (2D distribution without vertical spread)

- Each macroparticle interact with each other one throughout history

- Expensive: computation of retarded potentials and self fields \( \sim O(N^2) \)
  \[ \Rightarrow \] small number \( N \) \( \Rightarrow \) poor spatial resolution
  \[ \Rightarrow \] difficult to see small-scale structure

- While useful in obtaining low-order moments of the beam, *point-to-point approach is not optimal for studying CSR*
Coherent Synchrotron Radiation: A Computational Problem

- **Mean field approach with retarded potentials (2D)**: Terzić & Li, *in preparation*

\[
f(\vec{x}, \vec{v}, t) = q \sum_{i=1}^{N} \delta(\vec{r} - \vec{r}_0^{(i)}(t)) \delta(\vec{v} - \frac{\vec{v}_0^{(i)}(t)}{c})
\]

DF (Klimontovich)

\[
\rho(\vec{x}_k^*, t) = q \sum_{i=1}^{N} \int_{-h}^{h} \delta(\vec{x}_k^{*} - \vec{x}_0^{(i)}(t) + \vec{X}) p(\vec{X}) d \vec{X}
\]

charge density

\[
\vec{J}(\vec{x}_k^*, t) = q \sum_{i=1}^{N} \hat{\beta}_0^{(i)}(t) \int_{-h}^{h} \delta(\vec{x}_k^{*} - \vec{x}_0^{(i)}(t) + \vec{X}) p(\vec{X}) d \vec{X}
\]

current density

- Charge and current densities are sampled with \(N\) point-charges (\(\delta\)-functions) & deposited on a finite grid \(\vec{x}_k^*\) using a deposition scheme \(p(\vec{X})\)

- Two main deposition schemes:
  - Nearest Grid Point (NGP)
    (constant: deposits to \(1^D\) points)
  - Cloud-In-Cell (CIC)
    (linear: deposits to \(2^D\) points)

There exist higher-order schemes

- Particles do not directly interact with each other, but only through a mean-field of the gridded representation
Coherent Synchrotron Radiation: A Computational Problem

- Mean field approach with retarded potentials (2D): Terzić & Li, in preparation (continued)

  - Grid resolution is specified a priori (fixed grid) or changes as necessary (adaptive grid)
  - $N_X$: # of gridpoints in $X$
  - $N_Y$: # of gridpoints in $Y$
  - $N_{\text{grid}} = N_X N_Y$ total gridpts
  - Grid: $\mathbf{x}_k = [\hat{X}_{ij}, \hat{Y}_{ij}]$
    - $i = 1, \ldots, N_X$, $j = 1, \ldots, N_Y$
  - Inclination angle $\alpha$

- Grid is determined so as to tightly envelope all particles
  Minimizing number of empty cells $\Rightarrow$ optimizing spatial resolution
Coherent Synchrotron Radiation: A Computational Problem

- **Mean field approach with retarded potentials (2D):** Terzić & Li, *in preparation* (continued)

  - Computational cost:
    - Particle deposition (yields charge and current densities on the grid):
      - $O(N)$ operations
    - Integration over history (yields retarded potentials):
      - $O(N_{\text{grid}}^2)$ operations
    - Finite difference (yields self-forces on the grid):
      - $O(N_{\text{grid}})$ operations
    - Interpolation (yields self-forces acting on $N$ individual particles)
      - $O(N)$ operations
    - **Total cost $\sim O(N_{\text{grid}}^2) + O(N)$** operations (in realistic sim.: $N_{\text{grid}}^2 >> N$)

  - $N_{\text{grid}}$ and $N$ should be chosen *judiciously*

  - Favorable scaling allows for larger $N$, and reasonable grid resolution
    $\Rightarrow$ improved spatial resolution
Coherent Synchrotron Radiation: A Computational Problem

- **Point-to-point (P2P) Vs. Mean field (MF):**
  
  - Computational cost: $O(N^2)$ Vs. $O(N_{\text{grid}}^2) + O(N)$
  
  Fair comparison: P2P with $N$ macroparticles and MF with $N_{\text{grid}} = N$

- 2D grid:
  $N_X = N_Y = 32$

  ![Image of P2P and MF comparisons]

  **signal-to-noise ratio**

  \[
  \text{SNR} = \sqrt{\frac{\sum_i \tilde{q}_i^2}{\sum_i (q_i - \tilde{q}_i)^2}}
  \]

