23 Lecture 23: The Lane-Emden Equation

"Science is facts; just as houses are made of stones, so is science made of facts; but a pile of stones is not a house and a collection of facts is not necessarily science."

Henri Poincare

The Big Picture: Today we discuss the Lane-Emden equation, which describes polytropes in hydrostatic equilibrium as simple models of a star. We also derive the Chandrasekhar limit for the formation of a black hole.

The Lane-Emden Equation

Last time we introduced the polytropes as a family of equations of state for gas in hydrostatic equilibrium. They are given by the equation of state in which the pressure is given as a power-law in density:

$$P = \kappa \rho^{\gamma}, \tag{462}$$

where κ and γ are constants. The *Lane-Emden equation* combines the above equation of state for polytropes and the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho(r)\frac{GM(r)}{r^2}. \tag{463}$$

If we solve for the equation above for $M(r)$

$$M(r) = -\frac{r^2}{\rho G}\frac{dP}{dr} \qquad \Longrightarrow \qquad \frac{dM}{dr} = -\frac{1}{G}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right), \tag{464}$$

and compare it to what we obtain from considering the spherical shell in hydrostatic equilibrium

$$dM = 4\pi r^2 \rho dr \qquad \Longrightarrow \qquad \frac{dM}{dr} = 4\pi r^2 \rho, \tag{465}$$

we obtain

$$\begin{aligned} \frac{dM}{dr} = -\frac{1}{G}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) &= 4\pi r^2 \rho, \\ \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) &= -4\pi G \rho. \end{aligned} \tag{466}$$

After inserting the polytropic equation of state [eq. (462)], the equation above becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\kappa\gamma\rho^{\gamma-1}\frac{d\rho}{dr}\right) = -4\pi G \rho. \tag{467}$$

After defining quantities

$$\begin{aligned} \rho &\equiv \lambda \theta^n, \\ \gamma &\equiv \frac{n+1}{n}, \end{aligned} \tag{468}$$

the eq. (467) becomes

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{\kappa r^2}{\lambda\theta^n}\frac{n+1}{n}\left(\lambda\theta^n\right)^{1/n}\frac{d\left(\lambda\theta^n\right)}{dr}\right] = -4\pi G\lambda\theta^n$$
$$\left[\frac{n+1}{4\pi G}\kappa\lambda^{\frac{1-n}{n}}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n. \tag{469}$$

We now make this equation dimensionless by introducing a radial variable ξ

$$\xi \equiv \frac{r}{\alpha}, \qquad \alpha \equiv \sqrt{\frac{n+1}{4\pi G}\kappa\lambda^{\frac{1-n}{n}}}, \tag{470}$$

to finally obtain the *Lane-Emden* equation for polytropes in hydrostatic equilibrium:

$$\alpha^2\frac{1}{(\alpha\xi)^2}\frac{d}{d(\alpha\xi)}\left((\alpha\xi)^2\frac{d\theta}{d(\alpha\xi)}\right) = -\theta^n$$
$$\Longrightarrow \quad \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{471}$$

This is a second order ordinary differential equation, which means that it requires two boundary conditions in order to be well-defined:

1. Define the central density $\rho_c \equiv \lambda$. Then

$$\rho = \lambda\theta^n \qquad \Longrightarrow \qquad \theta(0) = 1. \tag{472}$$

2. At $r = 0$, $\frac{dP}{dr} = -\rho g = -\rho_c g = 0$, because $g_c = 0$ (there is no mass inside zero radius). Therefore,

$$\frac{dP}{dr} = \kappa\gamma\rho^{\gamma-1}\frac{d\rho}{dr} \propto \frac{d\theta}{d\xi} \qquad \Longrightarrow \qquad \left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0. \tag{473}$$

Analytic Solutions of the Lane-Emden Equation

The Lane-Emden equation can be analytically solved only for a few special, integer values of the index n: 0, 1 and 5. For all other values of n, we must resort to numerical solutions. However, it is beneficial from both pedagogical and intuitive standpoint to derive these analytical solutions, which is what we do next.

