16 Lecture 16: Light Element Abundances and Recombination

“We are just an advanced breed of monkeys on a minor planet of a very average star. But we can understand the Universe. That makes us something very special.”

Stephen Hawking

The Big Picture: Today we continue our exposition of the Big Bang Nucleosynthesis, by discussing the abundances of light elements. We also discuss the recombination epoch of the Universe, when the first atoms began to form, and the Universe became opaque.

Review of Processes Leading Up to the Big Bang Nucleosynthesis

In order to understand which processes were taking place early in the Universe, we need to compute the reaction rates and compare them to the rate of the expansion of the Universe $H$. Earlier, we have found that the expansion rate $H$ and the neutron-proton conversion rates due to weak reactions $\lambda_{np}$ become equal at $T \approx 0.68$ MeV [eq. (313)], and that the neutron decay rate $\tau_n^{-1}$ becomes equal to $\lambda_{np}$ at $T \approx 0.16$ MeV [eq. (314)]. The remaining equality between the neutron decay $\tau_n^{-1}$ and the expansion $H$ is easily found by solving:

$$\tau_n^{-1} = H(x) = \tilde{H}(x = 1)x^{-2} \implies x = \sqrt{\tilde{H}(x = 1)\tau_n} = \sqrt{(0.632 s^{-1})(886.7 s)} = 23.64$$

$$\implies T = Q/x = 1.293/23.64 \text{ MeV}$$

$$\implies T = 0.055 \text{ MeV}.$$  \hspace{1cm} (321)

<table>
<thead>
<tr>
<th>Temp.</th>
<th>Description</th>
<th>Reactions</th>
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<tbody>
<tr>
<td>&gt; 1 MeV</td>
<td>weak reactions on the right maintain the</td>
<td>$p + e^- \leftrightarrow n + \nu_e$</td>
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<td></td>
<td>neutron-nucleon ratio in thermal equilibrium</td>
<td>$p + \bar{\nu} \leftrightarrow n + e^+$</td>
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<tr>
<td>$\approx 0.68$</td>
<td>weak reaction rates $\lambda_{np}$ become slower</td>
<td>$n \rightarrow p + e^- + \bar{\nu}$</td>
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<tr>
<td>MeV</td>
<td>than expansion $H$; neutron-nucleon rate</td>
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<td></td>
<td>eventually “freezes out” at $\approx 0.15$</td>
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<tr>
<td>$\approx 0.16$</td>
<td>neutron decay rate $\tau_n^{-1}$ is equal to</td>
<td>$D + n \rightarrow ^3\text{H} + \gamma$</td>
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<tr>
<td>MeV</td>
<td>weak reactions rate $\lambda_{np}$</td>
<td>$^3\text{H} + p \rightarrow ^4\text{He} + \gamma$</td>
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<tr>
<td>$\approx 0.055$</td>
<td>neutron decay rate $\tau_n^{-1}$ is equal to</td>
<td>$D + D \rightarrow ^3\text{He} + n$</td>
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<tr>
<td>MeV</td>
<td>the expansion $H$</td>
<td>$^3\text{He} + n \rightarrow ^4\text{He} + \gamma$</td>
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<tr>
<td>$&lt; 0.1$ MeV</td>
<td>the only reaction that appreciably changes the</td>
<td>$D + D \rightarrow ^3\text{He} + p$</td>
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<tr>
<td></td>
<td>number of neutrons is neutron decay ($\tau_n = 886.7$ s)</td>
<td>$^3\text{He} + D \rightarrow ^4\text{He} + n$</td>
</tr>
<tr>
<td>$\approx 0.07$</td>
<td>deuterium nuclei production begins (BBN starts)</td>
<td>$D + D \rightarrow ^4\text{He} + \gamma$</td>
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<tr>
<td>MeV</td>
<td></td>
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<tr>
<td>$\approx 0.07$</td>
<td>helium nuclei production begins (with photon</td>
<td></td>
</tr>
<tr>
<td>MeV</td>
<td>emission); these reactions are slower because of</td>
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<td></td>
<td>the abundance of photons</td>
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<tr>
<td>$\approx 0.07$</td>
<td>helium nuclei production begins (without photon</td>
<td></td>
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<tr>
<td>MeV</td>
<td>emission)</td>
<td></td>
</tr>
<tr>
<td>$&lt; 0.05$ MeV</td>
<td>helium nuclei production finishes</td>
<td></td>
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<tr>
<td></td>
<td>(electrostatic repulsion of nuclei of $D$ causes</td>
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<td>it to stop); most neutrons in the Universe end</td>
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<td>up in $^4\text{He}$ nuclei</td>
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<tr>
<td>$&lt; 0.01$ MeV</td>
<td>deuterium nuclei abundance “freezes out” at $\approx 10^{-4} - 10^{-5}$</td>
<td>$D + D \rightarrow ^4\text{He} + \gamma$</td>
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</table>