  $\tilde{q}_i = \text{exact}$
  $q_i = \text{approx.}$

- MF approach provides superior spatial resolution to P2P approach
  
  $\Rightarrow$ Modify Rui Li's P2P CSR code into a MF
Coherent Synchrotron Radiation: Numerical Noise in the Mean Field Simulations

• There are the two major sources of numerical noise in MF simulations:
  • **graininess of the distribution function**: \( N_{\text{simulation}} \ll N_{\text{physical}} \)
  • **discreteness of the computational domain**: quantities defined on a finite grid

• One must first understand the profile of the numerical noise associated with the discreteness of the computational in order to be able to remove it

• Systematic removal of numerical noise from the MF simulations leads to physically more reliable results, equivalent to simulations with many more particles
Coherent Synchrotron Radiation:
Numerical Noise in the Mean Field Simulations

• If many random realizations of a given particle distribution have are
deposited onto a grid, the number of particles in each gridpoint is
Poisson-distributed (variance = mean) ⇒ noise is signal-dependent

• Wavelet denoising is at its most powerful (and mathematically strongest
ground) when the noise is Gaussian-distributed (signal-independent, white)

• Signal contaminated with Poissonian noise can be transformed to signal
with Gaussian noise by a variance-stabilizing Anscombe transform (1948):

\[ Y_G = 2 \sqrt{Y_P + \frac{3}{8}} \]

\[ Y_P = \text{signal with Poissonian noise} \]
\[ Y_G = \text{signal with Gaussian noise} \]

• After the transformation the noise in each gridpoint is (nearly) Gaussian-
distributed with variance \( \sigma = 1 \)

• Essentially, we have pre-processed the signal before denoising it

• This error/noise estimate \( \sigma \) is crucial for optimal wavelet noise removal

[For more details see Terzić, Pogorelov & Bohn 2007, PR STAB, 10, 034201]
Coherent Synchrotron Radiation: Removing Numerical Noise from Mean Field Simulations

• It is desirable to remove noise from the MF simulations
  less numerical noise ⇔ running simulations with more particles
  ⇒ increased sensitivity to physical small-scale structure

• Noise removal from the MF simulations can be done in several ways:
  • Particle deposition schemes:
    • Higher order deposition schemes serve as smoothing filters
  • Filtering:
    • Savitzky-Golay smoothing filter (local polynomial regression)
  • In Fourier space:
    • Truncating the highest Fourier frequencies
  • In wavelet space:
    • Wavelet coefficient thresholding

• Wavelets provide a natural setting for *judicious* noise removal
  (other methods indiscriminantly smooth over/truncate small scale structures)
Brief Overview of Wavelets

- **Wavelets**: orthogonal basis composed of scaled and translated versions of the same localized wavelet $\psi(x)$:
  \[ \psi_i^k(x) = 2^{k/2} \psi(2^k x - i) \quad k, i \in \mathbb{Z} \]
  \[ f(x) \approx \sum_k \sum_i d_i^k \psi_i^k(x) \]
- Each new resolution level $k$ is orthogonal to the previous levels
- Wavelets are derived from the scaling function $\phi(x)$ which satisfies
  \[ \phi(x) = \sqrt{2} \sum_j h_j \phi(2x - j) \]
  \[ \psi(x) = \sqrt{2} \sum_j g_j \phi(2x - j) \]
  (only finite number of filter coefficients $h_j$ and $g_j$ are non-zero: compact support)
- In order to attain orthogonality of different scales, their shapes are strange
  - Makes them suitable to represent irregularly shaped functions
- For discrete signals (gridded quantities), fast Discrete Wavelet Transform (DFT) is an $O(MN)$ operation, $M$ size of the wavelet filter, $N$ signal size
**Brief Overview of Wavelets**

- Wavelet transform separates scales
Brief Overview of Wavelets

• Advantages of wavelet formulation:
  - Wavelet basis functions have compact support ⇒ signal localized in space
    Wavelet basis functions have increasing resolution levels
    ⇒ signal localized in frequency
    ⇒ *simultaneous localization in space and frequency* (FFT only frequency)
  - Wavelet basis functions correlate well with various signal types
    (including signals with singularities, cusps and other irregularities)
    ⇒ *compact and accurate representation of data (compression)*
  - Wavelet transform preserves hierarchy of scales
  - In wavelet space, discretized operators (Laplacian) are also sparse and have
    an efficient preconditioner ⇒ *solving some PDEs is faster and more accurate*
  - Wavelets provide a natural setting for noise removal ⇒ *wavelet denoising*
Wavelet Denoising