Analytic solution for n=0.
After substituting $n = 0$ into the Lane-Emden equation [eq. (471)], we obtain

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -1 \qquad \Longrightarrow \qquad \int\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi = -\int\xi^2 d\xi$$
$$\Longrightarrow \quad \xi^2\frac{d\theta}{d\xi} = -\frac{1}{3}\xi^3 + c_1 \qquad \Longrightarrow \qquad \frac{d\theta}{d\xi} = -\frac{1}{3}\xi + \frac{c_1}{\xi^2}. \tag{474}$$

But, using the boundary conditions, we obtain

$$\left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0 \quad \Longrightarrow \quad c_1 = 0 \quad \Longrightarrow \quad \frac{d\theta}{d\xi} = -\frac{1}{3}\xi \quad \Longrightarrow \quad \theta = -\frac{1}{6}\xi^2 + c_2$$
$$\Longrightarrow \quad \theta(0) = 1 \quad \Longrightarrow \quad c_2 = 1 \quad \Longrightarrow \quad \theta_0 = 1 - \frac{1}{6}\xi^2. \tag{475}$$

From the equation above, we see that this configuration has a boundary at $\xi = \sqrt{6}$, where $\theta_0 \to 0$.

Analytic solution for n=1.
After substituting $n = 1$ into the Lane-Emden equation [eq. (471)], we obtain

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta \quad \Longrightarrow \quad \frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\xi^2\theta. \tag{476}$$

Introduce the variable χ

$$\chi(\xi) \equiv \xi\theta(\xi) \quad \Longrightarrow \quad \theta \equiv \frac{\chi}{\xi}. \tag{477}$$

Then

$$\frac{d\theta}{d\xi} = \frac{d}{d\xi}\left(\frac{\chi}{\xi}\right) = \frac{\xi\chi' - \chi}{\xi^2}, \tag{478}$$

and the Lane-Emden equation in eq. (476) becomes

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \frac{d}{d\xi}\left(\xi\chi'\right) = \chi' + \xi\chi'' - \chi' = \xi\chi''$$
$$\Longrightarrow \quad \frac{\xi\chi''}{\xi^2} = -\frac{\chi}{\xi} \quad \Longrightarrow \quad \chi'' = -\chi \quad \Longrightarrow \quad \chi'' + \chi = 0. \tag{479}$$

This is a harmonic oscillator with general solutions

$$\chi(\xi) = A\sin\xi + B\cos\xi, \tag{480}$$

or, in terms of $\theta \equiv \chi/\xi$

$$\theta(\xi) = A\frac{\sin\xi}{\xi} + B\frac{\cos\xi}{\xi}, \tag{481}$$

After imposing the first boundary condition, the general solution is obtained:

$$\theta(0) = 1 \quad \Longrightarrow \quad B = 0, \quad \text{because} \quad \lim_{\xi\to 0}\frac{\cos\xi}{\xi} = \infty$$
$$A = 1, \quad \text{because} \quad \lim_{\xi\to 0}\frac{\sin\xi}{\xi} = 1.$$
$$\Longrightarrow \quad \theta_1(\xi) = \frac{\sin\xi}{\xi}. \tag{482}$$

The second boundary condition $\left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0$ is explicitly satisfied, because, after applying L'Hospital's rule

$$\lim_{\xi\to 0}\frac{\xi\cos\xi - \sin\xi}{\xi^2} = \lim_{\xi\to 0}\frac{-\xi\sin\xi + \cos\xi - \cos\xi}{2\xi} = -\frac{1}{2}\lim_{\xi\to 0}\sin\xi = 0, \tag{483}$$

as required. From the eq. (482) above, we see that this configuration is has a boundary at $\xi = \pi$, where $\theta_1 \to 0$.