Figure 26: Rates of reaction between protons and neutrons in the early Universe, compared to the relative abundance of elements. $\lambda_n$ is the rate of reactions $p + l \leftrightarrow n + l$; $\tau_c^{-1}$ is the rate of neutron decay; and $H$ is the expansion of the Universe (top line is before and bottom after $e^-/e^+$ annihilation.)
Light Element Abundances

Nuclei of light elements are produced as the temperature of the Universe drops below $T = T_{\text{nuc}}$. The first to be produced are the nucleons of the deuterium, via the reaction $p + n \rightarrow D + \gamma$. If the Universe stayed in equilibrium, all neutrons and protons would form deuterium, which means that the equilibrium deuterium abundance is on the order of baryon abundance. From the eq. (291), we can see that the equilibrium deuterium-baryon ratio is of order unity when:

$$\frac{n_D}{n_b} \approx 6.77 \eta_b \left( \frac{T_{\text{nuc}}}{m_p} \right)^{3/2} e^{B_D/T_{\text{nuc}}} = 1$$

$$\implies \ln(6.77 \eta_b) + \frac{3}{2} \ln \left( \frac{T_{\text{nuc}}}{m_p} \right) \approx - \frac{B_D}{T_{\text{nuc}}} \implies T_{\text{nuc}} \approx 0.07 \text{ MeV}. \quad (322)$$

The binding energy of helium is larger than that of deuterium, which is why the factor $e^{B/T}$ in eq. (291) favors production of helium over deuterium. As can be seen from Fig. 26, production of helium starts almost immediately after deuterium starts forming. According to the figure, virtually all neutrons at $T \approx T_{\text{nuc}}$ are turned into nuclei of $^4\text{He}$. There are two neutrons and two protons in a nucleus of $^4\text{He}$, which means that the final abundance of $^4\text{He}$ is equal to about a half of neutron abundance at the onset of nucleosynthesis ($T = T_{\text{nuc}}$). If we define a mass fraction

$$X_4 = \frac{4n_{^4\text{He}}}{n_b} = 2X_n(T_{\text{nuc}}) = 0.22,$$

where we have used eq. (320): $X_n(T_{\text{nuc}}) = 0.11$. This approximates to the exact solution well:

$$Y_p = 0.2262 + 0.0135 \ln(\eta_n/10^{-10}). \quad (324)$$

One important feature of this exact result is that the dependence of the helium-baryon ratio has only a logarithmic dependence on the baryon fraction $\eta_b$. This means that the abundance of helium will not be a good probe in determining the baryon energy density $\Omega_b$. The value of the abundance of $^4\text{He}$ hinges on the presence of a hot radiation field which prevents the formation of deuterium before $T = 0.1 \text{ MeV}$. Therefore, the fact that presently most of the matter is in the form of hydrogen, i.e., not all the matter has transformed into $^4\text{He}$, is a strong argument for the existence of a primeval cosmic background radiation.

Figure 26 shows that a portion of the deuterium remains unprocessed into helium, because the reaction which does this $D + p \rightarrow ^3\text{He} + \gamma$ is not entirely efficient. It shows that the depletion of deuterium eventually “freezes out” at a level of order $10^{-5} - 10^{-4}$. The rate of this reaction depends on the baryon density: if there are plenty of baryons to interact, the reactions will proceed effectively; if the density of baryons is low, the depletion of deuterium will not be as effective. Therefore, abundance of deuterium is a powerful probe of the baryon density, as can be seen from Fig. 27. The measurements of primordial deuterium abundance show that the ratio of deuterium to hydrogen is $D/H = 3.0 \pm 0.4 \times 10^{-5}$, which corresponds to $\Omega_b h^2 = 0.0205 \pm 0.0018$.