- In wavelet space:
  - signal $\rightarrow$ few large wavelet coefficients $c_{ij}$
  - noise $\rightarrow$ many small wavelet coefficients $c_{ij}$

- Denoising by wavelet thresholding:
  - if $|c_{ij}| < T$, set to $c_{ij} = 0$

- A great deal of study has been devoted to estimating optimal $T$

$$T = \sqrt{2\log N_{\text{grid}} \cdot \sigma}$$

($\sigma=1$ after Anscombe transform)

Denoising factor ($DF$):

$$DF = \frac{\text{Error}_{\text{original}}}{\text{Error}_{\text{denoised}}}$$

[Terzić, Pogorelov & Bohn 2007, PR STAB, 10, 034201]
Wavelet Denoising and Compression

- When the signal is known, one can compute Signal-to-Noise Ratio (SNR):
  \[ \text{SNR} = \sqrt{\frac{\sum q_i^2}{\sum (q_i - \bar{q}_i)^2}} \]

- \( SNR \sim \sqrt{N_{\text{ppc}}} \)
  \( N_{\text{ppc}} \): avg. # of particles per cell
  \( N_{\text{ppc}} = N/N_{\text{cells}} \)
Wavelet Denoising and Compression

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2D superimposed Gaussians on 256×256 grid
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2D superimposed Gaussians on 256×256 grid

ANALYTICAL

\(N_{\text{ppc}} = 3\) \(SNR = 2.02\)
Wavelet Denoising and Compression

- When the signal is known, one can compute Signal-to-Noise Ratio (SNR):
  \[ SNR = \sqrt{\frac{\sum_i \bar{q}_i^r}{\sum_i (q_i - \bar{q}_i)^r}} \]

  \( \bar{q}_i = \text{exact} \)

  \( q_i = \text{approx.} \)

- \( SNR \sim \sqrt{N_{ppc}} \)

  \( N_{ppc} : \text{avg. # of particles per cell} \quad N_{ppc} = \frac{N}{N_{\text{cells}}} \)

2D superimposed Gaussians on 256×256 grid

**ANALYTICAL**

- \( N_{ppc} = 3 \quad SNR = 2.02 \)
- \( N_{ppc} = 205 \quad SNR = 16.89 \)
Wavelet Denoising and Compression

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2D superimposed Gaussians on 256×256 grid

- Analytical
  \( N_{\text{ppc}} = 3 \quad SNR = 2.02 \)
  \( N_{\text{ppc}} = 205 \quad SNR = 16.89 \)

- Denoising by wavelet thresholding: if \( |c_{ij}| < T \), set to 0
Wavelet Denoising and Compression

- When the signal is known, one can compute Signal-to-Noise Ratio (SNR):

  $$ SNR = \sqrt{\frac{\sum_i \bar{q}_i^2}{\sum_i (q_i - \bar{q}_i)^2}} $$

  $\bar{q}_i =$ exact
  $q_i =$ approx.

  $$N_{ppc} = \frac{N}{N_{cells}}$$

  - $N_{ppc}$: avg. # of particles per cell

  2D superimposed Gaussians on 256×256 grid

- Wavelet denoising yields a representation which is:
  - Appreciably more accurate than non-denoised representation
  - Sparse (if clever, we can translate this sparsity in computational efficiency)

ANALYTICAL

<table>
<thead>
<tr>
<th>$N_{ppc}$</th>
<th>SNR</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2.02</td>
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WAVELET THRESHOLDING

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<th>$N_{ppc}$</th>
<th>SNR</th>
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<tbody>
<tr>
<td>205</td>
<td>16.89</td>
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</table>

DENOISED

<table>
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<th>$N_{ppc}$</th>
<th>SNR</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>16.83</td>
</tr>
</tbody>
</table>