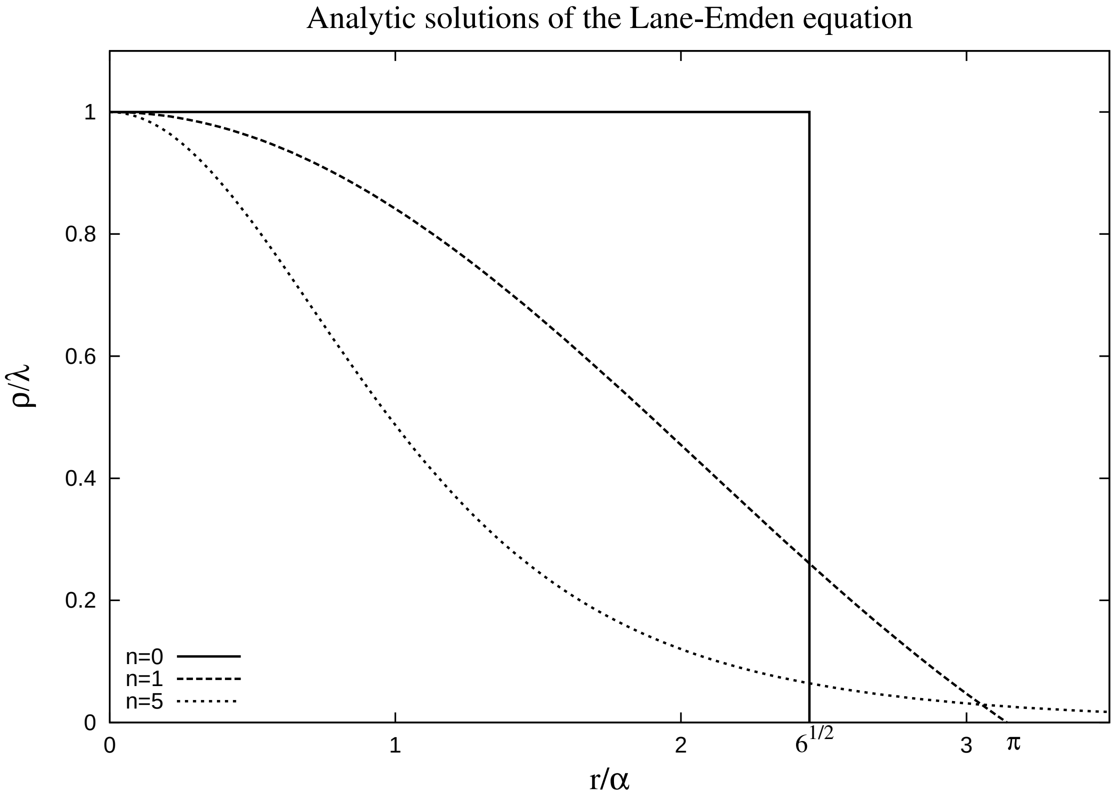

Figure 41: Analytic solutions for the Lane-Emden equation with $n = 0, 1, 5$.

Analytic solution for n=5.
The solution of Lane-Emden equation with $n = 5$ is analytically tractable, yet quite complicated to integrate. The solution is

$$\theta_5(\xi) = \frac{1}{\sqrt{1 + \frac{1}{3}\xi^2}}. \tag{484}$$

This configuration is unbounded: $\xi \in [0, \infty)$, and $\lim_{\xi\to\infty} \theta_5 = 0$.
[For explicit derivation, see S. Chandrasekhar's *An Introduction to the Study of Stellar Structure* (University of Chicago Press, Chicago, 1939), p. 93-94]

The Chandrasekhar Mass Limit

Consider a star which has, through gravitational contraction, become so dense that it is supported by a completely degenerate, extreme relativistic electron gas (*i.e*, $\rho > 10^7$ g cm^{-3}). The pressure in terms of the density is obtained by combining the eq. (436)

$$P = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} n^{4/3} \tag{485}$$

and

$$n = \frac{\rho}{m_H \bar{\mu}}, \tag{486}$$

to obtain

$$\begin{aligned} P &= \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{\rho}{m_H\bar{\mu}_e}\right)^{4/3} \\ &= \frac{\left(6.63\times10^{-27}\text{ erg s}\right)\left(3\times10^{10}\ \frac{\text{cm}}{\text{s}}\right)}{8}\left(\frac{3}{\pi}\right)^{1/3}\frac{1}{\left(1.67\times10^{-24}\text{ g}\right)^{4/3}}\left(\frac{\rho}{\bar{\mu}_e}\right)^{4/3} \\ \Longrightarrow\qquad P &= 1.24\times10^{15}\left(\frac{\rho}{\bar{\mu}_e}\right)^{4/3}, \end{aligned} \tag{487}$$

which is an equation of state for a polytrope with $\gamma = 4/3$ and $\kappa = \frac{1.24\times10^{15}}{\bar{\mu}_e^{4/3}}$. Corresponding value of the index $n = \frac{1}{\gamma-1}$ is $n = 3$.

The mass corresponding to this polytropic configuration can be computed as follows:

$$\begin{aligned} M_3 &= \int_0^{r_{\max}}\rho(r)d^3r = 4\pi\int_0^{r_{\max}}\lambda\rho(r)r^2dr = 4\pi\int_0^{\xi_{\max}}\lambda\theta^3(\alpha\xi)^3d(\alpha\xi) \\ &= 4\pi\lambda\alpha^3\int_0^{\xi_{\max}}\left[-\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)\right]d\xi \\ &= 4\pi\lambda\alpha^3\left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi_{\max}}, \end{aligned} \tag{488}$$

where we have used the Lane-Emden equation in eq. (471). The constant λ is defined in eq. (470), and for $n = 3$ is

$$\begin{aligned} \alpha &= \sqrt{\frac{n+1}{4\pi G}\kappa\lambda^{\frac{1-n}{n}}} \qquad\Longrightarrow\qquad \alpha = \sqrt{\frac{\kappa}{\pi G}\kappa\lambda^{\frac{-2}{3}}} \\ \Longrightarrow\qquad \lambda\alpha^3 &= \lambda\left[\frac{\kappa}{\pi G}\lambda^{\frac{-2}{3}}\right]^{3/2} = \left[\frac{\kappa}{\pi G}\right]^{3/2}. \end{aligned} \tag{489}$$