BBN Summarized

The BBN lasted for only a few minutes (during the period when the Universe was from 3 to about 20 minutes old). After that, the temperature and density of the Universe fell below that which is required for nuclear fusion. The brevity of BBN is important because it prevented elements heavier than beryllium from forming while at the same time allowing unburned light elements, such as deuterium, to exist.
The key parameter which allows one to calculate the effects of BBN is the baryon-photon ratio $\eta_b$. This parameter corresponds to the temperature and density of the early Universe and allows one to determine the conditions under which nuclear fusion occurs. From this we can derive elemental abundances. Although $\eta_b$ is important in determining elemental abundances, the precise value makes little difference to the overall picture. Without major changes to the Big Bang theory itself, BBN will result in mass abundances of about 75% of H, about 25% $^4\text{He}$, about 0.01% of deuterium, trace (on the order of $10^{-10}$) amounts of lithium and beryllium, and no other heavy elements. Small amounts of $^7\text{Li}$ and $^7\text{Be}$ are produced through reactions:

$$^4\text{He} + ^3\text{H} \rightarrow ^7\text{Li} + \gamma$$
$$^4\text{He} + ^3\text{He} \rightarrow ^7\text{Be} + \gamma$$
$$^7\text{Be} + e^- \rightarrow ^7\text{Li} + \nu_e.$$ 

(325)

Heavier elements are not produced in significant amounts, since there are no stable nuclei for mass numbers $A = 5$ and $A = 8$. The BBN is completed when all neutrons present at $T = 0.1$ MeV ($X_n \approx 0.15$) have been converted into deuterium (only a small fraction) and $^4\text{He}$ (dominates).

That the observed abundances in the Universe are generally consistent with these abundance numbers is considered strong evidence for the Big Bang theory.

![Figure 27: Constraint on the baryon density from the BBN. Predictions are shown for the four light elements — $^4\text{He}$, deuterium (D), $^3\text{He}$ and lithium (Li). The boxes represent observations. There is only an upper limit on the primordial abundance of $^3\text{He}$. (Burles, Nollett & Turner 1999, astro-ph/9903300).](image)
Recombination

When the temperature of the Universe drops to about \( T \approx 1 \text{ eV} \), photons remain tightly coupled to electrons via Compton scattering and electrons to protons via Coulomb scattering. Even though this temperature is significantly below the binding energy of the hydrogen electron of \( \epsilon_0 = 13.6 \text{ eV} \), whenever a hydrogen atom is created, it is immediately ionized again by a high-energy photon. This delay is caused by the high photon-baryon ratio, and is similar to the delay we have seen in production of nuclei of light elements.

The Saha equation for the reaction which forms hydrogen atoms \( e^- + p \rightarrow H + \gamma \) is given by

\[
\frac{n_e n_p}{n_H} = \left( \frac{n_e}{n_H} \right)^{(0)} \left( \frac{n_p}{n_H} \right)^{(0)}.
\]  

(326)

The equation above is simplified when we realize that the Universe is neutral in charge, which means \( n_e = n_p \). We can now define a free electron fraction:

\[
X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H},
\]

(327)

and rewrite the left-hand side of the eq. (326) in terms of \( X_e \):

\[
\frac{n_e n_p}{n_H} = \frac{n_e n_p}{(n_e + n_H)^2} \left( \frac{n_e + n_H}{n_H} \right)^2 = X_e X_p (n_e + n_H) \frac{1}{1 - X_e} = \frac{X_e^2}{1 - X_e} (n_e + n_H).
\]

(328)

The right-hand side of the eq. (326) is obtained from the eq. (276):

\[
\frac{n_e n_p}{n_H} = \frac{g_e \left( \frac{m_e}{2\pi} T \right)^{3/2} e^{-m_e/T} g_p \left( \frac{m_p}{2\pi} T \right)^{3/2} e^{-m_p/T}}{g_H \left( \frac{m_H}{2\pi} T \right)^{3/2} e^{-m_H/T}} \quad m_H \approx m_p
\]

\[
= \left( \frac{g_e g_p}{g_H} \right) \left( \frac{m_e}{2\pi} T \right)^{3/2} e^{-(m_e + m_p - m_H)/T} = \left( \frac{m_e}{2\pi} T \right)^{3/2} e^{-\epsilon_0/T},
\]

(329)

where we have recognized that \( \epsilon_0 = m_e + m_p - m_H \). Saha equation then reads:

\[
\frac{X_e^2}{1 - X_e} = \left( \frac{m_e}{2\pi} T \right)^{3/2} e^{-\epsilon_0/T},
\]

(330)