COMPACT: only 0.12% of coeffs
Harnessing the Power of Wavelets: The Past

• We have already used wavelets in mean field solvers and will greatly benefit from it in the current project:
  – Terzić, Pogorelov & Bohn 2007:
    • Designed a new 3D wavelet-based Poisson equation solver and optimized it for use in PIC beam simulations
    • Integrated the Poisson solver in beam code (IMPACT), benchmarked it and used to model Fermilab/NICADD photoinjector
      - First application of wavelets to 3D beam simulations
    • We provide a detailed treatment of noise in PIC simulations and implemented wavelet denoising
      - Roadmap to follow in the current project
  – Sprague 2008, Sprague & Terzić in preparation:
    • Tutorial of for wavelet use in solving PDEs
    • Enhanced the original solver by implementing adaptive grid
      - Will use this to further improve spatial resolution in our MF code
Harnessing the Power of Wavelets: The Present

- I am currently involved in two projects which bring CSR and wavelets together:
  - Collaboration with Rui Li on modifying her 2D CSR P2P code into a MF code:
    - Wavelet denoising of the representation is already implemented (can be turned on and off, enabling a clear comparison)
    - We already ascertained that only a small fraction of coefficients on the grid (<1% or so) is needed to accurately represent densities
      - Can this translate into a more efficient code?
    - Once the code is completed and tested, we will conduct a comprehensive comparison of the effects of denoising:
      - How much does wavelet denoising improve spatial resolution?
      - How accurate is the wavelet denoised representation?
Harnessing the Power of Wavelets: The Present

- Bassi & Terzić 2009:
  - Improved particle representation in Bassi’s 2D CSR code by replacing analytic cosine expansion with a wavelet approximation
    - Better spatial resolution (needed to study microbunching)
    - Appreciably more accurate (after wavelet thresholding)
    - Orders of magnitude faster

- How accurately can small-scale structures be represented by an approximation?
  - Analytic Monte Carlo cosine
  - Simple grid
  - Thresholded FFT (grid)
  - Thresholded wavelet (grid)

Flat-top with sinusoidally modulated frequency (FERMI@ELETTRA first bunch compressor)
Harnessing the Power of Wavelets: The Present

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Flat-top with sinusoidally modulated frequency (FERMI@ELETTRA first bunch compressor)

\[
N = 10^8 \quad \text{cosine expansion: } N_c = 40, M_c = 100 \\
\text{grid resolution: } N_x = 128, N_z = 1024
\]
Harnessing the Power of Wavelets: 
The Present

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Harnessing the Power of Wavelets: The Future

- In the future, we plan to further harness the power of wavelets:
  - Translate sparsity of operators and datasets in wavelet space to computational efficiency
    - Fast application of discretized operators
    - Efficient preconditioners for other operators?
    - Fast interpolation of discrete data from sparse wavelet representation
  - Use adaptive grid in wavelet-based methods to increase spatial resolution
  - Explore applicability of what we have learned about wavelets to other PDEs
Summary

• We presented two computational approaches to simulating CSR: P2P and MF
  – Demonstrated that the MF approach is better because of:
    • Better spatial resolution (a “must” for small-scale instabilities)
    • Better scaling with the number of particles $N$
  – We are now working on converting Rui Li’s P2P code into a MF code
    (We hope to start benchmarking it within the next few months)
• Compare with Bassi’s 2D CSR code for consistency
• Closing in on our intermediate goal: having an accurate, efficient and trustworthy code which faithfully simulates CSR
• Long-term goal: being able to quantitatively simulate CSR in real machines, as a first step toward controlling its adverse effects
Auxiliary Slides
Multi-Resolution Analysis and Wavelets

- Multi-Resolution Analysis (MRA) is a decomposition of Hilbert space $L^2(R)$ into a chain of closed subspaces $V$: $0 \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(R)$

- Define an associated sequence of subspaces $W$ as an orthogonal complement of $V_{j-1}$ in $V_j$: $V_j = V_{j-1} + W_j$. Also: $V_j = \sum_{j' < j} W_j$

- A set of dilations and translations of the scaling function $\phi(x)$:
  $$\{ \phi^j_k(x) = 2^{j/2} \phi (2^j x - k) \}_{k \in \mathbb{Z}}$$
  forms an orthonormal basis of $V_j$.

- A set of dilations and translations of the wavelet function $\psi(x)$:
  $$\{ \psi^j_k(x) = 2^{j/2} \psi (2^j x - k) \}_{k \in \mathbb{Z}}$$
  forms an orthonormal basis of $W_j$.