The term in brackets can be evaluated numerically (Table 4.2 of *Astrophysics I: Stars* by Bowers & Deeming) to about 2.02, so the total mass is

$$\begin{aligned} M_3 &= 4\pi\left[\frac{\frac{1.24\times10^{15}}{\bar{\mu}_e^{4/3}}}{\pi(6.67\times10^{-8})}\right]^{3/2}2.02 = 4\pi\left[\frac{1.24\times10^{15}}{\pi(6.67\times10^{-8})}\right]^{3/2}\frac{2.02}{\bar{\mu}_e^2} \\ &= \frac{1.16\times10^{34}}{\bar{\mu}_e^2}\text{ g} = \frac{1.16\times10^{34}}{\bar{\mu}_e^2}\frac{M_\odot}{1.99\times10^{33}} \\ \Longrightarrow\qquad M_3 &= \frac{5.81}{\bar{\mu}_e^2}M_\odot. \end{aligned} \tag{490}$$

Let us now compute $\bar{\mu}_e$ for a star with relativistic matter degeneracy. In such a star, it is convenient to define the matter density, due essentially to the ions, as $\rho = m_H\mu_e n_e$. Also, let us consider contribution from hydrogen (subscript H), helium (He) and elements with atomic weight greater then 4 (Z). Then, from the definition in eq. (439), we have

$$\begin{aligned} \frac{1}{\bar{\mu}_e} &= \sum_i\frac{m_H}{m_e}\bar{n}_i^e = \frac{m_H}{m_e}\sum_i\bar{n}_i^e = \frac{m_H}{m_e}\left[\frac{\rho_{\rm H}^e}{\rho}+\frac{\rho_{\rm He}^e}{\rho}+\frac{\rho_{\rm Z}^e}{\rho}\right] \\ &= \frac{m_{\rm H}n_{\rm H}}{\rho}+\frac{2\frac{m_{\rm H}}{m_{\rm He}}m_{\rm He}n_{\rm He}}{\rho}+\frac{\frac{A}{2}\frac{m_{\rm H}}{m_{\rm Z}}m_{\rm Z}n_{\rm Z}}{\rho} = \frac{\rho_{\rm H}}{\rho}+\frac{2}{4}\frac{\rho_{\rm He}}{\rho}+\frac{A}{2A}\frac{\rho_{\rm Z}}{\rho} \\ &\equiv X+\frac{1}{2}Y+\frac{1}{2}Z. \end{aligned} \tag{491}$$

Also, conservation of mass imposes that

$$X + Y + Z = 1 \qquad \Longrightarrow \qquad Z = 1 - X - Y \tag{492}$$

so

$$\begin{aligned} \frac{1}{\bar{\mu}_e} &= X + \frac{1}{2}Y + \frac{1}{2}(1 - X - Y) = \frac{1}{2}X + \frac{1}{2} = \frac{1+X}{2} \\ \Longrightarrow \qquad \frac{1}{\bar{\mu}_e} &= \frac{1+X}{2} \qquad \Longrightarrow \qquad \bar{\mu}_e = \frac{2}{1+X}. \end{aligned} \tag{493}$$

The stars that are undergoing extreme relativistic degeneracy of matter are highly evolved (near the end of their life-cycle), which means that it is reasonable to assume that most of their hydrogen fuel has been burned up, so

$$X \approx 0 \qquad \Longrightarrow \qquad \bar{\mu}_e \approx 2. \tag{494}$$

Finally, we combine this result with the eq. (490) to obtain the *Chandrasekhar mass limit*:

$$M_{\rm Ch} = \frac{5.81}{\bar{\mu}_e^2} M_\odot = \frac{5.81}{2^2} M_\odot \qquad \Longrightarrow \qquad M_{\rm Ch} = 1.45 M_\odot. \tag{495}$$

When a star runs out of fuel, it will explode into a supernova or a helium flash (see Fig. 16). The Schwarzschild mass limit implies that star remnants with mass $M > M_{\rm Ch}$ cannot be supported by electron degeneracy and therefore will collapse further into a neutron star or a black hole.