If we neglect a relatively small number of helium atoms, and recall that \( n_e = n_p \), then the denominator in the equation above is \( n_e + n_H = n_p + n_H \approx n_b \). A good approximation of the baryon number density \( n_b \) is found by combining eqs. (276) and (286):

\[
n_b \equiv \eta_b n_g = \left[ 5.5 \times 10^{-10} \left( \frac{\Omega_b h^2}{0.02} \right) \right] \left[ \frac{T^3}{2 \pi^2} \right] \approx 10^{-10} T^3.
\]

(331)

This means that when the temperature of the Universe is of the order of \( \epsilon_0 = 13.6 \text{eV} \), the right-hand side of the eq. (330) is

\[
\text{RHS}(T = \epsilon_0) = 10^{10} \epsilon_0^{-3} \left( \frac{m_e \epsilon_0}{2\pi} \right)^{3/2} e^{-1} = 10^{10} \left( \frac{m_e}{\epsilon_0} \right)^{3/2} \left( \frac{1}{e(2\pi)^{3/2}} \right)
\]

\[
= 10^{10} \left( \frac{5.1 \times 10^2 \text{ eV}}{13.6 \text{ eV}} \right)^{3/2} 2.34 \times 10^{-2} \approx 1.7 \times 10^{15}.
\]

(332)
Since $X_e$ is, by definition $0 \leq X_e \leq 1$, the only way that the equality in eq. (330) can hold is if $X_e$ is very close to 1. From the definition of $X_e$, this means that $n_H = 0$, i.e., all hydrogen is ionized. When the temperature falls markedly below $\epsilon_0$, a significant amount of recombination takes place. As $X_e$ drops, the rate of recombination also drops, so the equilibrium can no longer be maintained. In order to track the number density of free electrons accurately, we, again, use the Boltzmann equation for annihilation, just as we did for the neutron-nucleon ratio.

For the reaction $e^- + p \rightarrow H + \gamma$ (1=e, 2=p, 3=H, 4=\gamma) Saha equation is given by:

$$a^{-3} \frac{d}{dt} (n_e a^3) = \langle \sigma v \rangle \left[ \frac{m_e T}{2\pi} \right]^{3/2} e^{-\epsilon_e/T} n_H - X_e^2 n_b$$

$$a^{-3} \frac{d}{dt} (X_e n_b a^3) = \langle \sigma v \rangle \left[ \frac{m_e T}{2\pi} \right]^{3/2} e^{-\epsilon_e/T} X_e^2 n_b$$

$$\Rightarrow \frac{dX_e}{dt} = \langle \sigma v \rangle \left[ (1 - X_e) \left( \frac{m_e T}{2\pi} \right) \right]^{3/2} e^{-\epsilon_e/T} - X_e^2 n_b$$

(333)

After defining the recombination rate $\alpha^{(2)}$ and the ionization rate $\beta$:

$$\alpha^{(2)} \equiv \langle \sigma v \rangle$$

$$\beta \equiv \langle \sigma v \rangle \left( \frac{m_e T}{2\pi} \right)^{3/2} e^{-\epsilon_e/T} = \alpha^{(2)} \left( \frac{m_e T}{2\pi} \right)^{3/2} e^{-\epsilon_e/T}$$

(334)

the differential equation for $X_e$ above can be rewritten as

$$\frac{dX_e}{dt} = \left[ (1 - X_e) \beta - \alpha^{(2)} X_e^2 n_b \right]$$

(335)

The superscript (2) in the recombination rate $\alpha^{(2)}$ denotes the $n = 2$ state of the electron. The ground state ($n = 1$) leads to production of an ionizing photon, which immediately ionizes another neutral atom, thus leading to zero net effect — no neutral atoms are formed this way. The only way for the recombination to proceed is by capturing an electron in one of the excited states of hydrogen. This rate is well-approximated by

$$\alpha^{(2)} = 9.78 \frac{\alpha^2}{m_e^2} \left( \frac{\epsilon_e}{T} \right)^{1/2} \ln \left( \frac{\epsilon_0}{T} \right)$$

(336)

The Saha approximation in eq. (330) is a good approximation to the electron-baryon ratio $X_e$ until it falls out of equilibrium. It even correctly predicts the onset of recombination. However, as we have seen earlier, Saha equation is not valid when equilibrium is not preserved. The correct description of the evolution of $X_e$ in the presence of reactions leading to the formation of neutral atoms is accurately described by the full Boltzmann equation given in eq. (335).

We present exact solutions and compare them to Saha equilibrium solutions as we continue our discussion next time.