- They satisfy refinement relations:
  $$\phi(x) = \sqrt{2} \sum h_k \phi(2x - k)$$
  $$\psi(x) = \sqrt{2} \sum g_k \phi(2x - k)$$
  Quadrature Mirror Filters $H = \{h_k\}$, $G = \{g_k\}$
  used in the Discrete Wavelet Transform
  (only few of them are non-zero: compact support)

- Projection of function $f(x)$ onto $V_j$:
  $$(P_j f)(x) = \sum_{k \in \mathbb{Z}} s^j_k \psi^j_k(x) = \sum_{j' < j} \sum_{k \in \mathbb{Z}} d^j_k \phi^j_k(x)$$
  $$s^j_k = \int_{-\infty}^{\infty} f(x) \psi^j_k(x) \, dx$$
  $$d^j_k = \int_{-\infty}^{\infty} f(x) \phi^j_k(x) \, dx$$
How Do Wavelets Work?

Wavelet analysis (wavelet transform):

- **Approximation** – apply low-pass filter to Signal and down-sample
- **Detail** – apply high-pass filter to Signal and down-sample
- **Wavelet synthesis** (inverse wavelet transform): up-sampling & filtering
- **Complexity**: $4MN$, $M$ the size of the wavelet, $N$ number of cells
  - Recall: FFT complexity $4N \log_2 N$
The **continuous wavelet transform** of a function $f(t)$ is

$$\gamma(s, \tau) = \int_{-\infty}^{\infty} f(t) \psi_{s,\tau}(t) \, dt$$

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}} \psi \left( t - \frac{\tau}{s} \right)$$

$\psi(t)$ is the **mother wavelet** with scale and translation dimensions $s$ and $\tau$ respectively
Harnessing the Power of Wavelets: The Present

- Bassi & Terzić 2009:
  - Improved particle representation in Bassi's 2D CSR code by replacing analytic cosine expansion with a wavelet approximation
    - Better spatial resolution (needed to study microbunching)
    - Appreciably more accurate (after wavelet thresholding)
    - Orders of magnitude faster

\[ \lambda = 100 \mu m \]

\[ N = 10^8 \]

- cosine expansion: \( N_c = 40, M_c = 100 \)
- grid resolution: \( N_x = 128, N_z = 1024 \)
Numerical Noise in PIC Simulations

- In wavelet space:
  - signal $\rightarrow$ few large wavelet coefficients $c_{ij}$
  - noise $\rightarrow$ many small wavelet coefficients $c_{ij}$

- Poissonian noise $\rightarrow$ Anscombe transformation $\rightarrow$ Gaussian noise

- Denoising by wavelet thresholding:
  - if $|c_{ij}| < T$, set to $c_{ij} = 0$ (choose threshold $T$ carefully!)

- A great deal of study has been devoted to estimating optimal $T$

  $$T = 2\sqrt{\log N_{\text{grid}}} \sigma$$

($\sigma$ was estimated earlier)

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Terzić, Pogorelov & Bohn 2007, PR STAB, 10, 034201
Coherent Synchrotron Radiation: Numerical Noise in the Mean Field Simulations

• For NGP, at each gridpoint, density dist. is Poissonian:

\[ P = (n!)^{-1} n_j^n e^{-n_j} \]

\( n_j \) is the expected number in \( j^{th} \) cell; \( n \) integer

• For CIC, at each gridpoint, density dist. is contracted Poissonian:

\[ P = (n!)^{-1} (an_j)^n e^{-an_j} \]

\( a = (2/3)^{(D/2)} \sim 0.54 \) (3D), 0.67 (2D), 0.82 (1D)

[For more details see Terzić, Pogorelov & Bohn 2007, PR STAB, 10, 034201]

• Measure of error (noise) in depositing macroparticles onto a grid:

\[ \sigma^2 = (N_{grid})^{-1} \sum_{i=1}^{N_{grid}} \text{Var}(q_i) \]

\[ \sigma_{NGP}^2 = \frac{Q_{total}^2}{NN_{grid}} \]

\[ \sigma_{CIC}^2 = \frac{a^2 Q_{total}^2}{NN_{grid}} \]

where \( q_i = (Q_{total}/N)n_i \), \( Q_{total} \) total charge

• This error/noise \( \sigma \) estimate is crucial for optimal noise removal

• Signal with Poissonian noise can be transformed to the signal with Gaussian noise by Anscombe transformation:

\[ Y_G = 2 \sqrt{Y_P + \frac{3}{8}} \]

\( Y_P \) = signal with Poissonian (multiplicative) noise

\( Y_G \) = signal with Gaussian (additive) noise