Density Estimation Techniques for Charged Particle Beams with Applications to Microbunching Instability

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Introduction

- Motivation: efficient and accurate density estimation very important for reliable beam dynamics simulations.
- We discuss two new techniques to represent particle distributions:
  - Fast Cosine Transform (FCT)
  - Discrete Wavelet Transform (DWT)
- Application: 2D simulations of coherent synchrotron radiation (CSR) [1].

Monte Carlo Particle method

- Expand the 2D distribution density in Fourier series, \((x,y) \in [0,1]^2\):
  \[ f(x,y) = \sum_{n,m} \hat{f}(n,m) \cos(n \pi x) \cos(m \pi y) \]

- The Fourier coefficients are calculated with Monte Carlo integrations:
  \[ \hat{f}(n,m) = \frac{1}{N \times N} \sum_{i=1}^{N \times N} f(x_i,y_i) \cos(n \pi x_i) \cos(m \pi y_i) \]

Wavelet Thresholding

- hard thresholding: in which the coefficients with magnitudes below a certain predefined threshold \(T > 0\) are set to zero:
  \[ w_i = \begin{cases} w_i & |w_i| > T \\ 0 & |w_i| \leq T \end{cases} \]

- soft thresholding: in which the coefficients with magnitudes below a certain threshold \(T > 0\) are set to zero and the ones above it contracted by \(T\):
  \[ w_i = \begin{cases} \text{sign}(w_i) |w_i| - T & |w_i| > T \\ 0 & |w_i| \leq T \end{cases} \]

Here \(T = \sqrt{2 \log(N_{\text{part}})} \sigma\).

Filtered Fast Cosine Transform

- Deposit particles on the \((N_x, N_y)\) grid.
- Apply 2D FCT on the grid, thus yielding \((N_x, N_y)\) cosine coefficients.
- The high-frequency contribution to the density is then removed by filtering (truncating) coefficients higher than \(N_x\%y\):
  - this removal of the high-frequency components results in a smoother distribution, but it also removes small scale structures that may not be due to noise.
  - the truncation of the Fourier coefficients also restrict spatial resolution of the representation.
- Apply 2D inverse FCT on the grid, to obtain the smoothed distribution in physical space.

Spatial Resolution

- Argument of the highest order basis functions:
  \[ N_x x^i = \frac{N_x}{L} x^i + 0.5 \]
  \[ N_y y^j = \frac{N_y}{L} y^j + 0.5 \]
- Thus, the smallest wavelengths representable is
  \[ \lambda_{\text{min}} = \frac{2L}{N_x N_y} \]
- Grid resolution of cosine expansion
  \[ \lambda > \lambda_{\text{min}} \]
- Requirement for the accurate representation
  \[ \lambda < \frac{1}{2} \lambda_{\text{min}} \]

Wavelet-Denoised Density

- Deposit particles on the \((N_x, N_y)\) grid.
- Apply Anisoube transformation to convert the signal polluted by Poissonian noise to the signal with Gaussian noise.
- Apply 2D DWT on the grid, thus yielding \((N_x, N_y)\) wavelet coefficients.
- Perform wavelet thresholding on the wavelet coefficients in order to remove numerical noise and smooth the particle distribution.
- Unlike the filtering of the cosine coefficients in Fourier space, this procedure noise removal does not restrict the spatial resolution of the distribution.
- Small-scale structures which are not deemed to be noise during wavelet thresholding are retained in the denoised distribution.
- Apply 2D inverse DWT on the grid, to obtain the smoothed distribution in physical space.
- Apply inverse Anisoube transformation.

Accuracy

- Wavelet-denoising significantly improves the accuracy of the approximation.

Fast Cosine Transform vs Discrete Wavelet Transform

- We compare the accuracy and efficiency of the wavelet-denoised grid distribution and the filtered fast cosine approximation considering the initial distribution used in [1] to study microbunching instability.
- The amplitude of the initial modulation is \(A = 0.05\) and the wavelength \(\lambda = 100\mu\m\).
- All the results shown here are with \(N_x = 1024\), \(N_y = 128\) and \(N = 10^5\) particles.
- The number of cosine basis functions (highest order) in the Fourier expansion is \(N_x = 100\) and \(N_y = 100\), which determines the smallest structure representable by a finite cosine approxima.

Smoothness

- Small part of the cross-section of the particle distribution approximated with the various schemes.
- Notice that wavelet thresholding smoothens small-scale noise in the grid distribution, while maintaining high-fidelity signal.

Efficiency

- Execution times of the different methods and their constituent parts scale with the number of particles, \(N\).
- The Monte Carlo-based computation of cosine coefficients requires integration over \(N\), therefore scaling as \(\propto N\).
- Using cosine coefficients to approximate the particle distribution on the grid requires summing over all the cosine coefficients, thus scaling as \(\propto N^2\).
- In realistic simulations, \(N > N_{\text{part}}\), the computation of cosine coefficients will be by far the most time-consuming part of the two.
- Each grid-based method uses particle deposition, which scales as \(\propto N\), and a fast transform FCT and DWT which scale as \(\propto N\log(N)\) and \(\propto N\log(N)\), respectively, where \(M\) is the width of the wavelet family.
- For large \(N\) used in realistic simulations, the most time-consuming part of the approximation is particle deposition.

This is also why the execution times of the grid methods become quite similar for large \(N\).

Discussion and Conclusion

- wavelet method more accurate and efficient than Monte Carlo cosine expansion and fast cosine expansion.
- wavelet method uses particle deposition and then wavelet thresholding to remove radii- cally numerical noise.
- Future work:
  - exploit sparsity of wavelet-based method.
  - Store the entire history of the charge distribution as a small set of sparse wavelet coefficients to reduce simulation times.

References